Probability and Stochastics for finance-II

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Lecture – 08
Mean-Variance Portfolio Optimization (Part-III)

So, there is the part III on the mean variance portfolio optimization is a very important area of applications of optimization specifically Kuhn-Tucker theory.

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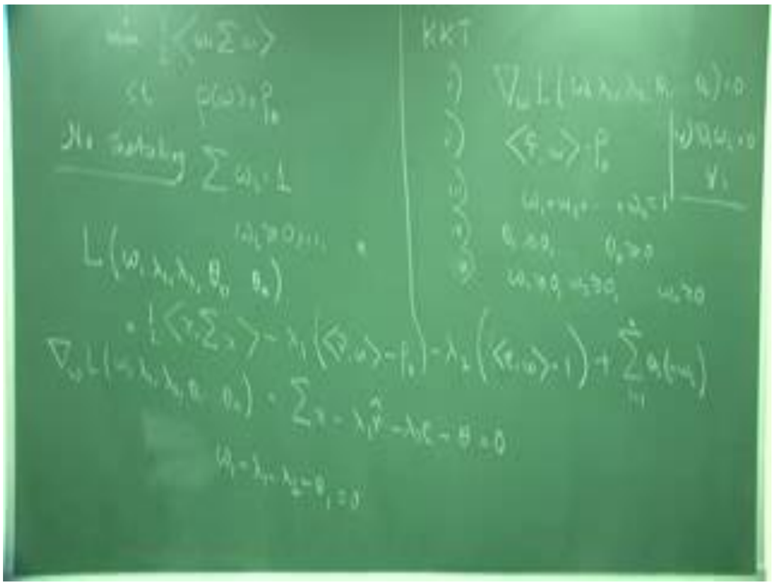

So, if I do not allow short selling then I just recall that my problem was of this form. Here the mathematics actually becomes very difficult. What essential is the Lagrangian in the case? And you have to take into account of the inequality constraints, n inequality constraints. So, constraints have increased from 2 to n plus 2.So, there is huge jump in the number of constraints.

And in this case writing the optimality condition is slightly nontrivial because your lagrangian would now have these lagrangian parameters and also would have a parameter which can a for the movement, let me write it as some theta 1 to theta n. And this theta 1 to theta n these pair of lagrangian would be associated with these n non negativity restrictions. And then this lagrangian in this case would become plus theta i minus omega i because I write it in the form of less than and equal to 0.So, that is your lagrangian function now here you change lagrangian function.

Here KKT conditions for this particular case, now becomes much more difficult. So, you have to first figure out what is the w which is giving you and so, this is one no short selling. So, once you do not allow short selling, the problem actually become very difficult you really a need machine to solve it. Once the data are given, you cannot solve it by hand even for small cases are not so easy doing this. Number two of course, you have to have this r hat omega equal to rho naught. Number three you have to have this as one. Number four you have to assure that these multipliers theta 1 to theta n, these are all greater than and equal to 0. Also you have to sure that and further there is another condition that would come which we do not write here as the complementary slackness condition that theta i rho equal to 0 for all i.

So solving this system of equalities and inequalities it is not a trivial matter. Even for the previous case uncorrelated case, the whole optimality condition would actually change right. So, it would actually change the whole system. So, I would suggest you to really go ahead and try out for the kip problem that we had just solved. So, if it is a short selling how would I actually go ahead and do that? So, it could be an exercise.

For example, now if you want to take the gradient a w. So, what will I get is the following. So, here I will get the vector theta back, so which is theta 1, theta 2, theta n and that would be equal to 0. So, if you look into what we have done in the last one, so what we are having is so if I write it down even with those uncorrelated aspects. So, if I go with the previous example in the last class then, what I will have for example, the first equation I have is lambda 1 minus lambda 2 minus theta 1. So, you have to have this and also should have theta 1 w 1 equal to 0. So, that is the way you are really going to figure out the stuff. So, it is not so trivial to figure this thing out and you would not have any time to figure the details. If there is a details I will hand it over to the student t a's with the solution of this. For this particular examples from Luenberger and I would also ask student who are listening to this to remind me that if I forgotten to put solution of this, at least show you how to analyze this situation.

So, once that is done. Let us now go into more looking at the problem more deeply right. So, will go and first we will take two problems.

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The one which I maximize the utility or minimize the loss. Or maybe I will go ahead with this problem without short selling. So, I have just held up the camera for a while and just wrote down this list of symbols. You might be wondering that I have suddenly it written an inverse of these this matrix. Positive semi definite matrix does not mean that it should be invertible. So, this Eigen values are nonnegative and it there could be a 0 Eigen value 2.

What we are now going to put in two important assumptions which are absolutely practical. Number A, assumption A says that this matrix variance covariance matrix is positive definite. The variance covariance matrix is positive definite. And second r hat is not multiple of e. So, if I write this in a more clear way then, this I should write this as this in equality constraint. So, what does this mean? If you take this function and if you take this function h 1 x, h 2 x, h 1 w h 2 w. So, the gradient here is r hat gradient here is e. If they are not multiples of each other they must be linearly independent. So, you are telling that the gradients are linearly independent.

Let us see what this condition tells me. It tells me that if these two are linear independent so, there are two constraints and they linearly independent and these are n dimensional chain. So, the maximum number of linear independent constraints in n dimensional of set up, in n dimensional space, in r n is n. So, here two of them are linearly independent. If n could be 2 also. So, that is fine, but n cannot be one right. Then this problem would be trivial. So, this condition says that n is here is always greater than or equal to 2. These are very very fundamental thing. So, you are to do something of meaning in the securities market. You need to know invest in at least two different aspects. And then what does this mean? This means that if it is not positive definite, but positive semi definite then, things could be different. Positive definite means so, matrices positive definite if this means that, but this can happen.

So, only if it is 0 this would be equal to 0. So, if you have non-zero allocation, this is always positive. But if it is just a positive semi definite, this would mean what? This would simply mean the following. This would mean that there are could be allocation omega hat such that they could be non-zero allocation omega hat sorry there could be a nonzero allocation omega hat such that this in a product is 0. So which means that if it is positive semi definite, there could exists an allocation omega hat, allocation of weights such the risk is 0. So, I will allocate in that way I do not care about that way. If risk is 0 and I am getting that reward. So, there could be w hat to satisfy these and they give me 0 things that is optimal. So, that is exactly the thing I will do it.

So, means there are could be the positive semi definite assumption also leave space for the existence of an allocation which is completely risk free 0, but in a securities market is no nothing is free of risk. So, this sort of things would never be allowed that you cannot have a risk free run in a securities market. That is why is much more naturalized these are positive definite matrices and every positive definite matrices is has positive Eigen values and determinant is the product of positive those Eigen values which is strictly greater than 0 and hence it is invertible.

Now once that is done, on an under these assumptions of course the Karush-Kuhn-Tucker condition holds naturally. And here what happens one can guaranty a solution and when this happens, this class of problems and now calls strictly or strongly convex problems and they have a unique solutions or the set of constraints here. They form what is called the closed convex set. If these term looks (Refer Time: 15:41) to you, please go and read some very basic optimization book.

So, you might be asking where is, your probability and stochastic coming in. See because already we have got into expectations and all those things. This essentially some sort of stochastic optimization, but we would also like to stress that within next 3 4 classes, we will get into hard core stochastic analysis and our aim is to go and prove stochastic approach to the Black-Scholes equations, Black-Scholes formula. To derive the Black-Scholes formula without re-coursing solution of a partial differential equation that is essentially the core.

So now, what we would do is that we would write down the result that if what is the optimal solution of this problem and that solution would be unique right. We will write down the result down and then we will try to analyze it. So now, let us write down what is the solution of the particular problem. So, here we consider only risky assets. We will talk about non risky assets to tomorrow in last classes, but we are essentially talking only about risky assets.

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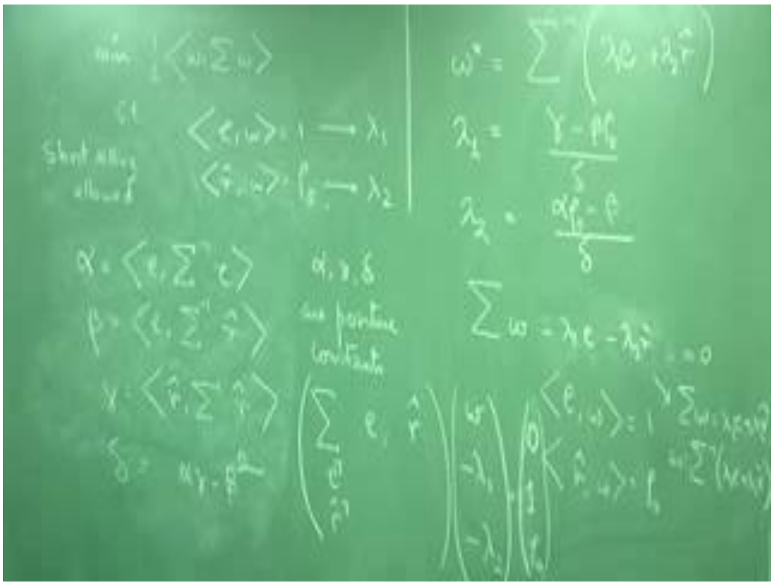

The first thing I want to tell you is that this, this and this. Alpha, gamma and delta are positive constants. And once that is you know that then, the solution of this problem is the following. Your solution which had as omega star is a unique allocation. So, under this situation your allocation would be unique, you cannot have any other choice right.

So, let us make that unique allocation. Of course, in the last example also you show that the allocation was unique even that rho bar of course, rho bar rho naught will of course, play a role. So, the allocation here is given as follows. So, what I will do that because just to save time I stop the camera write down the results then explain to you. So, here the solution w star would come to be this. We will see how it comes to be that lambda 1;

the first lagrangian multiplier comes to be this. Second lagrangian multiplier comes to be this, each at dependent on the choice rho naught that you have made. So, rho naught is be the parameter of the problem. So, we change that the solution changes. So, you plug in these values lambda 1 here and lambda 2 here, you get your unique solution. You put in here and then you take inverse of these matrices that is it. If it is uncorrelated metrics then it is much simpler right. It looks the w gets would nice.

Now how do I get the solution? Again going back to the Karush-Kuhn-Tucker conditions, the Karush-Kuhn-Tucker conditions would actually gave me this rho omega minus lambda 1 r hat minus lambda 2 e is 0. Of course, it is next equation is a e of omega is 1 and r hat omega is rho naught. Now I can put all this in the metrics form. If you look at it very carefully, I can put this in the metrics form and the metrics form, it looks like a partition metrics r hat e transpose and r hat transpose and here I would have w minus lambda 1 minus lambda 2. So, you are just clubbing all this in the metrics form and this would give me a 0 vector. I am not writing the 0 vector straight, so I am writing not given the 0 vector actually, first thing will give me the 0 vector then I will have the one and the rho naught. So, it will give me here 0 vector and then will give me 1 and then will be give me r naught n plus 2 vector.

So, now if you do that from the first equation it is obvious what you get. From the first equation is obvious that sigma w is lambda 1, I will just write I think I have written in the slightly different way. I have put the lagrangian I have put this as lambda 1 e minus lambda 2 r hat. It could be changed; the sequence can be changed this. I made I had just because in my writings I have just reversed this, I had actually this 1 e w equal to 1 and r hat w equal to rho naught. So, I have changed the lambda 1 will come respect to this and lambda 2 will come respect to this. So, lagrangian multiplier for this is lambda 1 and this is lambda 2. So, if you look at this, this is nothing, but lambda 1 e plus lambda 2 r hat and omega would be nothing, but sigma inverse lambda 1 e plus lambda 2 r hat.

So, it is your job now would be, once you know that to find the remaining that is all. So, you job would be find lambda 1 and lambda 2. Now how would I find the multipliers?

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So, I have got the omega. So, w so I have e transpose w is equal to 1 or e in a product to w is equal to 1 and I have r hat w is equal to rho naught. So, what does it give me? It will give me, if I write this it will have I will put w I have now got w star, so that I will put. So, it is sigma inverse lambda 1 e plus lambda 2 r hat that is 1. So, these are two components of a vector basically can be written in the form of metrics. And then I have r hat sigma inverse lambda 1 e plus lambda 2 r hat. This is rho naught. Actually if you club this whole thing in a metrics form then, one can easily write the following. That lambda 1 lambda 2 can be written as e transpose r hat transpose sigma inverse e r hat. This whole thing inverse because this is a metrics, please remember that into 1 rho naught.

So, once you do that once you can write this then, you can use these formulations. Actually you will see this formulation will come just let me take this. So, what will come here? e sigma inverse lambda 1 e plus sigma inverse lambda 2 r hat equal to 1. So, what I am getting? Lambda 1 e sigma inverse e right plus lambda 2 e sigma 2 r hat and that is equal to 1, but either you can write it in this form and immediately do it using the metrics thing which is a very compact form. So those who feel uncomfortable to put in this form just go and do it directly and write 2 equations. So, this is nothing, but alpha and this is nothing but beta. So, lambda 1 alpha plus lambda 2 beta is equal to 1.

So, similarly in the second case you will have lambda 1 beta and then if you look at this, it will give you lambda 2 gamma equal to rho naught. So, you just solve this out for lambda 1 and lambda 2. So, once you solve this out, you will immediately get this answers right. So, do not be worried. So, once you have done this, you are through. So you are now known how to actually handle the problem.

So, thank you very much for listening. So, tomorrow we are going to one of the next class. We are going to talk about how risk and reward actually interacts we will look at in a much deeper way.

Thank you very much.