Probability and Stochastics for finance-II

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Lecture - 07
Mean-Variance Portfolio Optimization (Part-II)

Here now want to formulate the mean variance portfolio optimization problem in the (Refer Time: 00:22) of Markov's.

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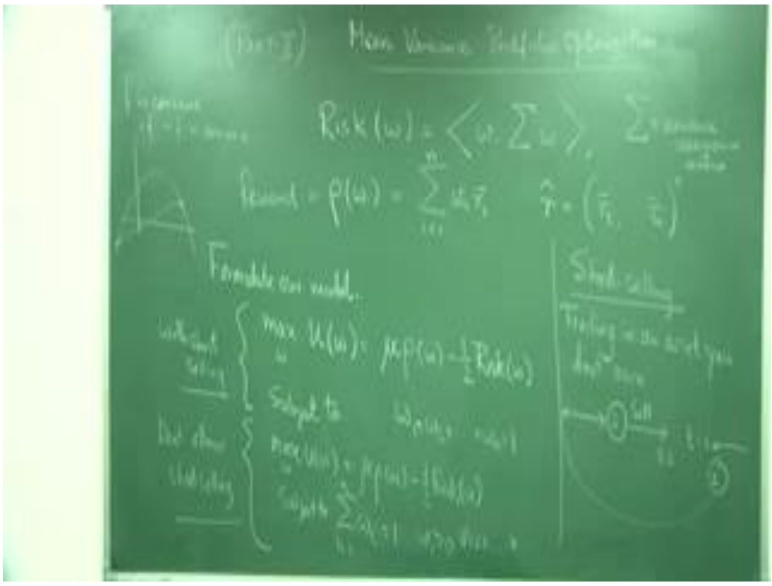

So, what Markov states we consider, that once you make an allocation w, the risk associated with w is the variance which we had computed yesterday, where omega this write as w is the allocation, is with the variance co-variance matrix. And the reward that is associated with the choice of omega, which we will symbolizes as rho x which we called reward. As this want you expected to get this expected return nothing else, is summation w i r i bar and often for the sake of convenience, rho sorry rho w is also written r bar so for the sake of convenience, we will write r hat is vector as r 1 bar r n bar instead of writing column vector, I am writing transpose to vector, is always meant to be a column vector.

Now how, so I know what is the risk, and I know what is the reward. So there must be trade of between the risk and rewarding. You cannot have lot of rewards without any risk. Now before I proceed further I need to make a comment. The comment is that and using terms like variance cohence matrix, and this is what I expect the students to know. In the sense that, if the students expect that every little thing about statistics that we use here, basics statistical tools has to be explain, then this course cannot be completed. These course for finance and hence people have to come knowing these basic things. Whatever details or extra things that was required, that was done in the first part of the course. To study quantitative finance or mathematical finance, but is important that, if you feel uncomfortable by such terms, you need to go back and looking do some books or some lectures on the web. So we are not going to go and to keep on explaining how this comes and why this comes and all these things.

So this is sigma. So do not confuse it with the, this some notation. So the once this is done. How do we formulate our module of, one is in terms of the utility? Or rather we define the utility as the trade of between the reward and the risk. So we write the utility my utility when I choose w, some sort of this is a trade of function, which some parameter mu, so this is something which I want. I want that this difference should be as large as possible. So what I want to do, I essentially want to maximize this utility function, subject to the fact that these vectors are equal to 1.

Now, here you might ask me the question. Are these not all summing up to 1, or these not all non-negative? The answers surprisingly are that it might not be non-negative always because here we keep in the idea of short selling. Short selling means that you are trading an asset which you do not own. So short selling very simply means, trading in an asset you do not own. Suppose what you do you take one asset from a broker, so you take an asset from a brokers, say the asset number 2 you have taken from the broker, what you did you sell in the market at whatever be the price right. So you sell at time t equal to 0, with some price. And then at time t equal to 1 you buy back that asset say asset number 2 and you return it to the main guy. So that is essentially the job so here, positive weight-age, means when your making in investment in the market we give positive variables. So you have some money and if you are making an initial investment that is view that a positive investment, if you are getting some money that is thought of as a negative investment. You are getting money you are not investing. So here it could be plus or minus anything. So you could be doing short selling. So if I do not want to a short selling so this is the standard model.

So, this is width short selling. So we will also use other form. So here we are trying to maximize your utility, or the trade of reward, and the risk, so one can also have multi objective to optimization view point which will not discuss. Another is what would happen if you do not allow short selling. If you are not allowing short selling, then you of course, have to know that you are making investments only you are not getting money. So all these w 1, w 2, w n has to be greater than 0. So your problem would now become subjective. See is a look at this, let us see what short of a problem it is. Or here he could have written minimization of the laws which is the risk minus the reward. That you also you could have done. Subject to summation w i equal to 1 i equal to 1 to n and w i is greater than equal to 0 for all i equal to 1 to n so. There 2 versions we shall just use this version for benefit for simplicity actually.

Now, is this only logical model? See this problem, if it has a minimum or maximum will have a global maximum because this is a linear function of w, rho w. And this is a convex function of w. So it is linear plus minus of the convex function. So minus of the convex function is called concave function. So f is concave, if minus f is convex. So concave function looks like this. If you take a function form r to r, that is if take any point 2 points r, can join them it at portion, the part of the graph line segment would lie below the graph. So that is and for convex was above the graph, so this is a concave function. So linear function is both convex and concave and this is a convex function negative is a concave function. So when we maximize the concave function over convex set, you have always a global maximizer. So this is one form of the utility one form of the portfolio minimization problem, using variance as the risk and the expected return as the reward.

Another important class of problem which we will now write down would ask us that, I simply own to minimize my risk. When and I want to get those omegas, sorry those ws, those allocations which will give me a certain level of expected return, and I want this expected return, and then one once I get that sort of expected return I am all though portfolio portfolios all the allocations, which will give me this level of expected return, I would like to know which minimizes my variance. So that is what was vary Markov model one form of the Markov, which model that we are now going to discuss.

So now, we will put in a second model, the second model is as in a following way.

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Now I have said that I only one those allocations which give me of fixed return rho naught. Of course, I know that is my allocations should satisfy the weights basically. So once I know that I can get rates of this form, and among those weights I want to know which minimizes my risk and that is exactly the allocation that I will make. So this the one where you allow short selling, and if you do not allow short selling, then the problem would now look like subject to here. I also should write subject to s t means, subject to or such that whatever you want to say rho of w is rho naught and summation w i is equal to 1.

Now, let us and of course, w i this is way no short selling. First of all, you have to understand as we have spoken about various types of optimization models in one-hour class. These 2 problems are optimization problems with have affine constraints. These this are a linear constraint there is no additional for, so these are all affine constraints so this problem are quadratic programming problem, q p with affine constraints. I would like to mention or result here which is usually known to optimizers, is that under this conditions any KKT point is a global minimizer.

So, the standard technique of writing the KKT point is to construct what is call the Lagrangian. So what I take this one only, look and do it later and there will be harder of course, so if I take this one, that affine constraint means that it is both equality and inequality. That is inequality plus equality. So I would now like to construct what is call the Lagrangian associated with my problem. So the Lagrangian means, the I would associate multiplied, this

Lagrangian multiplied rule is one which you have learned in high school or first year collage in calculus. So only mathematicians working optimization of do analyses to say that such multiplied would actually exist and the conditions would be true. So here I would have one constraint with this, and another constraint with so, I will have lambda 1 with this one and lambda 2 with other one, and then I will write down the things in the following way, is that, half of, so there is the way to write down the minus, if I look at this very carefully is nothing, but this rho w is nothing, but r hat omega, minus rho not, minus lambda 2. This can be written as e w minus 1.

So, what is e here, e here is the vector. So basically I have taken in the inner product with the vector w, w is in here w 1 w 2 w n. So that I am not writing I am sure that everybody understands this basic fact. So what essentially is the Karush Kunt and Taucket conditions in here, is first take off, so the KKT has a following form. So first take a gradient with the respect to l, and equal it to 0. And then take then of course, you need to satisfy this constraints, that is you first take with lambda 1 then take with lambda 2. So you have to satisfy the constraints that are it so this is the KKT system are you really need to solve.

Note that if I have uncorrelated assets, then this variance co variance matrix becomes a diagonal matrix consisting only of variance. So let me take an example now from Luenberger Investment Science. So my source has the example is Luenberger Investment Science. So the example consists of 3 uncorrelated assets. So n equal to 3, assets are uncorrelated. N equal to 3, assets are uncorrelated. And let the means, so each of variance 1, so sigma 1, so sigma 1 square is 1, sigma 2 square is 2, sigma 3 square is sorry 1, 1. And their r 1 vector is 1, r 2 mean is 1, 3, where is 1. So this data is given to me right? Now I want to find what is 1 optimal w. That is essentially what it is. That is my job.

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So, now we are going to write down this condition. See we are going to now write down this condition KKT. So you are taking the grad with respect to omegas. So what do I get? I get here half there the first condition sigma w, minus lambda 1, r hat minus lambda 2, e. That is equal to 0. Of course, first let me clarify these. So let me write sorry this is half is gone with, had I put half. If half was there, so in the derivative half was gone. So now, this is nothing, but sigma 1 square sigma 2 square omega 1 omega 2 omega 3. So here is the vector sigma 1 square omega w 1, sigma 2 square w 2, here even if I say omega in place of w, it is w. So I make the slip. So forgive me for this. Very difficult to correct it suddenly, unless I become conscious where maybe I should a started telling omega only, but does not matter. Now this is r 1 bar r 2 bar, r 3 bar; and this is 1, 1, 1. This is equal to the 3 dimensional 0 vectors.

So, once you have, now we would like to write then down terms of equations. Sigma 1 square this minus lambda 1 r 1 bar, minus lambda 2 is equal to 0. This is exactly or sigma 1 sigma 2, these are all 1, which is already given to you so I can just remove this part. So these all 1 which already given; what is r 2 bar? Sorry, here I have made a mistake, r 2 bar is given as 1 2, the data is given as 1 2 and 3. So r 1 bar is 1m r 2 bar is 2, 2 lambda 1 minus lambda 2 is equal to 0. And then I omega 3 minus 3 lambda 1 minus lambda 2 is equal to 0. These are the c equations. Because sorry this data mistake the data was 1 2 and 3. So these are examples which are worked out in worked out there, but I am giving the step by step explanation. So to omega that I found out here will satisfy the Karush Kunt other condition. That I will found out here will once it is satisfying the Karush Kunt other condition, I will, I am completely sure then, that this is my omega which I want to do the allocations, this is my solution.

Obviously now you will have another equation which is other second equation. So omega 1, 2, 2, w 2, 3, w 3, is equal to your expected level which is rho naught. And then you have w 1 w 2 plus w 3 is equal to 1. So how will you solve it find out w 1, from here w 2 from this equation, w 3 from this equation put it here. So once you do that you will get 2 equations 14 lambda 1, plus lambda 2 is equal to your rho naught. And 6 lambda 1, plus 3 lambda 2, could be equal to 1. So what you are doing w 1 is nothing, but lambda 1 plus lambda 2, you put it lambda 1 plus lambda 2, w 2 is 2 lambda 1 plus lambda 2, w 3 is 3 lambda 1. So just put it there and add and see this is the exactly what you will get. Because here will have lambda 2, sorry, they it cannot be like this, it has to be lambda 1. So lambda 2 is here 3 to 6 that becomes 6, because 3 lambda, that should be 3 lambda 2, 2 lambda 2 and 1 lambda 2 6 lambda. So this should be the equation and the next case; obviously, you will have lambda 1 lambda 2, lambda to will come 3 times, and this will come 6 times. Naturally, so these are the equations. So one this equation is done, then I can find out what is lambda and what is there is a lambda 1, and lambda 2. And that is exactly what I want to now write down. So here I have lambda 1, which is equal to, everything will come in terms of rho naught, the parameter you your expected you know return that you pretty want, this will be a lambda 1, and your lambda 2, would be 2 third minus rho naught.

So, once you have this, put in the lambda 1 lambda 2 here, and that will immediately give your omega 1 omega 2 and omega 3, so or w 1 w 2 and w 3. So I leave these things to your homework maybe a w 1 and w 2, w 1 is lambda 1, plus lambda 2 which would be lambda 1 plus lambda 2. So you should become rho 0, 2 rho 0 2 minus 1 plus, 2 third minus rho 0. Wait, you see this is what is coming. So fourth third, I think so sorry there was a small mistake in my calculation, standing in the calculation lambda 1 is all right, lambda 2 is not 2 third 7, 7 by third minus rho naught. So then your omega 1 is not lambda plus lambda 2. So lambda 1 plus lambda 2 means, this this will give me 4 third, so this minus this 4 third, and this is minus rho not by 2. Similarly, I leave you to find out the rest, as homework. So this is a very simple situation, uncorrelated situation. What about the general situation?

In our next lecture we are going to talk about the general situation, and this portfolio optimization would essentially now consist of 4 lectures, and a last one we have to look in more detail about interaction between the risk and the reward. And then will talk about the capital asset pricing model. So see you in the next class.