Probability and Stochastics for finance-II
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Lecture – 04
Optimization Models in Finance

Welcome once again to the fourth lecture of the first week. It rains outside it is really interesting thing to give lecture. And this time inside my own office, you must be observing the difference between the lengths of the boards compared to the first two lectures, we showed in a studio and here it is in my own office at the Indian Institute of Technology Kanpur. So, here is the topic of today's lecture optimization models in finance. Now optimization is the science or than mathematics of choosing the best among a given amount of things by following certain principles or laws or based on certain conditions. So, in financial models where risk is involved, we tend to use optimization two models, our problems.

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So, let us first write down what is the basic optimization model. So, it is I am writing it in the most general mathematical form. So, by an optimization model, we usually mean either to minimize or maximize I am more habituated with writing minimize because maximize is nothing, but minus of min of minus f x. So, maximize over subject to means, we will be our discussion considering the case were x belongs to a final dimensional space or n. And let us tell you what is this x and C and all those things. So, x of course it could belongs to vector space, it could be infinite dimensions. In many situations it is in infinite dimension especially, in the more deeper analysis of risk which is whose answer is such we search us very recently in the last 10 years. And here we will be logic answer were x is in R n is a vector in R n. And x is called the decision variable. This is this x is what we have to find out.

This set C which is a subset of R n is called as feasible set that is consideration of any x which is outside C is not really relevant to this problem. And f is called the objective function, a functional value there you tend one to minimize. Of course, you have heard minimization maximization you have done such problems in your first year calculus courses or in high secondary classes. But here we look into to the subject optimization standard on mathematical discipline which we analyze in a more detailed way. So, here is the basic structure of an optimization problem.

Now if C is equal to R n there is a whole space, we say and the problem is unconstraint that is exactly what you are learn in your high school an where high school C is equal to R actually n is 1 and in so, first year if can learn what n R n is 2, 3 like this. And when we C is the proper subset of R n that it C is a subset of R n but C is not equal R n, then the problem is a constrained problem. Constrained problem are of course harder. And in finance we have to deal essentially with constraint problems. Of course, here C is just an abstract set. It does not allow us to do much operation on C. So, in general C has a more detailed form.

So, that is what provides me with standard form of an optimization problem. Our aim here today is after we discusses a standard form that we essentially tell you about some important class of optimization problem and show two examples from finance how we can actually model the thing using optimization. So, let us observe this. That now let me write down C as a set of all x in some X which is subset of R n such that g i x is less than equal to 0 for i is equal to 1 to m and h j x, so these of functions. So, I only considered an x a vector x in R n to be in set C if it belongs to this given set x and also satisfying all these inequalities and equalities are simultaneously.

So, now the problem is this problem which I can now write as a problem P. So, the problem p in a standard form would look like minimize f of x subject to g i x less than equal to 0, i going from 1, 2 m. And h j x equal to 0, j going from 1, 2 k and x belonging to X. So, this is one of the most standard ways of writing an optimization a general optimization problem. So, these g i x these are called inequality constraints h j x are called equality constraint and this is called an abstract constraint inequality constraints and these are called equality constraints. And this is called an abstract constraint. Of course, you can say that you have not got rid-off this abstractness of C; you have plucked in that x. In most our applications is abstract constraint x would be R n plus. So, which would simply mean that x is a vector whose all components are non-negative sorry abstract constraint

So, one that is done, I has not to add that now I will start discussing various important class of optimization problem like linear problems, linear programming problems, linear optimization problems, linear quadratic optimization problem under linear constraints and the quadratic optimization problems under quadratic constraint. So, the type of problem that will arises in finance and also some class of problems very hard actually for integer programming problems. So, here if I, let us look into the whole thing. So, here by choosing very particular type of f right, very particular type g, very particular type of h, very particular type of x, we go to certain special classes of problems.

The type of problem which the finance world usually deals is called a convex optimization problem. The convex optimization problem arises when f is convex g i’s are convex h j affine and x is usually a close convex set.

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So, in this case on the problem P which now our standard problem, will now call this as P forget the other thing. We will be essentially dealing only with this structure. So, for this case, so will have f and g i to be convex of course, it goes without saying that both f and g I, if you a functions from R n to R. That is the statement I had not made in the last statement in the last description of the general problem. But f was there a function of R into R. Of course, because if you do not have the real values what are going to compare. I here assume that you know the difference between a local minimizer and global minimizer. So, those who do not know the difference between a local minimizer and global minimizer, just Google it and in Wikipedia and check the definitions or you have huge number of you know stock on optimization on the internet. So, you can really know the difference and I would not like to repeat the difference because I would rather go into the issues in finance.

So, here these a e convex. So, I am just writing in general a function phi from R n to R is convex if for any x, y in R n, phi of lambda x plus 1 minus lambda y is less than equal to lambda phi x plus 1 minus lambda phi y for all lambda belonging to 0,1. This essentially says that if you draw the curve or the graph of function, if just take the function from R to R and you just draw the graph of the function should look like this which essentially says that you will shade the area of a graph which is called the epic graph.

The epic graph must be convex; a function is convex if and only if it is epic graph is a convex set. So a convex set is the set where you take any two points and joined them by the line segment. Any two points on the set and joined them by the line segment that line segment lies inside the set. So, will not going to this detail it, this means that if you take this straight line a this graph of a convex function and if you join them by when if you take two points on the graph of the convex function, if you join them by a cord and this cord always lies over the graph of the convex function. So, this is the very very important idea very very important geometrical idea this is essentially geometrical idea.

Now, important examples of convex functions are phi x is affine or linear say a of x where a is given R n. So, this is called a linear function. So, linear functions I am sure most people would know what linear function that it a function f is linear if a for x 1 plus x 2 is equal to f of x 1 plus f of x 2 and f of alpha x is alpha of f x for any alpha. So, this a x is actually the standard dot product which simply means if you on to a dot dot dot a n is the components of the vector a n x 1, x 2, x n is the components of the vector x. And then this is nothing, but this quantity which is nothing, but extension of the notion of multiplication.

So, another class, so if you add another real number to this linear function then what you get is called an affine function. So, b here is the n R and what you get is an affine function. Another is these two functions are also these are all examples of convex functions, special class of convex functions. So, if the function f is convex g i's are convex h j h x are affine. These equality constraints have to be affine and this is a closed convex set or just a convex set, then we call this problem as a convex of optimization problem.

So, here for example, another important class of convex functions is that of quadratic of x Q x. So, this is the quadratic form and this Q must be positive semi definite. Whereas this quadratic form must be always greater than is equal to 0 for any x in or in that you choose, must be positive semi definite. Of course, in this just one or two lectures you cannot do justice to a huge on a subject of optimization, but this is just give you a basic idea because all these type of class of functions will start playing role in finance specifically the quadratic one. Because this will in the Markowitz which portfolio optimization, this will is a, this quadratic form of would be represent risk. Because variance of random variable can be expressed in terms of a quadratic form and that would be convex. And now we will write about linear programming problems will also write about quadratic programming problems.

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So, we are now going to start talking to our linear programming problems, linear optimization problems. There are lots of books which deal with linear optimization problem where we minimize C x which is nothing, but C1x1 C is in R n of course. Subject to a i x minus b i is greater than equal to 0. So, it is equality constraints of the greater than equal to type. You can represent it in the same way, you can put b i minus a x is less than or equal to 0, i equal to say 1 to m and each of x i is greater than equal to 0. This is as same as writing x element of R n plus.

So, this is the standard the canonical form of a linear programming problem. This is what is called a linear programming problem, sometime it an as LPP linear programming problem. So, these are the important class of problem and in finance in many cases, but this is quite a standard thing which will appear. It may or may not appear always but will appear in many cases. That could be another case where you can write this as minimize C x subject to a i x or you can just a i x greater than equal to b i, if you feel that is more comfortable or looks nice rather than writing this in a, it might look a bit clumsy books usually would like to this way. And then you can say the x i may not be all greater than equal to 0, it could be just integer 0 or 1. Or it could be that x i belong to z plus that is only belongs to positive integers or non-negative integers. So, this is I am just giving is not that is always either this or this or it could be x i's could be an integer. So, this sort of problems sometimes is called integer linear programming problems or linear programming problem.

Let me tell you that the linear programming problem is the convex programming problem while the integer linear programming problem is not a convex programming problem. Why convex optimization problems are important? Because for a convex optimization problem, every local minimizer is a global minimizer and we are searching for global minimizer. But for non-convex problem local minimizer did not be global minimizer and that is the major issue. So, this is probable instead of writing either linear programming problems, this is what will called as integer programming problems.

Now we come to 2 class of quadratic programming problem. So, the portfolio optimization problem that will start in the next to next lecture would essentially consist of quadratic programming problems. So, the class the type one is quadratic programming problem under this sort of affine constraint or linear constraint whatever you want to say. So, you minimize a quadratic convex function Q is positive semi definite plus an affine things C x plus d subject to the same thing. Here x i greater than 0, I would of course, go from 1 to m or other I should write this a j x may be that is better. So, that you do not confuse among the index. So, this is called a quadratic programming problem under affine constraints. In short QP quadratic programming under affine constraints and this is portfolio optimization problem is actually of this form.

So, these are the main classes of type of problem that will essentially deal in finance. You could have also these replaced by quadratic constraints. Constraints of this form is less than is equal to 0, but still I do not really bothered much because we will not be able have the occasion to discuss such issues. So, for our discussion this is enough, this is this problems are enough. And now we are going to show examples to how to model problems in finance with these sorts of classes of optimization problems. So, if Q here is positive semi definite, then this programming problem is the convex programming problems or it is not if Q is not a positives semi definite matrix.

So, here so we will be looking at the capital allocation problem. So, capital allocation problem is divided into two parts. One is called the capital budgeting problem. So, basically suppose capital budgeting is that you are funding agency holding a lot of fund a given of amount of fund and that you have to say fund to scientific projects. So, how will you find? How much you will allocate to which project? Which project would actually be given any funding? Which project would not be given any funding? So, that problem is called that capital budgeting problem. And the second one is called the portfolio problem which essentially his link to the sharing I mean the sharing put an allocating your given amount of money for buying the shares of various stocks or various financial instruments available in the financial market. So, when portfolio optimal problem is capital location link to portfolio security or not always securities, but it could be an imminent link to risky securities, but could be link to bonds etcetera.

So, portfolio optimization problem link to securities what we will study in detailed in next lecture. We will have two lectures on portfolio optimization compared to the capital budgeting which really does not, if look at traded assets. Capital budgeting problem is allocation of a budget between non-traded assets just like you know allocating budgets to scientific projects. So, which are our scientific organizations like DST etcetera keep on doing from time to time.

So, let me now give you an example of the capital budgeting model and let us see you what happens. So, here now suppose I am a scientific comedian I have m budgets to myself, m budgets been presented before myself I am in the funder. I am going to now decide what I am going to allocate. So, here we will analyze and show that such a problem of allocation of funds forces scientific problems can be modeled as a integer programming problem of this type.

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So, now we are going to talk about fund allocation to projects. So, now the m projects i is equal to 1, 2 m and each projects has a benefit b i. b i is usually the net present value. What is b i? b i benefit of the project i. So, what is the meaning of the benefit of the project i? It means every project would have cash flow stream associated with it. So, I have an initial budget which would be a negative cash flow my side and then it will be re-benefits which will be a positive cash flow. So, this difference between the positive cash flow the present worth of the cash flow minus the present worth of the negative cash flows which is the negative cash flows itself is called the net present value of any project.

So, that difference called b i which basically this is nothing, but the net present value or n P v; let us C i be the initial cost of a single project and I have a budget allocation. So, C is my total budget and C i with the initial cost to start project i. So, that is what I am supposed to pay to project i if I have to start. Now what is my objective? I want to have a scenario where my benefits are maximum I only give it to those projects where I have to maximum benefits. In fact, a government does this on a regular basis, this capital budgeting. So, my problem would be to maximize benefit. So, x i is the variable, now suppose x i is the variable which denotes either o or 1. So if x i is equal to, so x i is a 0 1 variables, 0 or 1. So, if x i is 0 means I do not give any money to project i. If x i is 1 then I give the money to C i project i.

So, my problem now is to solve this. I want to maximize my benefit. So, if I give is the project i selected then it gives in a benefit b i, if the project i not selected it gives me a benefit 0. Subject to the fact, if project i selected, I have to pay C i. If project i not selected I have to pay I do not have to pay anything. So, you can C i x i, but this cost cannot be exceeding C and of course, x i is 0 or 1. So, this is what is called a capital budgeting model. It is in the form of integer programming problem.

Now we are looking into portfolio selection problem on portfolio of bonds. We will not moving into portfolio security because that the Markowitz model of mean variance portfolio analysis would be dealt in details. So, we will not to speak about this right here. So, portfolio of bonds that is what how much bond I have to buy, in order for me to trade them buy so that in order to me to generate the funds with respect to which I can carry out my financial obligations. Suppose I have a pension fund. And every 6 months I have to make certain payment. So, let my payment vector, so I have to make that payment for say n period of time. So, these are the cash flows will that I will go from me. So, starting from now in the first 6 month I will pay y 1 then I will pay y 2, then the next 6 months then y 3, y 4 and y n till some given period. So, that is my promise.

So, the pension fund I will make this amount of payment right. It could be the one percent or to aggregate of the percent does not matter. So, how will I actually go about keeping my financial obligation or keeping the promise of the financial obligations? I will go and buy bonds. A bond with coupon or without coupon does not matter, but bond with coupon is essentially fine. So, I will go and buy some bonds and I will go and buy some m bonds. So, buy m bonds which I index by j, j would denote a bond. So, there are m bonds in the market and buy them. So, each bond j gives me a cash flow, say at the 6 months from now. This is giving me C 1 j, from further 6 months is giving me C 2 j and C n j. So, this is my cash flow be given by the bond C j. My question would be what is the number x j of bonds I should buy? So, what is the number of the bonds x j, I need to buy? That is exactly the question that I really have to answer that is issue.

Now, how will I get that number? Now p j is the amount of money, I have to pay to buy a unit of bond j. So, total money that I want to spend to buy the all my bonds is p j x j that j is equal to 1 to m. So, buy m bonds but, I want to minimize this, I do not want to pay so much. But this I have to do subject to the following, that C i j x j, j is equal to 1 to m must be greater than equal to i j and x j is an amount of bonds is greater than equal to 0, i is equal to 1 to m. I will explain this. So, what it means that a time set i equal to 1 right? i equal to 1 that is the first 6 months, from now the first time I am making the payment. So, there I have got money from all the m bonds.

So, from the first bond I have got money C. So, j is so what I have got? From the first bond I have got money C 1 1, from the second bond I have got a cash flows C 1 2, from the third bond I have got a cash flows C 1 3. This is the first one cash flow. The m-th bond I have got a cash flows C 1 m and each of them I have got x 1, x 2, x 3, x 4quantities. So, add them up. So, the in the first one, this is the amount of cash or this is the amount of cash I get from cash flows from all the bonds I have bought. And these cash flows must be at least equal to the amount my financial obligations and all these x j's that I buy must be greater than equal to 0. So, what we now get is absolutely a linear programming problem that we have studied.

So, here is the integer linear programming problem, here is the linear programming problem. The quadratic programming problem under affine constraints is exactly, what is a Markowitz portfolio model which we will study in detail later. In the next class we will talk about how to analyze optimization problem and what we can do with it. So, we will discuss that in the next lecture.

Thank you.