Probability and Stochastics for finance-II
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Lecture - 03
Term Structure of Interest rate

So, welcome back for the third lecture of the first week. We have been studying essentially interest rates. Today we are going to talk about term structure of interest rates so what is meant by term structure. Term structure means, that the behavior of interest rates or a given period of time that is for various levels of maturity dates, how would the interest rates behaves? We have been often seen in a market that if you take a bond and hold it for 10 years and if you take bond and hold it for 2 years the bond which you hold for 10 years give you substantially more interest rate then compares to the bond that you hold for 2 years so this idea this fundamental fact that, if you hold money for a longer period you get longer, you get more interest rate is the fundamental idea behind term structure.

So, for example, if you have taken fixed deposit for third years, you get a longer, you get a higher rate of interest or 10 years. For example, you get a higher rate of interest then you hold a fixed deposit for 3 years for example, you go to state bank of India hold a fixed deposit for 3 years you have lower rate of interest compare to is you hold for 10 years. For 10 years' money almost doubles.

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So, that has been fundamental idea that if you hold money for a longer duration for invest money, you have more you get more interest rate of it. I of course, I showed you that this is our last discussion on interest rates. Because and you might be wondering waiving to discuss so much our simple notions like interest rates. The idea is that interest rate is the most fundamental notion in a financial market. And that is we are studying it so interest rate is a fundamental notion and whatever you do, whatever computations you do for various financial instruments when you want to compute the price interest rate play of very fundamental role.

So, here today we are going to talk about 2 types of interest rate structures that is, 2 types. Now we are going to talk about 2 types of rates one is call the spot rate and another is call the forward rate. Now let me first talk about spot rate and what does it mean. Spot rate is interest quote that you get immediately that the moment you by a want to by a bond. And spot rates are usually evaluated by looking at 0 coupon bonds and especially government traded bonds, the government bonds which are not surely on default able.

So, the interest rate that you get on 0 coupon bonds usually banks could it yearly is called us spot rate, and that is call the spot rate so those in term structure theory, the spot rate is a very basic concept. That is the basic interest rate that you should have in a market so what is the spot rates so you come with some amount of money to buy a bond,

we have come with say some k rupees to buy a bond from me, then the interest rate quoted is the spot rate.

Now, how do I compute spot rates? That is very important so basically at t equal to time 0 time to equal t 0 invest the money, and at time is t, you want to take back what you have invested in the bond with the interest so t is equal to 0 you can quota spot rate s t, s0 and time t equal to t you will be essentially getting back money depended on that interest rates.

So, let us just look at the yearly quote so if I invest 1 dollar or say A dollars or rupees A, I have invested dollars are very common may has written in most books so rupees I have invested now with a spot rate s1 yearly spot rate s1 so 1 year might spot rate s1, then at the end of 1 year I, obviously, get this is exactly what you get. But now suppose I have been invested on if for 2 years then I quote a spot rate for 2 years not for 1 year, at that is very begin a quota spot rate for 2 years which I call s 2. And then the money I get by standard compute in the compounding, she is already knowing how to compute this thing so this is very important. So under the yearly the convention, under the convention of quoting yearly spot rates, so at time t so my money grows to now here is very important that t has to be an integer. If t is a fraction, then some adjustment has to make. Because that is what you can exact you got bank to do or you can may money anywhere we you where you put your money they will do, we otherwise very difficult to make such computations.

So, t has to be and integer. You can do the compounding by dividing the whole year into m periods. So see if you look at the spot rate for the second year if I invest for 2 years. Then even for the first year I am mean get getting interest rate that is short for the second year for to holding for 2 years, not that I get it first for the first one, S1 for the first one and second S2 for the second one. It is not that, so you quote so because this quotes this S1 or S2 they are all quoted at time t equal to 0. Now if you want whole period you want to keep your money hold your money till time t, but you want to have every year to be divided into m equal period, then the money that you will get, we invest a rupee. Now it is very important banks can also continuously compound and sorry s t here so you know even if am using the same quote, that I got for the tth 1 at every stage. So here also it is to be very important that m t must be integer or some adjustment has to be made.

So, this is what you get of course, you can continuously compound and take the limit has m times to infinity, so as m times to infinity, my spot rate this would become e to the power s, t, t. S t is given thing it become it to be called s t, t is the spot rate per every time t and that into A. So this is the money that I will get back so this is money compound. So once you have structure or how do you would actually how much money would actually give me, you can actually find what how much money I should pay now, to you if you are selling me the bond so that you give me the one this amount of money at the end. So that is what is called discounting or present value for present worth of that instruments which we are already been studying for quite some time, but I would just recall, so that you recall again all the thing that we have studied and we get there. So you must be observing that the math it is very simple, but as you go on it reaches the first week, so you will see from tomorrow the math would slightly take on some level.

So, let we first start of what the discounting factors associated with this which you can easily figure out right, but still let me write out. So now, will write down the discounting factors so of course, we are only making 1 yearly code.

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So, the discounting factor for the A-th year, the discounting factor again when you do m periods compounding, with m period compounding. So let me write down discounting factors. So when you write down m period compounding what you have is the following that, then this thing becomes 1 by, so this is the amount of 1 year I should pay that this

into A is the amount money I should pay to get back A amount, so this into A. So which means what fraction of money has to pay now what fraction of A I have to pay now, so that I recovered A at the end of time t? So that is of course, writing. So this is the yearly discounting, and then this is the continuous discounting. And that is of course, at any time t is given as e to the power minus s t, t. Now of course, now once you know these given cash flows suppose you are given a cash flow.

So, this the amount of money that would your investment would generate. So suppose you have the cash flow is 0 x 1, x 2, x 3, x n, you note thus know the spot rates, you know the spot rates. Once you know the spot rate what is the present value of the cash free. So how much have to invest now in order to get that cash flow in in these time periods. So the present value of the stream is, so this calls the present value of the stream, so you have already you how to compute present value of cash flows which were studied earlier. But of course, you can say that how do you determine your spot rates. We said that just look at the behavior of what sort on interest, or what sort of yield a 0 coupon bond is giving and say that is the 1-year, and then that take it as a you know your spot rate. How to determine spot rate, how to determine spot rate? That is key question.

So, to determines spot rate notice the yield values determine from yield values of 0 coupon bonds. Now let me come to the idea that let me tell you the idea that, the 0 coupon bonds are very rare. It is not that in the market in a huge amount of 0 coupon bonds, so you see here some sort than every get behavior and then try to determine something. 0 coupon bonds are rare and as a result of which you do not have reliable 0 coupon bonds unless you take government bond on which the yield would be very less. Suppose you have a coupon paying bond. So and which say pays a yearly coupon how see so can I determine spot rates from even knowing, even if I have a coupon paying bond.

So, the question is, can I determine spot rates from coupon paying bonds. So I am going to give a very simple situation. I am talking what a very simple situation. So what is that situation? See the idea is this suppose am looking at only a yearly compounding, sorry yearly spot rates. Suppose S1 is known. So it me invest for 1 year your money than S1 is suppose known to you, but you know let us considered a bond which pays a coupon c, coupon value yearly. So it pays you this amount of money yearly, not at certain given

intervals that we had learned in the previous lecture what here we are going things like in a different way.

So, once that is known, it is clearly important to do the following. How would I now find s 2? What is it what is s 2? Now how much is a priced that I should now, pay if I want the bond the coupon paying bond at the end of 2 years should pay me a face value f. So the price which I pay should be this. So should be the first coupon payment discounted by 1 plus s1. And the second year where I am at the end where I have to pay both this coupon plus get the coupon as a face value of the bond.

So, if I know how much paying if I know the coupon value and what I should get at the end, if I know s 1 s 2 is known, so that is one way of findings spot rates. So now, one says 2, is known you can go for other ones. This idea of short rates brings us into very nice idea forward rates. Forward rates or interest rates that you speak about not from a time 0 to a time t, but between 2 intermediate time values t1 and t 2 and how is it that? How do you compute that, what are the issues we shall now discussed?

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So, now we are going to study what we said is forward rates. And will explain why we are calling it forward rates. So forward rates are interest rates that are agreed upon by 2 parties at this time for a future time interval t1 and t2, so at time says t equal to 0 I agree for interest rate that should be held during this time period, t1 and t 2. And I agree on an interest rate f. So any money that you accumulated t1 and that the interest rate that would

be levied on that is capital between the time t1 and t2 will be f. So that this is this is what is called the forward rate. Forward rate I am just writing a short form of forward the SMS generation short forms are much more, WhatsApp generation short forms are much more relevant I guess. Now what does it mean what I am supposed to do what is the forward rate. Suppose at time t equal to 0. I am looking at a scenario like this. Suppose I want to invest time equal to 0 say 1 rupee, rupees one at time t equal to 0 I give it to the bank for hold it for 2 years and spot interest rate is or spot rate is s 2.

Now at the end of the 2 years the money that I will get is 1 plus S 2 whole square which we already see. Now instead of doing that we do something else, that now we want it doing do something very funny, we might be more our business type and is it let me do something, t is equal to 0 t equal to 1 t equal to 2, so I have 1 rupee which have invested now and the bank and with a spot interested 1 I said I will just hold it for 1 year, I will not bother out 2 years. That t equal to 1 I have 1 plus s money given to the bank and I said, but I actually want to use this money over a period of 2 years this 1 rupee I could you 1 lakh of rupees that is the different issue, but just want used it.

So, 1 plus s 1, I will now loan it someone. Right I loan it someone. And I agree that at the time t equal to 0, I agree with someone that I will loan you this amount of money at an interest rate f, at the end of period 2 years, I have given somebody money 1 plus s1. Basically somebody ask me for a money 1 plus s 1 at time t. I said what I will do is that I placed this rupee 1 at time t equal to 0 spot rate S1 get this money, and I agreed that I will give you this money between time t1 and t2 be will that time t equal to 1 I will give you that money at time t equal to 2 t2 it give me that the money with interest rate, but that interest rate is a forward rate that I we decide upon.

Sometime t equal to t2 the money that you will get is 1 plus s 1 into 1 plus f. For the question is how would you know how to determine the forward rate? How would you know how to determine the forward rate f? How would say to somebody this is my interest rate, I cannot arbitrary make a statement like that the reasons for which you should make, to determine the f. So how to compute the f we invoke what is call the comparison principle in the financial market, see it says that the forward rate should be a such a rate that the money that is generate by using the forward rate between t1 t2 should be same as the amount that would have generated if we are put into if we are put the

money in the bank for 1 year at a spot rate to and held it for 2 years bond bank or bond whatever 0 coupon bonds.

So, the comparison principle then says that 1 plus S1 is equal into 1 plus f, and then must equal 1 plus S2 square. So then that will tell me that 1 plus f is equal to 1 plus S 2 square by 1 plus S 1. So that is that simply tells me that f is equal to 1 plus S 2 square by 1 plus S 1 minus 1. So here I computed the spot rates while forward rate once I know the tool spot rate.

Now, why this comparison principle was invoked? What was the use of the comparison principle is comparison principles invoked to stop what is called the arbitrage in the market? So market should be free of arbitrage. The comparison principle tells us that if you evaluate f in this way tell me no arbitrage in a market. Market should be free arbitrage. I will tell what arbitrage is. Who will use this concept much in may much detail later on? In a financial market largely the goal is that those who invest get paid, you may lose money also. The main goal of the financial market is to stop people from getting money or making positive amount of money without making any investment.

So, arbitrage is the art of making positive income without making any investment. So what would have been the comparison principle is not invoked. Suppose I have for example, that this is not equal, but strictly bigger. So whatever power time I can if I choose forward rate f as this is bigger than this, so I will have this right. So if I am allowed to code such a forward rate and not this forward rate such that this is strictly bigger than, I will show you will argue that we can actually make money without doing any sort of investment. How is that possible? Let us argue on for this this this particular case. What I do is a following I borrow 1 rupee from a bank borrow for 2 years at spot rate S 2.

So, at spot rate S 2 right; meanwhile, I will take that 1 rupee and invest in the same way, invest in some other bank up to 1 year and then give me it as loan to somebody else 1 1 plus S 1 as a loan to somebody else. So at the end of 2 years I have to give back to the original bank, from which I took rupees 1, 1 plus S 2 square amount of money, but then because of these different between this 2 is positive, I will have. This is what I will get, so from 1 plus S 1 1 plus f which I greater than the 2 years while putting investing that 1 rupee that I have taken on loan have to give out the loan amount and this difference 0.

So, even after give have the loan amount have positive money left with me. Which means that I had no money with me I came to a bank, ask for 1 rupee they gave it to me at a spot rate of 2. Took that 1 rupee invested in this way got this money by coding forward rate as this is bigger and hence I get. So prevail would see everybody can do that and that have a fun, have lots of money. Answer is that the people try to do that, the market straights to balance themselves such that, the only allowed forward code would be this one. So of course, you can argue in the opposite way. You can completely argue in the opposite way and that I that I leave to you forward argument will make it is strictly license you.

So, unless you are this equal arbitrage you all assume in and in this arbitrage operation is gradually observed in the market and the market will only has accepted forward rates would only allow forward rate codes and which will be provided by this formula. So here today we also introduced to you very important notion arbitrage and how interest rate notions are linked with this very fundamental notional arbitrage the most fundamental notions of market and modal of financial market. So with this I close today stop and tomorrow we are going to start discussing optimization modals in finance. And then we would talk about (Refer Time: 32:16) conditions and portfolio optimization and capital as a passive modal.

So, from the next class we will essentially we are looking into the role of optimization in finance. And then of course, with this optimization not free of stochastic, in some sense is stochastic optimization, but we can still handle things you know much different way. We can make it look as if it is generally the non-stochastic optimization. So we are going to talk about the use of optimization and finance and we are going to talk about the capital as it prices you modeled which is one of the most central concept of whole financial theory. The man has great auspicious black has spent all is life thinking about get the capital as at pricing modal.

Thank you very much.