Probability and Stochastics for Finance II

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Lecture – 19
Pricing European Call Option (Count. Time)
The Black – Scholes Formula

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So, here we are starting the (Refer Time: 00:18) of finance the Black-Scholes formula for pricing of European call option in continuous time that paper was published in journal of political economy actually they could not publish elsewhere. The Ito's calculus approach that we have used here is largely due to Merton - Robert Merton of MIT, who is student of Samuelsson. So, it is also called the Black-Scholes Merton formula. So, Black was a mathematician and he is master in mathematician, he is in physics actually. And (Refer Time: 01:05) Black was possibly one of the greatest genius is all over possible the only genius finance as ever seen. And nowadays to honored people in finance, there is a price called the Fisher Black Award and his first recipient was Professor Raghuram Rajan.

So, again we go back to the pricing formula. It is exactly like European call option is exactly everything is same. Only I do not say it is up and down, this is I do not say this is price will go out and price will go down, price will take some value at a given time t, the price will take some value S t. So, the price process which has described, it is a continuous times stochastic processes. It does not tell me anything else that is now the difference. Now, once that is done that how do I know, so I can still have options, which call option? And I agree at time t equal to 0, I make a contract to sell or share of a stock at the price k, which is the stake price. Here, obviously, ah this is between the upper price and lower price that we saw in the binomial case, E is not valid that is what of statement can all be done in. So, you do not know what is S t?

So, at time capital T, the worth of the option is done in the same way. So, at time capital T, so worth of the option. So, at time capital T of the see we where the price is bigger or smaller then k and then you exercise the option or you do not exercise the option fine. Now, what would you do as before you will charge of price for the option which you would be the Black-Scholes price? At time t equal to 0, and then you will use that money in the market in the market why trading keeping some part of the money in the money market, use some part of the money to buy some shares over that given stock. See, again I reiterate, we are operating in a one stock and one stock market, which consist, so one stock and one and the money market it does not consist of anything else is the very, very ideal as situation very expect compute has in complete safe guarding of your risk. But this is not truly the scenario in the market which we do not have time to study in the separate course if you want to study in complete markets.

So, what I want to do is now, in a continuous time setup, if I start with x 0 amount of money, and at every time here there is no time period in a continuous time how would I actually go on building up my portfolio. So, my portfolio would now vary continuously as a continuous function. So, how will I do it? So, at time t, so what happens I have written this portfolio, what things this calculation we have done in the last part, but here we are doing it in a very simple way. So, at time t, so worth of my port folio that is here I am looking at the change of my portfolio value on a very small interval of time.

And then and I am assuming that over that small interval of time, I am holding the delta t number of shares of the stock. This is so small that I am assuming that there almost side by side and of course, there cannot be side-by-side in a continuous time framework that is just (Refer Time: 06:03) a talk. But this so small that again I mean is I cannot trade immediately, nobody can ever trade buy now sell now, buy now sell now that same instant, it is not possible. Because in a continuous time set of given t, there is no next time, there is no time t which is in just next to t. So, there is always of very, very small gap. So, we are holding and because that the gap is so small that means, say that ok the whole process is continuous in a theoretical point of view. That is why the binomial process is very handy when you are actually doing the gain doing the real computations.

So, now what you do you hold it over that is very small time period where on which the price changes by this amount. And during this time with your interested R t u at held. So, what is your interested now interest you are given your S t amount of money at time t and you are bought delta t shares with the price S t. So, the remaining money you put in the money market and you are holding it for a very small amount of time d t. So, it is R t into d t. So, for all the small periods that fraction over period, so you do not charge as we are discuss out interested earlier you do not charge the interested over the whole some big period, you just charge the interest rate over that fraction of time. Assuming that on that fraction of time or over a slightly larger interval on that, the interest rate actually is almost same.

So, this is the portfolio wealth equation, this is the portfolio wealth equation. So, I write down the formula, this is you already know. When it comes to actual Black-Scholes pricing all these would be constant, it will must simpler, so do not bother about it now plus the same part. Now, once that is done you can now right this as so you club the d t term club the d W t d t term. So, here you multiply R t delta t S t d t, so that you can pick up with delta t alpha t S t, so an R t S t d t is separate plus what you have here this into this delta t sigma t S t d W t.

Now, if you look at that I can write this term in terms of sigma t theta W t. So, I can write this thing as R t because I can write this term in terms of the theta from the formula which we have. Because delta t, so it will this will become sigma t into theta t this alpha t minus R t term because by the definition. So, this means now here I will get delta t sigma t S t into theta t d t plus d W t. You see I have now expressed the wealth equation also in terms of various neutral measures and that is the fundamental thing. So, what I have got here is d X t R t X t d t plus delta t sigma t S t d W t, this is what I have got finally, this is what I have got. So, express the wealth equation the wealth process in terms of the risk neutral measure.

Now, what about the discounted wealth process, so, everything is very important that we have to look at wealth process we are to look at the discounted wealth process. And the discounted wealth process is the martingale that is what we really have to show what we have done in the previous lecture in the previous week. So, once we show that the discounted wealth process is the martingale then we are on track. Let me rub this part of and look at what is the discounted wealth process. Now, if I write D t S t, so I will write discounted wealth process the differential of the discounted wealth process. Again I will go by the Ito's product rule, if I use Ito's product rule, it will simply come out to be delta t, I am just writing the result because I do not have so much time to D t S t. So, if you remember this is exactly what we had in the discrete case, the wealth process is nothing but the amount of shares into the discounted stock price, this is the exactly what we just remember that result. So, this is exactly what we have.

Now, what is this, we know thus this is nothing but sigma t S t d t and so simply means X 0 is X 0. So, it is D t is D 0, it is see simply means this D 0 is 1. So, X t is x 0 plus 0 to t delta t sigma t D t d W t. So, what we have proved, because Ito integral is the martingale, so we have proved again that the discounted wealth price D t X t is a martingale under the risk interval measure P tilde. [FL] cut.

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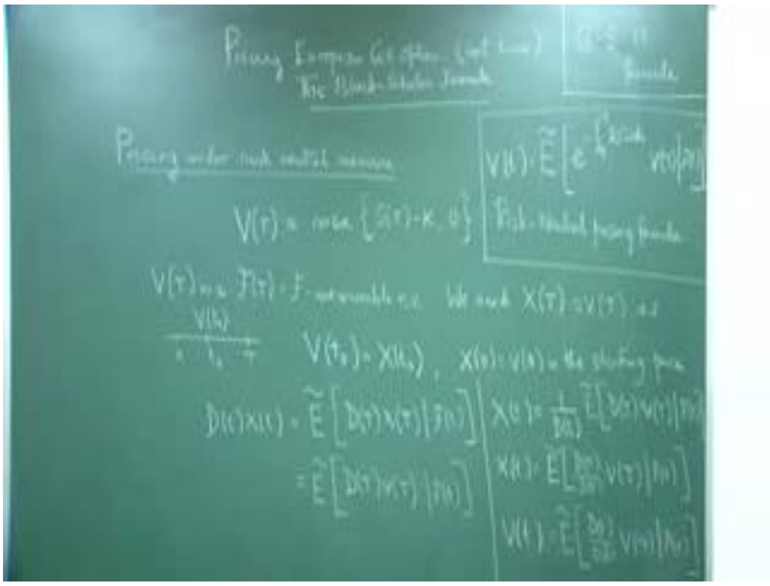

Our next call is to show how do we price in a continuous time setup under risk neutral measure. So, pricing under risk neutral measure is now what we are going to study. So, what happens that when you look at the worth of the option and time T - capital T the terminal time when you have to you are near suppose to exercise the option, it is this simple fact of course, of you change the strike price this will be a function of T, there is random variable may be. So, if you change the S, your time limit we can vary the t capital T, so it will be it will essentially we are on smooth function, will be 0 and then to be all. That is there will be convexity of V t, we will not get into all those things, so convexity.

So, what are the thing that we are no looking for because V t will depend on the all the information till time capital T. So, V t is a F t, which is the F measureable random variable. Now, by the fact, there it is a complete market and we do complete hedging. So, we need that my final wealth process X t must be equal to V t almost surely. See all these equations very ride equations between random variables in probability theory, all these equation whatever we are writing in finance all these equations are in almost surely; there is no other sense basically.

Essentially, what it means that in between these time intervals 0 to t at any time, you can want to buy the option. Suppose, avoided sometime T naught then T naught minus T is a time to expiration your time to expiration of the option is less. But here what would be the price that I should ask for you if you come to me at time T naught and not time 0, then the price that I should be asking is V t naught. But what should be V t naught, V of t naught just like we have done in the discrete case should be same as my portfolio value that is the whole idea my portfolio value is equal to the worth of option that is exactly the principle and there is nothing about it.

Now, so if you choose initial capital V t is V 0 that is an X 0. So, X 0 is your initial capital, V 0 is the price. Actually the Black-Scholes price tells you at any time what should be the price, starting price the price will change. So, I announce that I am selling this option, the option contract is announce that time t equal to 0, and expire a time t equal to capital T, you make come any time and buying between that mean the price will depend on the choice T.

Now what happens is that how do I now actually price it. Look at the interesting thing. We know that the process D t X t the discounted wealth process is the martingale under the risk inters probability. So, D t X t is conditional expectation of D t X t given F t. But what we have discussed, if you want a complete perfect hedging and because the market is complete, we have perfect hedging D t V t, even F t. Now, I can now write this equation as X t is equal to 1 by D t E tilde D t V t F t. Now you understand that at time t, everything is own our D t. So, basically if you can take whatever is known out of the expectations sign, you can bringing bring nothing inside also. So, this simply means X t is equal to D of capital T by D of small T. So, it is X potential D of capital T by D of small t V t F t, but then we have also said that I have be a such any point that you take between capital 0 and capital T, X t was equal to V t the price must be equal to the worth of my replicating portfolio. So, V t must be equal to E tilde D t by D t V t.

So, let me write down more fairly here. So, my option price under risk neutral pricing is V t is E to the power minus small t to capital T R to the power s d s V t given F t. So, this is the risk neutral pricing formula. Of course, t is between 0 to t that, I do not have to tell you every time, I am telling t is between 0 to t, just do not forget get that. So, this is the risk neutral pricing formula, and this would be used to actually figure out the Black-Scholes price.

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So, now we will start the Black-Scholes formula. In this case d S t is fixed means if you write it down under the risk neutral measure it will be there r is R t is equal to r, it is fixed, what happens when you are changing form a standard measure to risk neutral measure, the market probability to risk neutral probability, you change the drift you change the rate of return, but you do not change the volatility that is the key idea to keep in mind. That is now what volatility to be constant here. If you take risk neutral price in here, I am just talking about risk neutral price. So, this would be the (Refer Time: 24:53). So, r is constant, sigma is constant.

So, r sigma, if you want that alpha also if you just take in terms of d W t. So, if you write it in terms of d W t d S t is alpha S t d t plus sigma S t d W t. So, in alpha these are all constants in the Black-Scholes pricing, all constant. So, then your D t - D of capital T is nothing but e to the power minus r small t. So, because of this you are d t, now let us look step-by-step at the pricing formula that we are going to do for the Black-Scholes pricing formula. Let us also remember the Black-Scholes pricing formula at V t, it just not only depend on t, it essentially to be write V t value at what that time t which depends on the c - this is a cost which depends on not only t further price of the stock at that particular time.

So, that is strike price should be actually fixed basically now over then the value the worth would be, sorry, it is not strike price, I make a mistake. Because based on this, this V t would be fixed; the V t needs a dependent on this. Now, but we are computing this with risk neutral measure, and I write E tilde e to the power minus r capital T minus small t, where we are tilde also the integral V t V of capital T. So, V of capital T here is max of S t minus K and 0 given F of T. So, this is my formula. Now if I go by this formula then I know that may S of t, if I go by this formula in the risk neutral pricing right where your r is of course, the standard fix interest rate, S t is S 0 into exponential sigma W t. Because this is the constants to integral 0 to t d W t d W t was W 0 0 plus r minus half sigma square d t, this is the formula.

Now, what about S of t so, you call these formulas as star. And then S of T, naturally this is the terminal time here it is any time and here is a terminal time exponential sigma capital T plus r minus half sigma square. Now if I divide this line, if I call this star - double star. So, dividing double star by S t, and dividing this expression by dividing S t dividing. So, basically nothing you are writ in. Now you have to S t in terms of small S t. So, divide this side by S t divide both sides of star by S t and then use divide both sides of double star by S t and use star to get S of capital T is S of small t, because S this 0 this 0 will cancel exponential sigma W tilde capital T minus W tilde small t. So, sorry it is not D t it is t, here it is t naught d t is a integral 0 to t d t is t, so plus r minus half sigma square capital T minus small t that is what you will have. So, here also there is the change mean. So, here this will be capital T, this will be small t.

Now what would the interesting case is now let us write capital T minus t, that is the time to expiry of the option. So, that let us put it as tau. Let tau is equal to capital T minus small t. Once I do that then let us see what my expression becomes. And let me put why as W tilde capital T minus W t by root over T minus t that is root over tau, let me put this variable. Now, first we have to decide W tilde t is a normal random variable with mean zero and standard deviation variance T. This is so W t minus W tilde t; this difference is a normal random variable. So, W tilde t minus W tilde t is a normal random variable. So, I made this change is normal random variable with mean 0 and variance T minus t.

So, standard deviation is root T minus t, which is tau that is from the basic fact about because T is strictly bigger than t, and because of the independents of the non-overlapping intervals. So, this comes on way you basic definition of the Brownian motion that this difference should be satisfying this. We have to then go back to much detail in the Brownian motion in which is in the first parts. So, kindly we have look at the first part. So, this is what you will have. Once you have this, so you take a normal random variable and scale it. So, you will get the random variable. So, why would be a normal random variable which mean 0, but the variance is root over t minus with variance is 1.

Because, here if you take the variance of this thing; it will b. So, if you take variance of y is mean would be 0, if you take variance of Y, and it will become 1 by tau time variance of W tilde T minus W tilde t, which as by a basic formula variance, because this is the independent random variables W t minus t. So, this is independent random variables right. So, there is no on there is no other variables. So, it just one variable. So, this is just one variable, W tilde t minus W t. So, it is 1 by variance of these variance of this is of tau t minus t, so it is 1. So, Y is now a standard normal variable. So, this expression has to be kept in mind.

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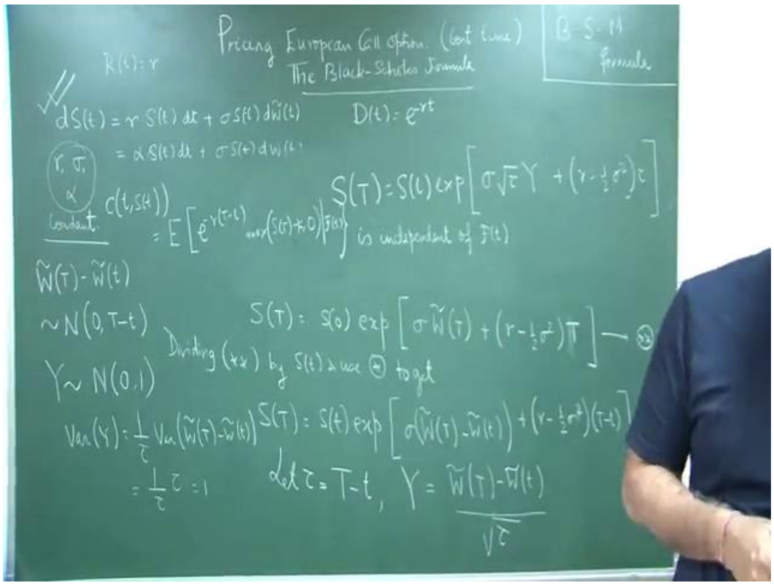

So, which means we can now write down S t, so S t can be written as S of small t into exponential sigma, this 1 sigma root tau y plus r minus half sigma square tau, this is the expression. Now, observed that this expression is independent of F t, because this is looking at everything beyond the time t. So, this expression is independent of F t. So, this is the very, very important fact, because when we do the risk neutral pricing formula which I again write here, which is c t x which is c t, S t equal to exponential e to the power minus r capital T minus t max of S t minus k and 0, F t. See, I am computing the S t, because I have to put it here that takes the game that is what I am writing this. So, keep this S t, now I will rub this part and do the further calculations. So, we have to understand that this part this expression is independent of F t, because here everything in terms of tau which is beyond small t. Now, what we are going to do, we will treat the random variable S t as if it has some value x, the way you compute things you write density function the etcetera and probability.

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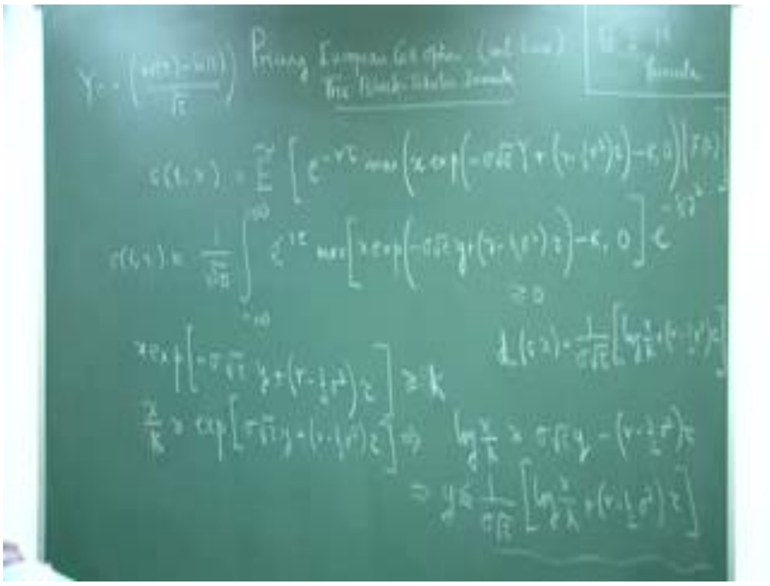

So, let us just to do that, and let me write it down c t x is equal to exponential e to the power minus r tau which is capital T minus small t. So, max of x of exponential, now let me make I think there was a small correction I want to make, Y was minus because there will be some sign problem t by root tau. So, here they would a minus sign. So, in the previous expression of S of capital T, so just check this expression S of capital T S t into exponential minus sigma root tau y plus r minus half sigma square tau that would be the expression. So, would this minus is been taken by (Refer Time: 39:01) because there will some sign issues, but this is not a very week thing, when take it in the same way which I had taken. The sign issue can be handled at the end, for some books take it like that on some take the minus sign.

So, we will take the minus sign go which free, because I have already announced that I am doing thing some this book root tau Y plus r minus half sigma square t, this S t minus K 0 max of this slash F t. So, I can write this as c t x using the fact that Y, so here a random variable is Y, so this is the function of y. So, I am just writing down the expression. So, 1 by root 2 pi, I am assuming that you know very basics facts about statistical distribution which is the normal distribution the most important one, e r tau max of x into exponential minus sigma root tau y plus, now should not put root tau Y here it just to be root tau y minus plus r minus half sigma square tau, this is tau sigma square tau. See, it is only the complication the expression that makes the whole thing look so damn bad, max of this e to the power minus half y square. So, 1 by root to pi we have taken out.

So, basically we have to find, so this expression is meaningful when this is greater than equal to 0 otherwise it is 0, to find when is this expression, greater than equal to 0. So, this expression, we have to find whether this expression is greater than equal to 0. So, this expression is greater than equal to 0 when x into exponential of minus sigma root tau, so the sign issues come here actually in this. It does not matter, so it just is a matter of convenience that is all r minus half sigma square tau, this should be greater than K.

So, I can take x assuming that x is non-zero of course, positive for the time I can also bring with this here x of k is greater than equal to so e to the power minus sigma root tau y plus r minus half sigma square tau. So, you put the minus here, so it becomes minus. So, you take log of both sides, so this would imply that log of x by k is greater than equal to sigma root tau y minus r minus half sigma square tau. So, you take log is increasing function. So, you take the exponential when is taken off- by the log, log is log to base c. So, this implies y is less than 1 by sigma root tau log of x by k plus r minus half sigma square tau.

So, I will call these expressions, this whole expression as d minus tau x d minus tau x as (Refer Time: 44:03) writes, we will call this expressions sigma root tau, sigma root tau comes here and the denominate to come here. Of course, T cannot be equal to t then it is has no meaning because we are tracing it at some T which is strictly bigger strictly less than t, if it is 0 nth S t, then there is no meaning then it will be 0 then it right then it will be different calculation. So, here so this is these expression log x by k plus r minus half sigma square tau. So, this is the expression. Now, once this is done, we will go to a next step. So, basically then my integration has to be there y has to be less than this. So, my integration, so when once y is between minus infinity to this number this is get of n equal to 0; y strictly bigger than this number then this less than equal to strictly less than 0 in this whole thing become 0 the max of whole thing become 0.

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Now what you will do is that we just have to integrate find the integral from minus infinity do that and that is exactly what we are going to do now. So, I will be slightly faster. So, I am just writing down everything the way it was. Here I can make this. So, now, I can take the minus this into this and separate the tau with the k and separate others, d minus tau x, x exponential, so I am pulling in this inside minus y square by 2 which is these being pulled inside. And e r tau e r tau gets canceled e r tau minus e r tau. So, e to the power 0 is 1, so that is what happens minus sigma root tau y minus sigma square tau by 2, this is one thing into d y minus 1 by root 2 pi.

So, here in contrast, see here also task spoken over Black-Scholes things in the part one and the variant, but there will we found we wrote down the differential equations who solution out give me the Black-Scholes price. There here we are going to explicitly calculate the solution without Ito's to differential equation, but just by here it is a minus sign it is not equal to this whole thing minus this was. I am separating the k part also e r tau k into e to power this one e to the power minus half y square d y.

Now, this thing can be taken out because this does not depend on y. So, here you will have if you do some little calculations. So, we will have x by root 2 pi infinity to d infinity tau x and it can be written exponential of minus half y plus sigma root tau whole square this whole thing d y minus e to the power minus r tau k into minus infinity to d, so that of this one by root 2 by minus infinity to d. So, it is a distribution function cumulative distribution function, cumulative distribution function of the normal distribution which we write as it is standard all over the world to write this as phi of tau, x. So, probability at y is less than equal to d minus tau x. So, here also you can put z equal to y plus tau s i ah sigma route tau then d y equal to d z and then you put here this plus sigma root tau in this limit. So, these equation would become x root tau minus infinity would d tau x plus sigma root tau into e to the power minus half z square by e to the power half z square d z minus e to the power minus r tau k phi of d minus tau x.

So, this things, suppose I write d minus of so d plus of tau x is equal to d minus of x tau sorry tau x plus sigma root tau. And that would give me, if you do that if you do the simple calculation that would give me 1 by sigma root tau log of same thing only the minus changes to plus log of x tau k plus here tau the minus here will become plus r minus half sigma square tau just a plus minus adjustment. So, what I can write here if I write it in this sense way, then I can write finally, c t x. So, this is nothing an n by the distribution function of the standard normal distribution x is of course outside x of phi of d plus tau x minus e to the power r tau t minus t K phi d minus tau x. So, may be so this is exactly the Black-Scholes formula, and we are at the end of the course and so let me now write down the Black-Scholes formula in all it is glowing that you want.

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So, the price the Black-Scholes price or Balck-Scholes made on price at time t, which is V t which is nothing, but c of t, S t is equal to S t x is now replaced by S t, x was playing role of S t phi of d plus t minus t S t minus e to the power minus r capital T minus t K into phi of d minus t minus t S t. So, we just are rewriting this. So, this is exactly what we required this is our Black-Scholes formula which we are computed. Of course, we have to use the standard normal tables, normal tables that does not mind. So, here we have essentially finish the course. So, in the next class, which is the last class which is possibly mandatory to take, who will basically some of what we have done and maybe I will give a hind to find the delta t. Of course, you can ask me that if I am really the guy which doing things in the continuous time that I was hold option what is this.

Thank you very much for attending this ten our long course in finance, we will have a bye, bye class in the next one and finish the course.

Thank you.