Probability and Stochastics for finance-II
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Lecture - 18
Stock Prices under Risk Neutral Measure

So, we are now going to discuss about the nature of stock prices, how the stock prices behave. You might think that ok just because the stock prices can do zigzagging, it can be modal by Brownian motion. But it important to remember is a Brownian motion can take negative values while stock price downed. So, their model what is called the geometric Brownian motion? And let us now write down a geometric what is the form of geometric Brownian motion, which you have already written down in the previous part of this series of lecture in part one.

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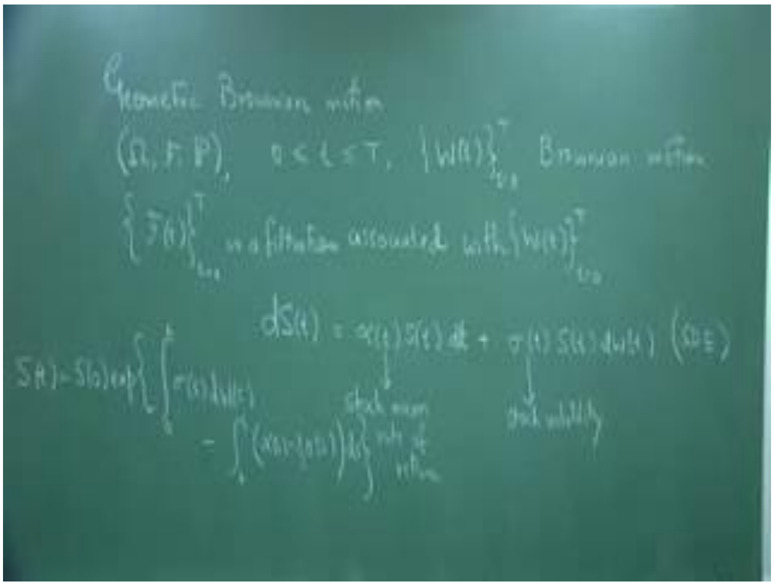

So, stock prices are assumed to follow geometric Brownian motion. Of course, geometric Brownian motion is built out of Brownian motion and basically in this sort of situation you can take an exponentiation will actually give you the non-negativeness. So, what you would have is your component should as before your probability space, your trading time of the option 0 to T; 0 is a starting time, T is a expression time, just very general point of view just a time interval. W t is Brownian motion, and F t is a filtration, obvious F of capital T is F, actually now sigma algebra is of filtration associated with W t. Say in the sequence also we have making this I am writing in that you know that these are actually stochastic process defined over given time interval.

Now, a very general form of a geometric Brownian motion is defined in terms of what it is called a stochastic differential equation, which we are already discussed in the last part, which is given in this form. Of course, these are short hand for a stochastic differential when integrate equation rather in Ito integral. Alpha t is call in this case stochastic I am writing stock for short stochastic mean rate of return - return on the stock under the market probability is a mean rate of return and here is a stochastic volatility. So, these measures how much zigzagging or how much variation one has in the price. It tells you that over certain given a period of time how much we can expect stock to give us back.

In case of the Black Scholes model, where alpha t is fixed assumed to be constant and sigma is assumed to be constant, largely such estimations of the constant estimation of the constant is done based on historical data. And for this particular equation, you do have a solution, this can be written in the integral form as s of t is s of 0 into exponential means e to the power basically not writing e to the power this side I am just writing is 0 to t sigma s d w s minus 0 to t alpha s minus half sigma square s. This term is might just forgot this term petty often d s. So, this is the solution essentially of this stochastic differential equation or SDE. Now, once this is known we shall write down what is call that discounting factor. The discount factor just like we are discount that 1 by 1 plus r here to more precise, we will assume are interest rate R t is itself a stochastic process.

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So, interest rate stochastic which I am writing in short stochastic process which is adapted to this filtration F t that we have stated earlier or these are the t is equal to 0 to t. So, R t where is very less possibly when you looking at very small time interval or else it can vary considerably. So, R t is the stochastic process in the money market. So, money market here is exactly not a bond; in bond, you know if interest rate can change abruptly, very sharply you know very short time, but this is a just essentially a bank, we can think it as a bank. So, discount process, so the discounting here is also stochastic process the discount process c to the power minus R t type things but the discount process D of t. So, at the anytime t, the discounting process is given by instead of sending R t, because you know how the process you integrate. So, this is the definition you have your discount process on the discounting mu or whatever you want to call, the discounting process or discount process.

Now, how can we write down this in terms our stochastic differential equation that is exactly what we will learn now? So, how do we right down these thing in terms of a stochastic differential equation and here again we leave it was calculus. So, trick is this same it is e to the power minus something when you have process is e to the power something always take your F that we will use in the Ito's formula as e to the power minus x. So, f dash x is minus e to the power minus x, and f double dash x is equal to e to the power minus x that is a f dash x in this case is minus of f x if you observed.

Now, if you look at if I write this as my I t this process has I t then from the fundamental theorem of calculus assuming that these have nice properties it will imply that I dash t or d I t d t, this is this is not a sophistic integral this is ordinary integral please remember it. This is ordinary integral, so I am applying the fundamental theorem calculus for that if you have a sophisticate integral I cannot. So, this is nothing but R t. And hence this implies that d of I t the differential is nothing but R t d t. Now, what if I write d of d t, if I want to calculate the differential of the stock process - this process, it is actually t of f I t. So, I will ask you to ponder and write what is I t into I t, d I t into d I t. So, d I t in to d I t the quadratic variation would be obviously, equal to 0, because will R square t into d t d t, d t d t is a ordinary function, it is quadratic variation is 0.

So, then again by applying Ito's formula, we come to the following conclusion. So, this is f dash I t d t plus half f double dash I t d w t is a d I t d I t and d w t, actually this is not need at when d t is a stochastic process, while this is just normal integral. So, f dash I t, if you put it here is minus or f x. So, minus f of I t minus f of I t d t plus f double dash I t is again e to the power minus I t. So, it will be same as f of I t. So, here it is again f of I t d I t d I t. So, d I t d I t is 0, this gives you, so what is f of I t, f of I t is nothing but d t of there some mistake I am committing somewhere. So, this is not d t, I written mistake it should be d I t that some mistake is committed in the Ito's formula. So, this is by Ito's formula. So, then what happens is that, what is d of i t it is R t d t. So, it is minus R t d t. So, minus R t into d t, so D of d t is equal to minus D t into R t into d t.

Now, we will look at the discounted stock price. So, what is the discounted stock price D t into S t? We would like to see whether under there is neutral measure the discounted stock price is martingale or not just like we have seen in the case for the discrete scenario. So, in the discrete time scenario whatever you have seen whether they work for the continuous time scenario. So, now we will write down the discounted stock price I am not do the complete calculation I will ask you to check the calculation which is a good exercise, first I will write down the discounted stock price.

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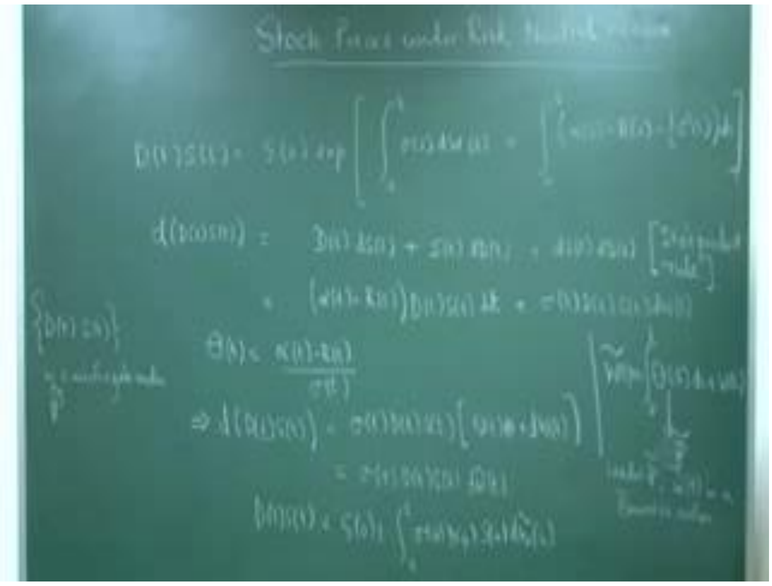

The final answer D t into S t there is of course, S 0 into exponential the term inside the exponential is to be calculated and check. The first term is obviously plus 0 t alpha s, now we have an additional term due this D t has it is R t exponential R t in the definition. So, we will have this term also because this will get added. So, there are two exponential terms the exponent thing here and exponential get added to the thing here in the exponential, the constant term really does not matter it goes out. So, it is an obvious thing. So, I am not going to explain to you how it come this is just very, very obvious thing multiplying exponential that that is all how you add the power.

Now, I want to see what is the differential. So, what we do we go by the Ito's product rule. By the Ito's product rule if you go, you will have D t into d S t you know what d S t is plus s t into d D t which you already know, you calculate d D t you know what is d S t plus d S t into d D t that is Ito's product rule. This is called Ito's product rule. It just like the product of the derivatives got additional quadrant term, which comes from because of the quadratic variation.

So, now once you have this is a complete calculation will show that you have the following? So, please check this calculation not hard just, but just too many of algebra in manipulation, which we are not doing at this stage. Now, construct the function consider the function theta t is alpha t minus R t by sigma t. So, if you do this, and here you can replace this by sigma t theta t, then it implies that the differential of D t S t is equal to sigma t D t S t into theta t plus theta t d t plus d w t. So, what is this? This is, if now I consider W tilde bar as theta t W t is theta t and it is 0 to t theta s d s plus W t. If I consider this and then I consider theta, I construct the process Z t in the as in the Girsanov's theorem in the last class putting that theta. Then in the same way using Z as Z of capital T, we can construct the risk neutral measure. We can now construct not this neutral measure or measure P tilde which we will call the risk neutral measure.

From here we can create P tilde such that under P tilde W tilde t is a Brownian motion is it not amazing. So, you have now constructed at a new thing. So, what you have got sigma t D t S t. Now, what is D t S t? If you look at, so what is D t S t, D t S t is nothing D 0 into S 0, D 0 is 1, S 0 is S 0 is S 0 plus 0 to t sigma s d s that is not sigma u D u S u d w tilde u. So, these are in Ito integral. Under in terms of the w tilde - the Brownian motion which is Brownian motion under the new probability measure P tilde which we called risk neutral measure.

So, what we have proved here because every Ito integral is a martingale, where proved here that D t S t - the discounted stock price is a martingale under P star. Actually P and P tilde as I told you yesterday are equivalent measure that is a agree on set some measures 0, if a set is null set in terms of P, it is null set in terms of P tilde vice versa. Martingale under P tilde; this thing would be have big help to us we will soon see. Actually in writing of risk neutral pricing this will be a fundamental help. If you remember what we did in the last week class where we are done the discrete time thing this is exactly what we need to look into.

So, once this is done then it is important that we should know under there is neutral measure, how does the stock then how does the standard stock price would like and this is a very important thing. So, we will now write down under the risk neutral measure how the stock price looks like. Now if I want to write down the stock price then what should I do.

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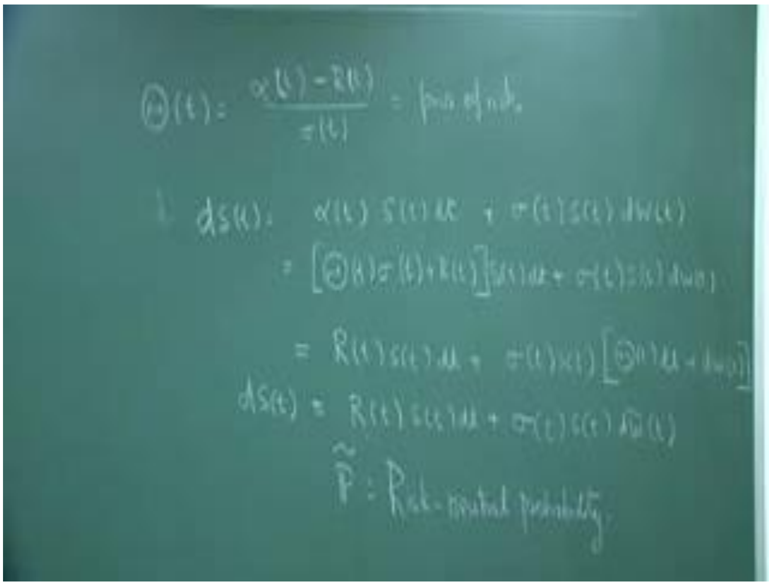

So, I do it in detail. Now, this term theta t, which is alpha t minus R t by sigma t, this is always a non-negative function because in general see alpha t is a rate of return under the market probability, which is always bigger than the 1, under for the money market return for the money market, this is usually strictly bigger than 0. Now d S t, so we have already seen d of S t is equal to alpha t S t d t plus sigma t S t d w t. So, let us see from here let me write down what is alpha t. So, it is theta t sigma t plus R t, this into S t d t plus sigma t S t d w t. So, this quantity is also called the price of risk that if you are taken the risk of investing in the stock market this is the price of risk. Suppose alpha t goes below R t then you are losing. If alpha t is bigger than R t you are winning having it better. So, these sometimes called up this quantity call a price of risk, sigma t S t d w t, so this is what we have got.

Now, what I do, now I take this part sigma t S t and add up with this part, keep this part separately. So, I will have R t S t d t plus sigma t S t theta t d t plus d w t. So, what is this, this is nothing but d w tilde t. So, you have R t S t d t plus sigma t S t d w t. So, this is now to this equation. So, if I write it down in more detail, so d S t. So, this is the stochastic differential equation governing the stock price movement, it is stochastic Brownian motion under the risk neutral probability of this Brownian motion, where the rate of return of the stock is same as rate of return of the money market. So, under there is neutral measure P tilde, the rate of return of the stock is the rate of return in the money market of course, so that is why it is p tilde because here you are neutral to the risk.

Because even if you put in stocks whether you give it in the money they have the same rate of return that is why P tilde is often call the risk neutral price, it is neutral to risk, it is call the risk neutral probability.

So, here now we have a fairly good idea how to handle things. So, we will stop our first lecture here; and go to the second lecture, actually computing the Black-Scholes price, computing the price of European call option, which we will again start in detail. Thank you very much for observing taking a careful look at this talk, but we very carefully calculate all the calculations of I have done in detail, all the things that I have written down should be checked in detail.

Thank you very much.