Probability and Stochastics for finance-II
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Lecture - 17
Girsanov's Theorem (Statement and Proof)

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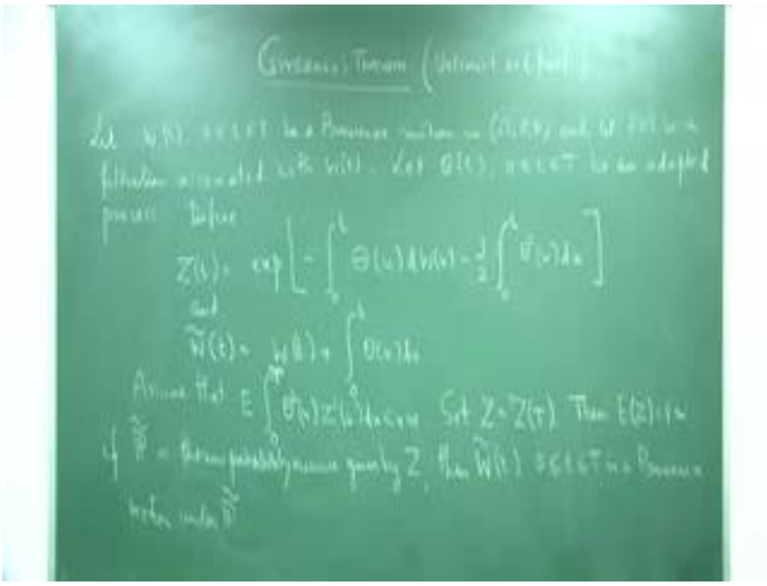

So, now we are stating this very, very important result that Girsanov's theorem, but in the one-dimensional set up not to in the multi-dimensional setup. We will now go to the multi-dimensional set up in this course other it takes a long time. So, we will states there is a theorem and then we will prove it step-by-step. So, let W t be a Brownian motion in the probability space and write F t be of filtration associated with the Brownian motion. So, all these terms do come the first part of the series. Let theta t be an adopted process means adopted to this filtration that is theta t is F t measurable for every each t be an adopted process. So, we define Z t a new process given as follows, exponential you would feel that is almost the type of thing that we wrote for that normal distribution essentially related with geometric Brownian motions.

Let us not though we define this. And we also defined a new process from the Brownian motion which is given by W t plus addition of the integral. So, we essentially have to show that these are Brownian motion. Now, we are getting, we have putting condition which guaranties the existence of an Ito integral which will come in a proof of the theorem. See those who have not done it those who have not done the details just learn the statement that ok; from old Brownian motion you can construct a new Brownian motion that is enough.

Set Z equal to Z t, so Z t that the terminal random variable of the stochastic process Z t is given the name Z by which you transfer the measure of P to P tilde. Then E of Z is equal to 1; and P tilde is given is the new measure or new probability measure I should write new probability; obviously, here z is obviously, greater than or equal to 0 and all those things because of the explanation function is actually is strictly greater than 0. Probability measure given by Z and p tilde is the new probability and if P tilde is the new probability measure given by Z, then W tilde t is a Brownian motion under P tilde. So, here it is here is important thing that it is a Brownian motion under p tilde.

So, what are the things that we have done? We will now do prove after rubbing this, because we have one board system here we have do not any other board. So, we define a process Z t which will allow you to create a variable Z, which we can show the E to the power z is equal to 1. And then if P tilde is the new probability measure given by z that is integral Z d p is P tilde a and integral a Z d p is P tilde a then W tilde is a Brownian motion under P tilde. So, what it is says that give me a Brownian motion under P then I can create a new Brownian motion by certain translation and under not P, but P tilde. This P tilde plays role a risk neutral measure then we start trying to prove the theorem of black and show the result of black and shows.

Let us now rub this and then do the proofs step-by-step. So, first thing how do I know that W tilde t is a Brownian motion? There is a very famous result of Levy’s which says that if W t is a stochastic process which starts at 0 that is W 0 is 0 or W tilde 0 is 0. And it is a martingale and has a quadratic variation equal to t, at a any t, then that one such a stochastic process has got to be a Brownian motion. So, this is something. So, again I repeat that the key to the proof of this result is a Levy’s theorem. So, Levy’s theorem states that if there is a stochastic process which starts at 0 is a martingale; and at every time t the quadratic variation is t then such a one such a stochastic process has got to be Brownian motion.

Of course, I have not been very clear when I am not told it in detail of course, you have to say that that is stochastic process must a continuous path because Brownian motions must have continuous path. So, that is exactly we have to show. So, we will show first the W tilde start at 0, at 0 it will be 0; obviously, we would 0 here, we would 0 here, this goes to 0 w 0 is a Brownian motion, so it is 0, W tilde is 0. Then we have to compute it is quadratic variation, so that it is t.

So, once you show that that thing then you. And of course, because of the continuity of this path, because of the continuity of this path and this integral has a function of t assuming that theta is you know adopted process is a continuous function that this would result in a continuous function that is the function of t, it will be continuous. So, ultimately this is a continuous function. So, every sample path would be continuous. So, what would be now that we have to show that quadratic variation is d t then you have to show W t tilde t is a martingale. If the whole machinery of the crew are all the things that we have developed in the last lecture is to show that W t tilde t is a martingale.

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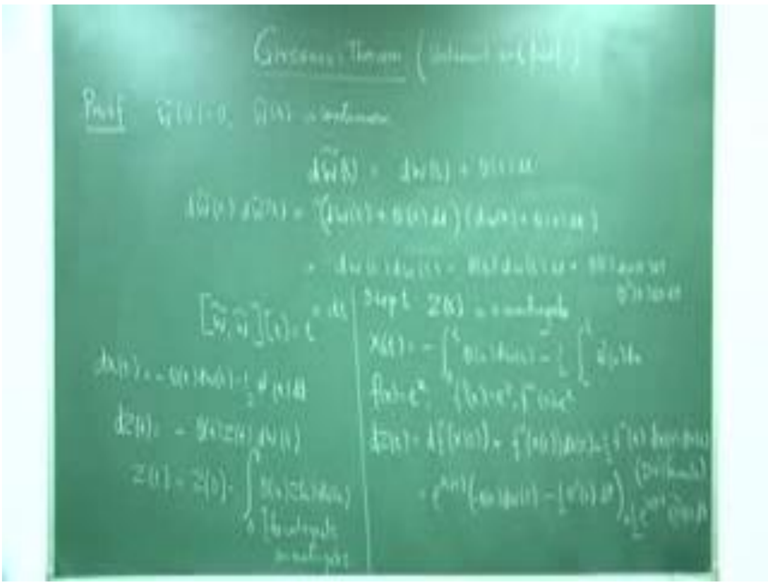

Now, we will remove this proof and start removes this sorry statement and then writes down the proof. Now, further proof W tilde 0 is 0 by definition W tilde t is continuous, if you take theta t be continuous and we are taking it to be continuous. Without theta does not matter, theta to have continuous function. But even if it is a Levy integral function by the measurable function by the first fundamental theorem of integral calculus levy calculus, we can show that the function in t is continuous whatever it is does not matter even that think that theta is continuous nice, nice function.

Now, if you look at this the expression of W tilde t is W t plus something in short hand we can write d that is in d form stochastic differential equation form we write this as theta t d t. So, if you want to compute the quadratic variation, let us just compute this. These are the tricks we have learned in the last previous part, so it will become W t plus sorry d W t plus theta t d t into d W t plus theta t d t. Finally, if you have these this will be d w t d w t theta t d w t d t plus theta t d w t d t plus theta squared t d t d t. So, all this is 0. So, the second variation of a normal function is 0, this cross variations are all 0 which you have done and this is d t. And this shows basically so continuity that so W starts with 0, W is continuous, W second variation t at every t.

So, what we have to just prove is the (Refer Time: 13:24) that is the fact W tilde t is the martingale, and then you see the analysis now starts. So, we will first show that Z t is a martingale. So, step one, Z t shows us that we have defined is a martingale. Of course, you might said that why you are having the Z t that would need further more details which you not like to give in this course as why you are chosen Z t the choice of Z t is pretty important. Why you are always choosing exponential functions; obviously, do keep positive, but why this sought of choice. These essentially come from the certain structure of the stochastic differential equations that is used in case of the stock prices the geometric Brownian motion. See, once we start doing more about that we will get a feeling why it is coming like that.

So, whatever X t which read write as X t is equal to what was it an inside that is minus 0 to t theta u d w u minus of 0 to t theta square u d u. So, now, write f x is equal to e to the power x. So, f dash x is equal to e to the power of x, and f double dash x is equal to e to the power x. Now, d of Z t is d of f of X t. Now we can apply the Ito’s formula and by Ito’s formula we have this to be nothing but f dash X t d x t plus half f double dash t d x t d x t, which is the quadratic variation. So, this is why it Ito’s formula.

Now you know what d x t is, so what is d x t here d x t, x 0 is 0. So, d x t is minus theta t d w t minus half theta square t d t this is your d x t. So, I would ask you to find the variations yourself just to try it out I am writing down the answer always you short hand. So, whenever you agree to Ito integral you can write them in the form of stochastic differential equations. So, I would ask you to, this can be written as so e to the power x t is of course Z t plus half, you find out the quadratic variation I am not going the detailed calculation. So, this would be a good exercise to find out the quadratic variation and I ask t s to actually put it put it in the assignments for you. So, once you know this you note that this into this and this part cancels out. So, finally, what you have is that d Z t is minus of theta t e to the power x t is Z t into d W t.

So, from here what you can write you write down in the form of the Ito integral Z t is Z 0 minus integral 0 to t theta u z u d W u. This is just a constant and this is a random variable and this is a constant. What is this? This is an Ito integral, and every Ito integral is a martingale, so Z t must be a martingale. This is this is I am using the fact that Ito integrals are martingales, so that is one we solve it. So, once that is proved, now E 2, now expectation of Z is a expectation of z of capital T, now because this is a martingale expectation of every level remain same expectation of Z of 0. So, Z of 0 is 1, because this is expectation of 0, so it is same as Z of 0 is 1.

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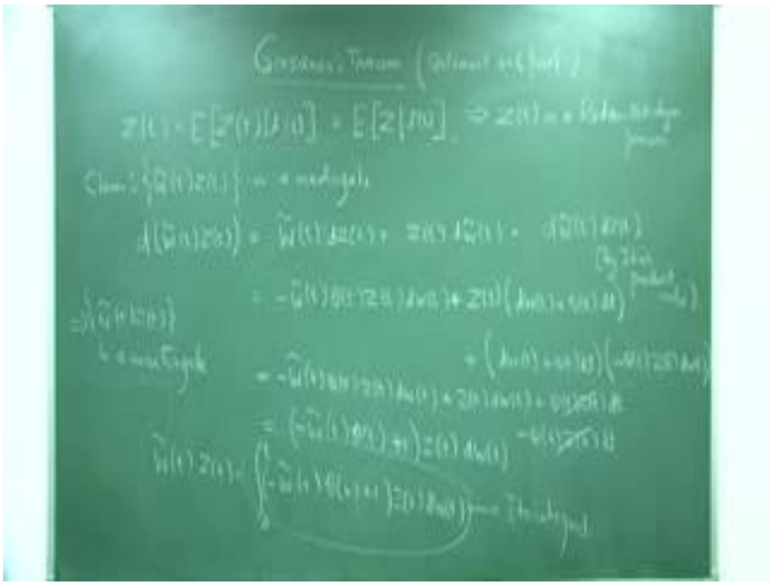

Now, seen Z t is a martingale, we can write Z of t is expectation Z of capital T F of small t, t is the largest one largest time. So, this is because Z, we have defined as Z t as Z and e to the power Z is expectation Z is 1. So, what does it mean that Z t is a Radon-Nikodym process this implies at so we can now apply things I would lemma a, and lemma b, Z t is a Radon- Nikodym process. But before doing so, we will first show that our claim would be to show that W tilde t and Z t, this process is a martingale. Actually when you write everything in a for most stochastic differential equation and if you do not want of the d t term plus d t the only of the d w t term it becomes the martingale because that is only adjust in Ito integral.

So, here also we are going to do something like this. So, if you look at Ito's product rule, so by Ito's product rule, so derivative of this into this plus derivative of this into W t plus derivative of this into derivative of this. So, not really derivative, but ok in that sense, so Ito's product rule, we have d Z t plus Z t d W t plus d W t d Z t, this is by Ito's product rule. So, you write down what was d Z t, you already knew what was d Z t. So, if you write down from the previous case what was d Z t, and E become minus W tilde t theta t Z t d W t plus Z t. Now, if you write down d W t in detail, so it will become d W t plus theta t d t. What about this thing, this is a cross variation and this cross variation will go to 0. Here you can write it down in detail and write down the cross variation writing Z t in this, you can show at the cross variation should go to 0.

So, this part will now what I will write here, I will have d W t plus theta t d t plus Z t which was already known here minus theta t Z t d W t. So, here you see, here you take the first term multiply with this term you will have d t here, but this term and this term this will be a cross variation which will give you 0. So, what is happening is let me write down in detail minus W t theta t Z t d W t plus Z t d W t plus theta t Z t d t. Now, here if I multiply this one, it will become minus theta t Z t d W t d W t is d t. Now, plus theta t into this right, so minus theta square t Z t d w t d t will cross variation will give me 0. Now, this term we cancel out. So, finally, I will be having minus W tilde t Z t plus sorry theta t Z t or just write theta t plus 1 into Z t d W t.

Now, W tilde 0 is 0. So, basically we can write W tilde t. So, W tilde also this, if I would have 0, here this whole thing become 0, Z t is nothing but 0 to t minus W tilde t theta t plus 1 Z t d W t which is the Ito integral. So, this is an Ito integral this whole thing is an Ito integral which is the martingale. So, this would imply now that this process is a martingale that is the beauty of the mathematic actually. So, once I know this, I can go back and apply my theorems, but lemmas A and B. Now, my job would be to apply lemma A and B and finish off with this.

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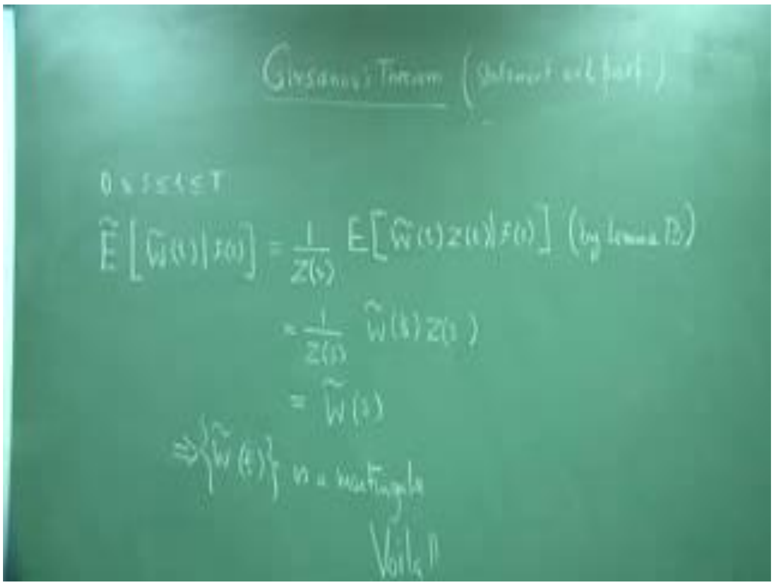

So, we are in the final phase of the process. And we are now going to prove that W tilde t is a martingale under the probability measure P tilde. So, let us show this we have to show that this is W s - W tilde s. So, now, if I go back then from my lemma B, because Z t is a F t measureable random variable if you look at the definition of Z t is clear that it is an F t measurable random variable because theta is F t measurable. So, Z t itself is a F t measurable. So, I will have by lemma b Z s is equal to exponential w tilde t z of t F s. So, this is by lemma B.

Now, W tilde t and Z t we just have proved it to be martingale. So, by very definition of martingale this simply means W tilde s Z s and that is simply gives me W tilde S that is what we are started to prove this you would imply W tilde t is a martingale and that competes the requirements of Levy’s theorem and hence the result. And as the French would say you write voila that is it. Thank you. So, we have completed one of the most central results that you required to study mathematical finance. And tomorrow, we will go more deeper in to the study mathematical finance and move towards a Black-Scholes formula. See, a lot of details cannot be done here because of the time; otherwise if there is a lot of very, very small detail all these things are actually all most surely. All the equality all in here all the equalities this is the expectation, all these equality expectation where in the almost surely setting.

Thank you very much and we will see how things precede in our next classes three more classes to go.

Thank you very much.