Probability and Stochastics for finance-II
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Lecture - 16
Girsanov's Theorem (Basic Tools)

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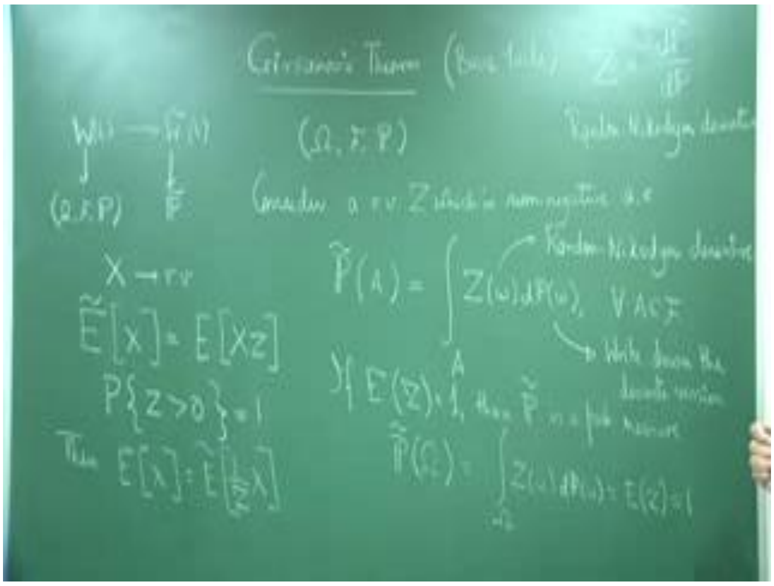

Almost everyone almost sure in right, whichever we want to do. Now, you construct for any A, which belongs to this set of events, so basically on the same samples phase on the set of events, I am constructing a new measure. And this is given as I should write this way. All ‘A’ element of the sigma field this is what I should write. I construct this, but I additionally assume that if the expectation of Z is equal to 1, then P tilde is a probability measure. Observe that in that case when e to the power Z is 1, you will get for example, if I have this is nonnegative, so this is always greater than equal to 0, if e to the power Z is equal to 1 when I can write P tilde of omega that should give me 1. So, that is integral omega Z omega d P omega and this is nothing but e to the power Z, sorry expectation of Z, and expectation of Z is equal to 1 which we are assume. So, under that assumption, you will see that this gives me a probability measure.

Now, this Z is often called the Radon-Nikodym derivative. We will write it in slightly more details. So, if you go and look into the discrete version and this would be nothing but the ratio of P tilde by P, give me any x random variable. So, x is any given random variable. Now, we can find it is expectation under the probability P or we can find it is expectation under the probability P tilde. So, what would be that how would you compute? If you use this formula, and you can show that for any such random variable, the expectation under the risk sorry under the measure P tilde probability measure P tilde is same as expectation of x into Z in terms of the probability measure P.

Of course, if probability at Z is strictly greater than or equal to 0 that is set of all omega as said Z omega ids strictly greater than 0. If it is equal to 1, that is Z is equal to 0 the Z over omegas that Z is equal to 0 that probability is 0 that is the non event. Then you can write the following then E to the power x is equal to E tilde 1 by Z x. So, these are all of fallout of this formula, all fallout of this formula. You can just calculate the amount, it is not a very big issue I think this is a good exercise you should really try it.

So, Z as I told you is called the Radon-Nikodym derivative, this random variable is often denoted in this form. There is a deep breath Radon-Nikodym derivatives has a very deep role in measured theory, because there we talk about equivalent measures and all those things that is P, P tilde and P are equivalent probability measures if they agree on the null sets that if a is null for P a is null for P tilde and vice versa. Just if we use a calculus inside it looks pretty often d P tilde by d P you know then it is just d P tilde and that gives you this. I leave you to so those with a little bit of feel for mathematics I ask you to write down the discrete version.

So, as I told we are from this moment, we are in a continuous time set of we are no longer be dealing with this discrete things, discrete timeframe. So, this change of measure is a very, very important thing. So, suppose you have or a normal random variable under given probability measure and then you make change on the variable. Suppose, you have a stander normal variable in which make change or translate the variable then your mean changes, so it does not remains standard normal. But under probability change, if you can derive a new probability measure in this way, so such that under that particular new probability measure a new variable also remains standard normal. We will just describe this here.

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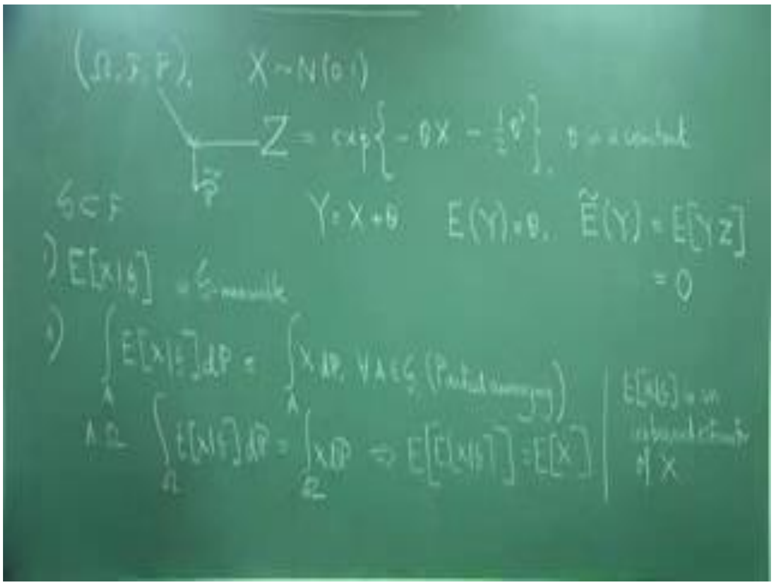

Now, consider as an example of what I just told you consider now let this P is the probability space and X be a random normal standard normal variable. So, X is N 0, 1 under this probability measure. And let me define Z, so I am going to know make a transformation, Z equal to exponential minus theta X, where theta is a constant minus half theta square, theta is constant, real numbers, any value to be positive note. If so under this Z from P, you can come to with this, Z and using this P you can come to a P tilde.

Now, if you construct a variable Y, which is X plus theta and under this probability measure P, the expectation of Y is equal to theta because expectation of X is 0. But you can show that if you take P tilde then expectation of Y under P tilde ,which will be nothing but expectation of X Z and that would give you 0, which will be expectation of Y Z, so that would finally, if you use this formula now. So, P tilde you know what to do, now this is a expectation of Y Z, and this will give you 0. So, under this new measure Y remains a standard normal variable. So, this is just an example.

We will now start deriving two lemmas, which would be very, very useful for our discussion and proof of the Girsanov's theorem. Just to remind you about the notion or definition of conditional expectation, so conditional expectation as you know is a random variable, which takes a particular value on a particular piece of the sigma algebraic if it is countably generated. So, in general the definition is following. So, what is conditional expectation of a random variable x given sub sigma algebra g of f, it is actually trying to estimate x itself.

So, in that case just remember that what we have is that so if G is sub set algebra of F and the conditional expectation of X given G, X is a random variable if G measurable. So, this random variable G measurable that is inverse image is inside G. And number two is called the partial averaging that is for any A in G, this is called the partial averaging formula. I assume that you would go back to part one, and remember the properties of conditional expectation.

Now, what I want to tell you is the following. From here, you can have a very important result. You see for all A element of this G or calligraphic G, you have this result. Now, this G be itself A sigma algebra, the whole space omega must be inside G. So, this would imply that I can put here omega. So, if I put A equal to omega then I will have sorry, so what does this give me, it tells me this implies that the expectation of the random variable conditional expectation x given G is same as expectation of X. So, E x of G is an unbiased estimator in the language of statistics it is unbiased estimator of X. So, E x by G is an unbiased estimator of X. I think this is this was I do not remember whether it has been stated in the previous part. So, I have just told it other laws had be completely stated and this formula also been stated. Now, we are going to define something called Radon-Nikodym process idea which will be useful in proving the Girsanov's theorem. So, we remove everything from here and then we would start this.

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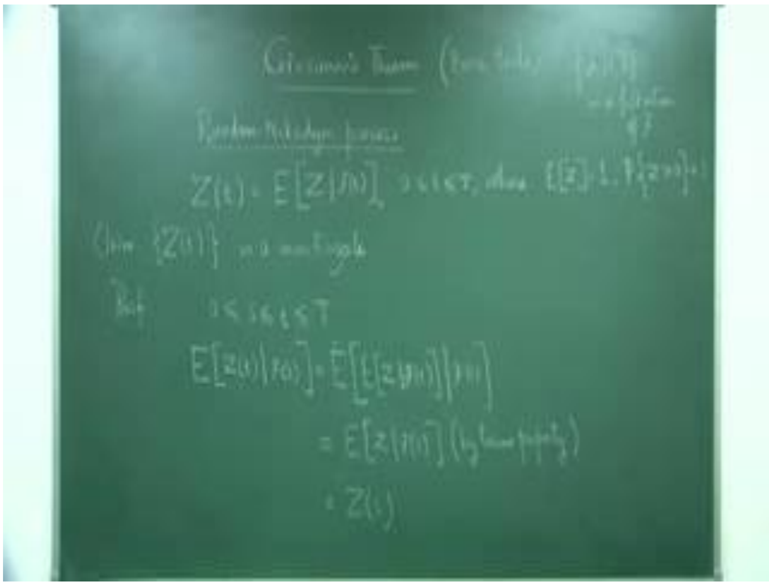

So, we will start with the definition Radon-Nikodym derivative process. So, let us just write it down see, we are into depart territories, where all from the simple algebra thing that we have been doing for long time, and we are really into advance territory. So, in Radon-Nikodym process, we define a process means stochastic process as Z is a random variable with respect to with which do the change, change of measure probability measure. And F t, so F t is a filtration of F, I am just writing of F means it is as preceded with F. Of course, Z is a nonnegative random variable I, it is better to have rather also let us assume that that it is a strictly positive random variable almost everywhere that will be needed because we will have to use this in the denominator. So, this is conditional expectation generates this random variable Z t and the change of t this generate a process. So, 0 is the basically starting time of the contract and T is the terminal time of the European call option that is it from a finance point of view.

So, we are going prove that first step for claim is that Z t stochastic process is a martingale. So, how do you prove that? The proof goes as follows. So, you take expectation of Z t given F s and Z t a by definition so this under the same just same probability measure P Z t is expectation of Z given F t, T is always bigger than S. Now, applying the tower property by application of the tower property, we will now prove that this is nothing but now once you know from the definition that this is nothing but Z of s and that is exactly what we need. So, this proves that this is a martingale. Now, once this is done, we need to prove two important lemmas associated with this sort of processes.

So, first we will be writing down the statements and proving them separately. This lecture, this is one-theorem basic tools lecture would be culminated with the proof of these two lemmas and then in the next lecture. We will prove Girsanov's theorem itself. So, we will do the lemma A and B, so lemma A will start with I will just found the rewind that all these things are done from this wonderful book of Shrive, we will now follow Shrive for the remaining part of the lectures.

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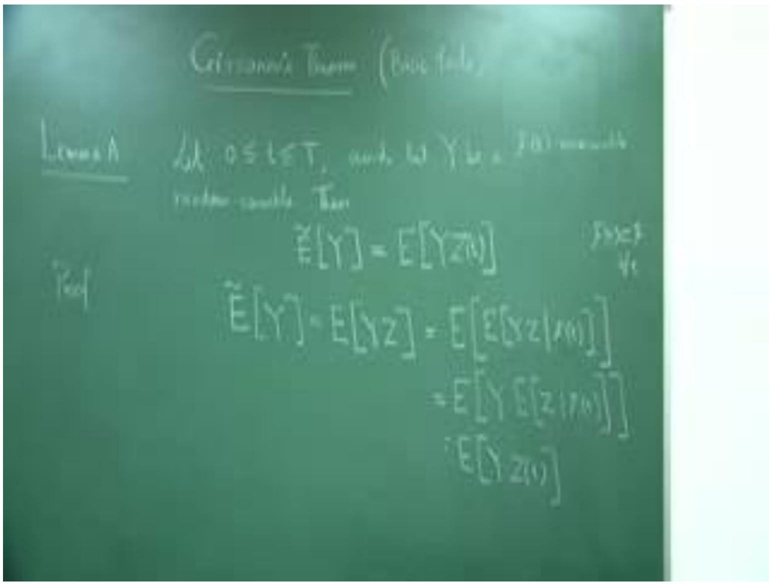

So, lemma A says that, so let us consider this trading period 0 to T, and let Y be F t measurable random variable. Then under the new probability measure, so you will have a Z etcetera everything is now assume here that you have this. Under the new probability measure, expectation of Y is same as expectation under the old probability measure Y Z t. So, take any T this is beauty of the whole point, whatever T you take it does not matter. So, when you take put a T, Z t, this Z t is the when you fix a T it is a random variable. So, Y into Z t is another random variable. If you take it is expectation under the usual probability measure and always divides where of course, Z t is Radon-Nikodym process. If me take the Radon-Nikodym process at any point T and multiply it Y and take the expectation is same as taking the expectation under the changed random variable that is slightly interesting part can be used in several things.

So, using what we have earlier stated expectation of Y. So, I am doing a proof expectation of Y is expectation of Y Z, where Z is the random variable with e to the power Z equal to 1, with which you do the change of measure. Now, this because conditional expectation itself is an unbiased estimator of the variable itself this is the unbaisness thing. Now, the first law taking out y is what is known because Y is F t measurable at time T equal to T you can take Y out. So, it will become expectation of Y of expectation of Z given f t conditional expectation of Z given F t. So, because Y is F t measurable at time T, you know everything about Y, you would taking out Y. So, expectation of y and by definition this is F t that is exactly. So, whatever be your T, does not matter this is the answer. This is the answer.

Here when once you fix T - the time T, F t becomes sigma algebra sub sigma algebra of F because here F t of course, is sub sigma algebra of F for all T. So, these now acts as G and that just do that that is it. So, whatever be your T, if you do compute this it always give you this that is the beauty of the Radon-Nikodym process. Radon-Nikodym process whatever point you choose from that, it always gives you this it is not that this is also T dependent no, so amazing. So, ultimately this expectation becomes independent of T once you take the new changed measure. So, then we are now going to write our second lemma and will do the proof and finish it.

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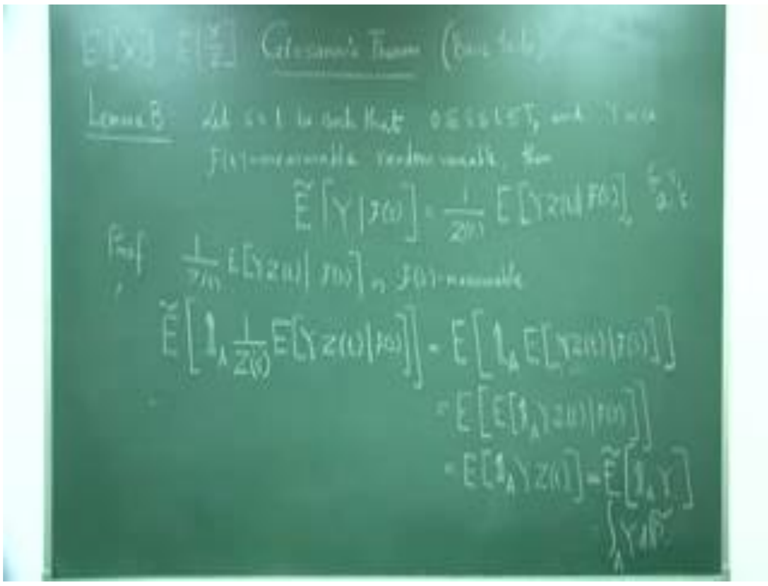

So, now we are going to prove lemma B, which is important in the proof of the Girsanov's theorem both of these actually, lemma B would be slightly looking more complex, but does not matter. So, let s and t be such that 0 less than s less than t less than capital T and Y t be sorry not y t, I make a mistake, just y like the one before. In lemma y be an F t measurable random variable let Y. And you see this we are assuming that the variable Z is a strictly positive variable that is Z is positive almost everywhere. See all the statement that we are making is always, almost everywhere statement. And Y is a F t measurable random variable. See all these terms those who are new to this course have to be seen from the first lecture, measurable random variable.

Then E tilde Y F s is 1 by Z s and that is why you needed that positive assumption, expectation of Y Z t F s almost surely. So, this is true for only those points where Z s would be strictly bigger than 0, so for s almost surely. So, this is what we have to prove because if Z s equal to 0 of course, it does not have any meaning. So, with only for those points where Z s is strictly bigger than 0. So, most of these statement that we are making, even when you are writing the equality of the results or the properties of conditional expectation then all almost surely there are. And there are essentially because these all are integrals you just forget about the set of measures 0 when you integrate.

So, proof, if you look at the very first statement this is what. So, suppose this is equal to this, but this by very definition is an F s measurable random variable by definition of conditional expectation, and then this should be also F s measurable random variable under the new probability measure P tilde. Basically what happens is that to show the equality of this, we know from this fact that we will have for any A in F s, this will be true, but we essentially have to show basically then we will have, any random variable which satisfies this in terms of Y is the conditional or expectation of y with respect to that sub sigma algebra.

So, essentially what we have to show, if we show that for all A belonging to F s, if you can show that 1 by Z s expectation of Y Z t F s d P tilde, this must be, suppose if I can show this, so then this must be itself the conditional expectation of Y in term with given the F s. Means these two variables essentially have no difference only they can vary over mirror 0. Otherwise, this is also the any, any random variable, which is F s measurable and does this is the. So, obviously, we can show that this is F s measurable right and then that is it proving the remaining is nothing. So, once you can prove this you are essentially proving this. So, we will start proving this, I am removing this part and I am starting to prove and I want to prove this fact.

Now, this one - this is the conditional expectation in terms of the thing F s. So, this expectation this and this random variable F s is measurable by very definition of conditional expectation. Take [FL]. So, start. So 1 by Z s expectation of Y Z t F s is by very definition F s measurable because this a conditional expectation with respect to F s is F s measurable by the very definition of conditional expectation in the continuous setting F s measurable.

Now, once I know this then I have to now do take this integral this integral 1 by Z s expectation of Y Z t F s, where A belongs to F s d P tilde. I can instead write this as omega with a indicator function characteristic function A this random variable 1 by Z s expectation of Y Z t F s conditional expectation this d P tilde. And this actually gives me conditional expectation in terms of the measure P tilde of 1 A, means if when omega belongs to A, this will be one otherwise it will be 0 is a characteristic function. So, I can write everything in terms of single expectation. Expectation of 1 by Z s expectation Y Z t

F s, so this is what I have, so this integral is actually this, just I have to compute this and show that this is integral A Y d P that is all, so that is what I am now going to do.

So, once you have done this, so I am just now going to compute this. So, basically now I am writing tilde characteristic random variable I A or 1 A whatever. So, again this random variable if you takes the value 1 in A, omega is in A; takes the value 0 when omega is not in A. 1 by Z s expectation y Z t f s this is what we have to compute. [FL] When Z is strictly positive almost everywhere then we have read this thing now E of X is right. Once you have once you know this, so what would happen? So, Z is strictly positive almost everywhere this one. So, this is nothing but this can be written as expectation of the indicator function, the indicator random variable expectation. So, I am the expectation is now just like this expectation tilde x by Z is expectation of x.

So, I have now taken of the tilde because now I am back to the old measure. So, Y Z t F s because A is element of F s, since, A is element of F s, I can take out if it is inside I A or 1 A, the indicator form of random variable you can taken out. But it could be taken in also right, because this is element of F s, it is an F s measurable random variable. It is expectation of expectation Y Z t F s. By the previous result, so what is this now? This is again the unbiased results. So, this conditional expectation is an unbiased estimator of this. So, this would be expectation of 1 A the indicator function Y Z t, and this by the very previous lemma A. So, by lemma A, we again go back to the E tilde where it shows that E tilde is A Y and this is nothing but integral a of Y d P tilde and that is what we had set to prove out. And hence, we are proved the results, this is exactly what we were intend to prove.

Thank you. In the next class, we state and prove the Girsanov’s theorem which is pretty involved it will take the time, it will take, the whole class will be gone to actually to do the proof. Thank you very much.