Probability and Stochastics for finance-II
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Lecture - 15
The Binomial Model-IV

So, here we have binomial model part IV, I guess binomial model part IV. So, we are now going to discuss various natures of the stock prices itself, in terms of risk neutral probability, if you have observed going by what we have studied in the last lecture.

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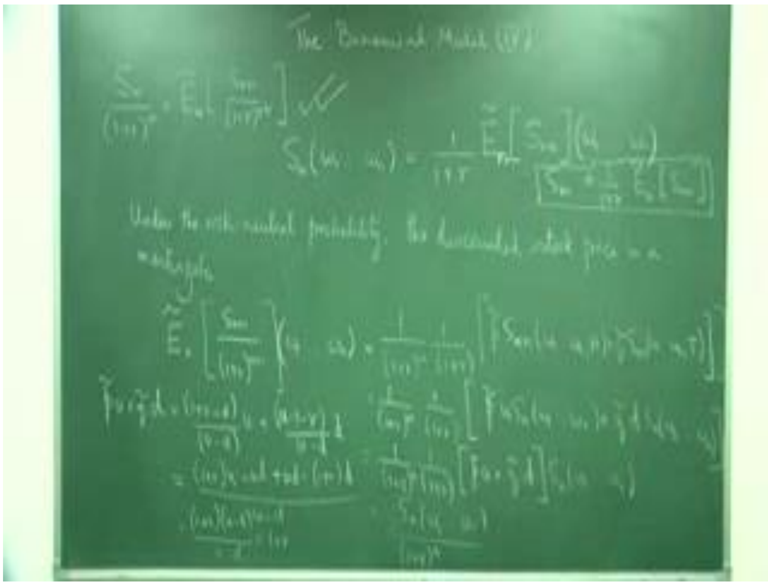

I get that nth stock price can be given like this at time n when you know what has come out; nth stock price can be written as, conditional expectation. This is what we have discussed in the last class. You can do that do it for V n and whatever everything as the same format.

So, we are now going to prove that the discounted stock price, we are going to prove that of course, p tilde, q tilde given I am not repeating p tilde, q tilde. Under risk neutral probability, the discounted stock price is a martingale. How do I prove? What is the meaning of martingale in this case? So, what I have to prove, discounted stock price. So, discounted stock price at the nth stage is this one, where r is the simple interest rate. This is nothing, the conditional expectation under the risk neutral probability; we will write it E n for short. This thing I can write this as, this expression as 1 plus r I can write this expression in short. So, when I am taking meaning, when I am writing this, I am actually meaning this. I am actually meaning this thing.

So, this random variable operates on this, this is the meaning right, exactly what it is. Sorry mistake in the signs. I think I also put n here. So, this will mean that S n is equal to this right, n is means conditional expected at the nth stage. So, n is just to n, I am just using n here also because it means at the nth stage, what is happening right. So, discounted stock price, this thing would be same as S n plus 1, 1 plus r n plus 1. So, we will prove this fact. So, this is a very important thing and this idea will be of primary importance, even in the case of the continuous time model because this idea would be used in the repeatedly in many ways in final. These are very fundamental idea and (Refer Time: 04:12) pricing theory.

So, these are all pricing for complete market. So incomplete market things are completely very different and the lot of things, if suppose you want to talk about interested you talk about interested nothing else. So, let us see how we go about it. The two different proves. I am just giving one proof quite a direct one with what I have written. You can use formulas or properties of conditional expectation, but let me just do it directly. I can write this as 1 plus r to the power n 1 by 1 plus obliviously it is a fix thing and take it I out of the conditional expectation and this means, this is nothing, but 1 by r to the power n, 1 by 1 plus r. Thus this expression is nothing, but E tilde S n plus 1 omega 1, omega 2, omega n or rather I can just write S n plus 1 does not matter, omega 1, omega 2, omega n write after omega 1, omega 2. Here I am writing everything in a functional form.

So, just for gravity S n plus 1 basically S n plus 1 H plus q tilde S n plus 1 t. Actually I should not write this basically I have to be very frank with you my actual writing should be more precise. So, if I want again still write it more precisely. Let me just do it. So, that you do not get confuse at the end. So, conditional expectation is the random variable, operates on this outcome at the nth stage. So, the nth stage the sigma algebra is the sample space is omega n, there is the sample space consisting of these sorts of points all possible n out comes and now I can write this as 1 by 1 plus r to the power n, 1 by 1 plus r writing the formula directly p tilde S n plus 1 omega 1, omega 2, omega n H plus q tilde S n plus 1 omega 1, omega 2, omega n.

So, again 1 by 1 plus r to the power n into 1 by 1 plus r, you know this is p tilde u S n omega 1, omega 2 omega n plus q tilde d S n omega 1 omega 2 omega n. So, this up down business and then S n is fixed to take it outside, S n you know, so you have 1 by 1 plus r n to 1 by 1 plus r into p tilde u plus q tilde d into S n omega 1, omega 2, and omega n.

Now let me compute, what is p tilde u plus q tilde d. What would be this? If I did, p tilde if you just put the value of p tilde and q tilde, you will see what would happen that this will become p tilde was let me see what I can remember. 1 plus r minus d y u minus d into u plus u minus 1 minus r into u minus d into d, what are we getting? We are getting 1 plus r u minus u d plus u d minus 1 plus r d by u minus d. So, basically we are getting 1 plus r u minus d by u minus d. So, we are getting this to be 1 plus r.

So, this is 1 plus r which cancels with this 1 plus r to give me S n. So this occurs for what? Omega 1, omega 2, omega n and all possible combinations, this random variable exactly is equal to this random variable that is exactly the type of thing that we had wanted. So, shows that it is a martingale.

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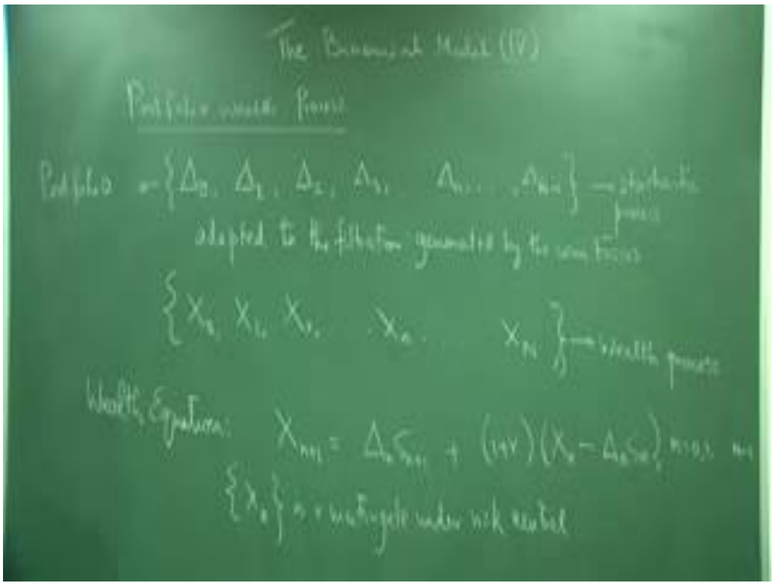

Now, we talk about the portfolio process, portfolio wealth process. So, we are not talking about at essentially talking about stochastic process. If you look at the price S 0, S 1, S 2, S 3 capital S n that is also a finite times stochastic process. So, I am not going to tell you what is stochastic process etcetera, etcetera, because already discussed in part 1. Those who are not studying this thing in the part 1 should go back and look at the things in part 1. So, portfolio wealth process, what is portfolio wealth process?

So, portfolio wealth process means, what is happening? At time delta not the investor buys my time t not buys delta not amount of shares of the stock. At time t 1 he basically sells them whatever money he is also getting for the money market uses that, to use whatever he gets and also form the money market uses that to finance. What is the money is used to finance delta 1? And the delta 1 depends on the outcome that he has. So, delta 1 depends on the first outcome, delta 2 depend on the first 2 outcomes and so on so forth, delta 3 delta n to delta n. So, delta n minus 1, this process is also a stochastic process, is a stochastic process what, it is adopted to the filtration because thus delta n depend on the first n coin tosses, delta 3 depends on the first t.

So, the information available up to m time period n is what are the n coin tosses up to that time and this depend on that. So, this is the adopted process, is the process which is adopted to the filtration. So, the portfolio stochastic process is adopted this called the portfolio process, so portfolio of the guy who has sold option and is now trading the market to hedge his exporter to risk.

So, risk is essentially the whole issue. To a market there is only one issue that is risk. All things are done just to safeguard your risk there is nothing else. Risk is the fundamental issue, adopted to the filtration that is generated by coin tosses. So, what is going wealth of the port folio at each n? So, at time t 0, you invest X naught and t 1 you have X 1 and time t 2 you have X 2, time t n you have X n, time here capital X. So, this is V n sorry, sorry. So, each of these random variables here that you see that make of this stochastic process this is called a wealth process and again this a adopted process. This is the adopted to the filtration. It depends on whatever information that you have till time small n. So, X n o depend on that called a wealth process. That gives us something called a wealth equation. So what is the wealth equation? That is what is the worth of the portfolio, at the period n, so which is called the wealth equation.

So, here the wealth equation at time n plus 1 is nothing, but I have to sell what I held at time delta n. I am selling it and making this amount of money plus I am looking at what I had invested in the money market X n I had in the earlier thing or which I have bought delta n share being S n price and remaining I have invested in the money and then here n starts with 0, 1, 2, dot, dot, n minus 1. So, this is called my wealth equation. What sort of a process is this? So, this process is the martingale, that the discounted, the expected value of the discounted process at time n plus 1 is same as the discounted value at S n. So, this is a martingale. Basically this is a no arbitrary scenario. So, martingale essentially arises in that sort of scenario.

So, now we are going to show, that under the risk neutral probability. So, under the risk neutral probability X n is the martingale. So, this is my wealth process X n is the martingale under risk neutral probability. So, this a very important information about the behaviour of this processes because what we are telling is that when you have n period models, we are generating finite time stochastic process and here stochastic process idea of the stochastic process comes in. If you really want to know, what is it? This is an E 2 integral. What I have written here is an E 2 integral that is all. E 2 integral in this discrete times (Refer Time: 17:01) this is an E 2 integral that is all.

This is actually some sort of difference between S n plus 1 this is d of S n basically. Which we will not discuss right now, but rather we would try to prove that this is a martingale, under these neutral probabilities. So, these are not martingale under the market measure, market probabilities ever. The only market the martingale under risk neutral probability and this is a very very fundamental thing because this would allow us to compute the prices in this complete market set up where, you can do perfect hedging. Otherwise this market probability would not allow us to gain anything. This is a very very fundamental leap in the imagination. These are amazingly brilliant idea to get this neutral probability measures, so the martingale under risk neutral probability measure.

So, our goal here finally, we are now look show that the wealth process is the martingale discounted wealth process. So, we are goal here is to really show that how can we do the computation of this option price at every step. That is an option is been told that, this is the selling time of the option. Somebody declares that I want to sell now option now and which will have an expiry time at capital N and 0, I am selling the option, but suppose you want to buy the option at time period say 3. But you are expiry is still at capital N. What would be the price you have to pay then? So, that sort of things can be answered by using the risk neutral frame work. So, there are lot of issues which we are not getting into. So, we are again going to show that the discounted wealth process is the martingale under risk neutral probability and then we are going to write down the risk neutral option pricing formula which will end our discussion.

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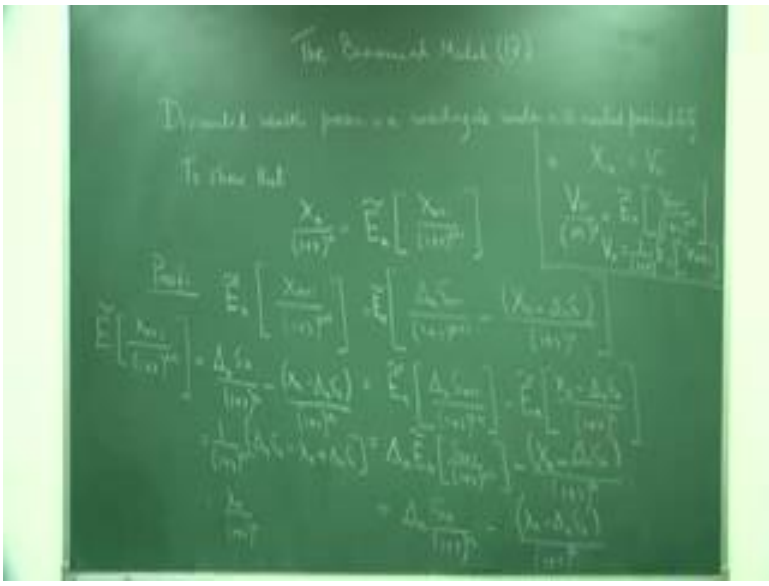

Discounted wealth process is a martingale under risk neutral probability. So, what we have to show? This is the very very important pricing formula.

So, we will come to that, we will show that X n. So, how do we do anything about it? Let us start looking in to it, so the proof. So, this is again a key to the pricing formula of V n because V n, X n is equal to V n that is the fundamental fact. That at every n the worth of the portfolio, the wealth of the portfolio must match the worth of the option at n that is it and worth of the option at n is the price of the option at that time. So, you may buy the price, or the option at any time in between, time on the option is announced and time on the option is finished buy at any time in between, but actually you want to know the V 0 price, but at anytime in between what is the worth option is actually the price of the option.

Proof, let us write down. So, we will now use a wealth equation basically. So, this is written as delta n. So, whatever you have got at time n is now sold a time n plus 1 with S n plus 1 minus 1 plus will now be get cancelled and you will have X n minus delta n S n. So, that is the amount you have in the invested in the money market at time n. Now by the basic norm of sorry you have forgotten to write expectation. So, by basic facts about expectation or conditional expectation because they are linear, you can write that because

I used p tilde, q tilde I am giving this tilde sign. So, say that this is under risk neutral measure, risk neutral probability measure. Now at time n X n is completely known, S n is completely known. At time n if I am selling time n, I know what is the price at time.

So, I know what is the portfolio? What that time n? So, whatever is known it has to be taken out right. At time n delta n is also known. If I am because the things are done at time n plus 1, but at time n right, delta n is also known. I know how much I have to buy now right. So, delta n I am taking out what is known. Let see how things are beautifully coming. Now here also it is like taking out, what is known this into the variable 1, constant random variable which will you give 1.

So, basically this is at n this no constant basically. This is completely known this just a constant. So, the expectation of that constant is. So, this is same as taking what is known. Now what does this tell us? We have already known that the price process, the discounted price process is the martingale under risk neutral probability, this will give me delta n S n by 1 plus r to the power n minus X n minus delta n S n 1 by r to the power n. This I will use the fact that, this is the martingale the price process is a martingale. So, what I have now got is E tilde X n plus 1 1 plus r to the power n plus 1 is equal to delta n delta S n by 1 plus r to the power n minus X n minus delta n S n by 1 by r to the power n. So what I am getting 1 plus r to the power n delta n S n minus X n plus delta n S n. So this is giving me X n into 1 plus r to the power n. This is exactly what I have wanted.

Now, this has a magical conclusion because at every stage n, for the risk to be completely hedge or perfectly hedge, the worth of the portfolio, the value opened at wealth of portfolio must match the worth of the option which means, now from here I can write that V n by 1 plus r to the power n is E n V n plus 1 1 by r to the power n plus 1. So, basically then from here I will get V n, if I take this 1, bring this out is 1 by 1 plus r, which way we had been studying all along. So, this is called the risk neutral pricing formula for an option.

So, if you put n equal to 0, it is V 0, V 1, E 0 V n means nothing, but the expectation E 0 is n, but expectation of V 1. So, and V n would bought again. So, here you have to go back recursively. This whole process, this process when we write this formula in continuous time would actually generate the black (Refer Time: 28:06) model using probability theorem. That is one of the fundamental things that we had been trying to tell though this discussion. So, here in this discrete set up we have got finally, into a frame work by which we can now move in to continue set up. You see once you have got an idea about the discrete set up, a lot of things in the continuous set up would look similar, you will ok. Of course, there you are discounting or discounting price would be something like this.

This would be a discounting parameter instead of 1 plus r n all those things. So, except those things the continuous version the limit versions of them, everything else would look like similar. So, learning the discrete time part is absolutely a fundamental importance. You will think, forget the discrete time, and just do the continuous time. That is absolutely a wrong thing to do.

So, next week last session, we will use things which we have already spoken in the part 1 and so those who have done the part 1, it is fine those who have not done please have a look at them.

Thank you.