Probability and Stochastics for finance-II
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Lecture - 14
The Binomial Model-III (Multiperiod model)

So, here we are now going to talk about multiperiod model. That is my expiration date is not at time t equal to 1, may be a time t equal to 2, time t equal to 3, time t equal to 4, or time t equal to n. So, here we start and the very basic structure when time t is equal to say 3 rights, it is a 3 period model how would it look like?

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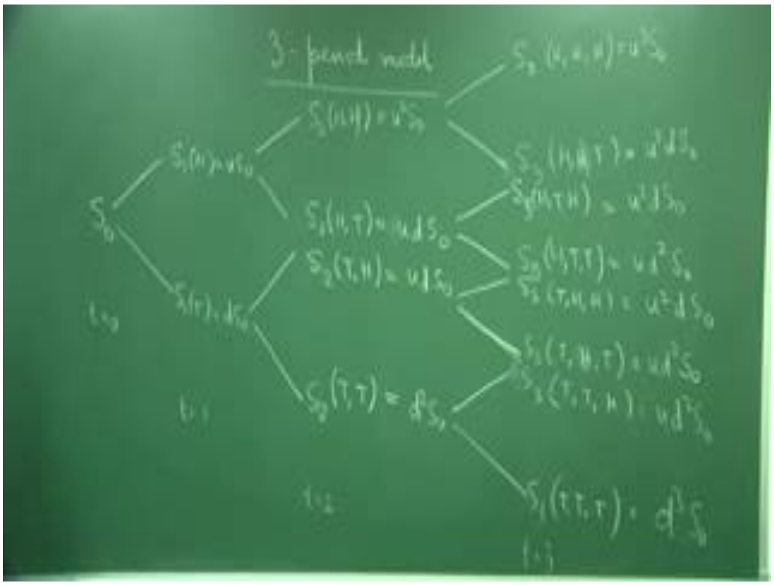

Let us draw the diagram. First will get the stock price we will get very soon from here. So, suppose I start with S 0 that is time t equal to 0 and it goes up and goes down with S 1 H and S 1 T. So, this is u S naught, d S naught. From here if I am here, I am on this branch and I have two possibilities right. So, which is here S 2 H, S 2 H H means head and then in second toss also I said and S 2 head and the second toss is tail. These are the possibilities. So, this will become u square S naught and this will become u d S naught. From here also I left two possibilities, which will be S 1 T H, that is now first tail has come then head has come, when S 1 sorry S 2 T H the time this index is represented putting on a representing the time, this time t equal to 1, this time t equal to 2.

So, here tail has fallen, and then again tails has come. So, this is nothing, but again u d S 0. So, you see this is same right and then this is d square S 0. So, these are the same. This is equal to this, these two things. Now, from here again if I am on this branch I have [FL] branch [FL], there will be two things, which will happen. So, from here I can just take, take more branches out, but lot of the values would be similar. So, here what would happen? It will become S 3 H H H and that will be u q S naught or it will be S 3 H T T sorry H H T. So, head has fallen, head has fallen and tail has fallen. So, this will become u square d S naught. Similarly if you look at this point, you can also make from here two choices and from here you can make two choices, from here you can make two choices, 8 choices in.

Now, let us see what is this choice? If I want this branch, the two choices are S 2 H T H S this is nothing, but what happens if I toss 3 coins simultaneously or a coin 3 times. So, these are the outcomes that are all S 2 H T T. So, if this happens then from here it goes up. So, it will become u square d S naught and if it goes down it will become u d square S naught. So, you see these are equal. So, it not S 2 S 6, so S three H H T S 3 H T H if there are two H 2, two heads or two tails and if the basic structure not the permutation, but basic structure is same you basically have the same outcome right.

So, here again you will have, from here you will have S 3 T H H S 3 T T sorry T T H T. So, this will again become u square d S 0 and this will become u d square S 0. So, this was u d S 0, u d S 0. So, here you go up. So, you become multiply by u, u square d S 0 here again. So, u d square S 0, you see 2 tails 1 head. These are the same values. Similarly here you will have S 3 T T H S 3 T T T. So, what you will have, here will have u d square S 0, when you go down you have d q S 0, u d square S 0, u d square S 0, u square d S 0, u d square S 0, u square d S 0, u square d S 0. You see that three batches of same value and these are with the cubes. So, this is the case for time t equal to 3.

Now, this is one particular setup. Now I will draw this with very with lot more you know it will very difficult to draw, but with lot more periods and then I will show you a strange and interesting pattern emerges, which we have spoken about the movement of the stock price in part one, that they follow some sort of geometric Brownian motion, so sort of zigzag thing comes. So, we will study this here. So, let me remove all this. I am measuring that this is clear to you just I remove all this and then I can do the other one.

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Here i have done a very bad job, bad features as. So, basically I am trying to increase the number of periods, I am trying to draw the binomial lattice, binomial diagram. Suppose you come from here go down there, go down here, then go here and go here and if you look at this path here you a zigzagging path.

Of course this is a very bad diagram you can make it much more simpler by having spread outs in a much more simpler way. So, it is always like this there will be equalities. You know you have to actually this sort of thing actually if you should is what happen. So, I could make a much more. So, what you see actually have a zigzagging thing coming. So, the stock price as moved in a zigzag way and very well understood by drawing a binomial lattice. Now how do you do an option pricing in this case.

So, here my price or the worth of the option cannot be seen suppose I take a 2 period model right. If I take a 2 period model, after that will general write the general version of (Refer Time: 09:12) period model. Suppose we take a 2 period model how do you we price the option? How do I get V 0 now, I have to first come back to the period 1 and back to period 0. Now, I am taking 2 period models.

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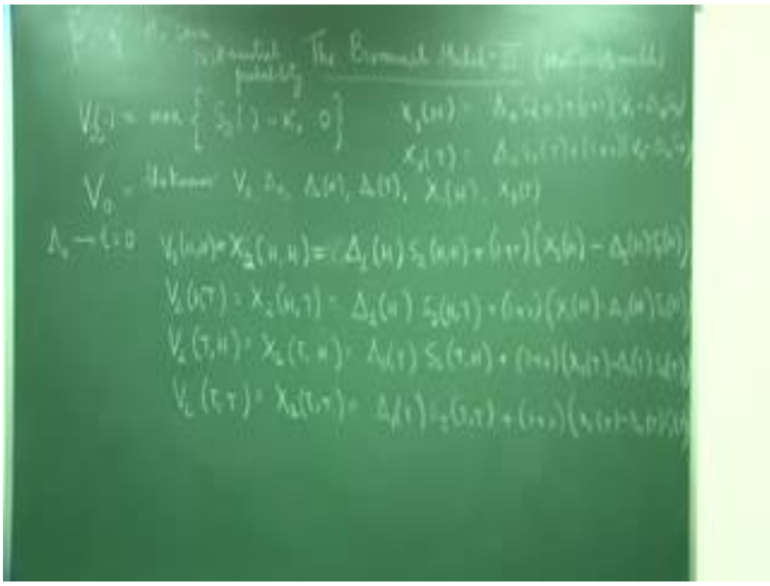

In a 2 period model my worth of my option is V 2 which is maximum. So, I am talking about only a European call options nothing else S 2 whatever it is there and the strike price. This is my worth of the, these are random variables I am keeping this again to it look like a function.

So, random variable is of course, the function. So, here this is my worth of the option at time period 2. Now I can be much more straight forward now at. So, why have to find V 0 option price. So, when you have periods this, if your expiry time is after say 20 period or 30 periods or say 100 periods of trading periods, then this binomial lattice gives you a very good approximation of the black (Refer Time: 10:37) price. So, V 0 is an option price. Now what I will do? So, they are investing that V 0. So, what I have done? So, what is my portfolio? What that time t equal to 1, it could the H or tail depends. So, I have bought delta naught shares of the stock at time t 0 sold it at this price H or T of course, and then the remaining I had invested I have actually used V 0 to buy it because I am directly writing these are not writing S 0 anymore because now I know that this option price V 0 and S 0.

So, then this is what I get time t equal to 1 right. Of course, you can write them in H and T form that is you could put an H here and you could put a T here also because V 0 delta naught S naught is the fixed quantity. So, do not have to bother about that. Now this is the worth of my portfolio. This is the money I get wise trading the portfolio at time t equal to 1 then at time for time t equal to 2 I buy delta 1 amount of shares, I buy delta. So, delta naught share is border time t equal to 0. So, I had delta naught share is border time t equal to 0 at time t equal to 1 you buy delta 1 at that though buy guy who has sold the option and then is using option price should take when I trade in market will buy delta 1 shares of the stock. So, at time t equal to 2, what is the value of it is portfolio? The time t equal to 2, the value of it is portfolio is X 2 and X 2 can have many options X 2 can have H H, it can have H T it can have T H it can have T T right.

So, value of the stock should be is delta 1 and S 2 this is the amount I can sell delta 1 S 2 this is the amount I can sell and I had X 1 amount of a money at time t equal to 1 with that I have bought delta 1 shares of stock and remaining I put in the money market. So, from where I will get X 1 minus delta 1. So, delta 1 was brought with S 1 amount of money could be H or tail, that does not matter and then when I come to time 2 t I sell that delta 1 shares and then this is the money I get from the money market. So, this is my total wealth. So, this is written on the random variable. Now I will put this T T T H and also those things. Now what I will do, I will write down what are the four possible cases with X 2. So, I am just now rubbing everything, I am now writing down what are the four possible cases with X 2 and we will use the same p tilde and Q tilde. The same risk neutral probability which we have got with the data of the problem to solve this problem also. So, our data is nothing, but u r and d nothing else that is the main data in S naught.

So, this is same writing, but I am just writing it down. Now delta 1 this share is now also depends on what head and tail has come. The delta 1 share that you get, you have. So, X 2, now if 2 is the final period X 2 must be same as the worth of the option. That is the whole thing. So instead of writing this I just write V 2 H H X 2. So, delta 1 share that I want to buy will depend on what is the outcome of the coin toss at time t equal to 1 you cannot just fix up something. So, that is also random (Refer Time: 15:28) for delta itself is a random variable, you see how uncertainty has entered into the picture without you are realizing it possible because it looks very simple if you are actually and uncertain zone and so which pretty complicated stock actually. So, how do I write this one? So, it is delta 1 H see if H has come in the first 1 and I bought delta 1 H S 2 H H plus 1 plus r X 1 H because X 1 is what I have in the worth of my portfolio, when H has occurred in the first step. First time H has occurred so, minus delta 1 H into S 1 H.

Similarly, you can write V 2 H T is X 2 H T is delta 1 H into S 2 H T plus 1 plus r X 1 H minus delta 1 H S 1 H. So, S 1 the first outcome was head. Now again the first outcome was tail and the second outcome is H. So, it will become delta 1 T into S 2 T H they are same, but so, this is different (Refer Time: 17:31) 1 plus r X 1 T minus delta 1 T into S 1 T. Same story will come, it will be X 2 T T is delta 1 T S 2 T T plus 1 plus r X 1 T minus delta 1 T S 1 T. So, essentially you have here 6 equations because we have equations for X 1 H and X 2 H the X 1 T. So, worth of the portfolio at time 4 equation or 6 equations, what are the portfolio at time is delta naught S 1 H plus 1 plus r into V naught minus delta naught S naught delta naught S 1 T plus 1 plus r V naught minus delta naught S naught, the 6 equations.

And what do you have to find? And you find your unknowns. What are your unknowns? V 0 is your unknown right, V 1 is your unknown; V 0 is your unknown. So, V 0 is your unknown, delta naught is your unknown. Then you have to find. So, let us write down the unknowns. So, you have 6 equations, how many unknowns are there? So, my unknowns here, which I have to figure out is V naught, I1 have to figure out delta naught. I have to figure out delta 1 H; I have to figure out delta 1 T. So, the unknowns are V 1 delta naught, delta 1 H delta 1 T this is I have to know and I have to also know what is my worth of my portfolio at time 1. So, there are 6 unknowns in 6 equations. So, these are already known to me, this V 2 is known, but because V 2 is equal to X 2, X 2 is actually known, this is not unknown.

So, now how will go about solving it? So, again I would not go into the detail, but just write down the results of this by using the same risk neutral probability. Use the same P tide and Q tilde, the same risk neutral probability. So, once you take the, with P tide (Refer Time: 20:56) tilde is given by the same formula, that we have done in the last lecture right. So, 1 now we will rub this part and I am assuming that you have already taken it down, then we can just solve it. So, give you the solutions because it is no use for me going to the algebra and wasting the time. You can do the algebra yourself rather giving that solution directly to you. I want to tell you that delta naught would be same as what we have done earlier. We not of course, will take it is different form but.

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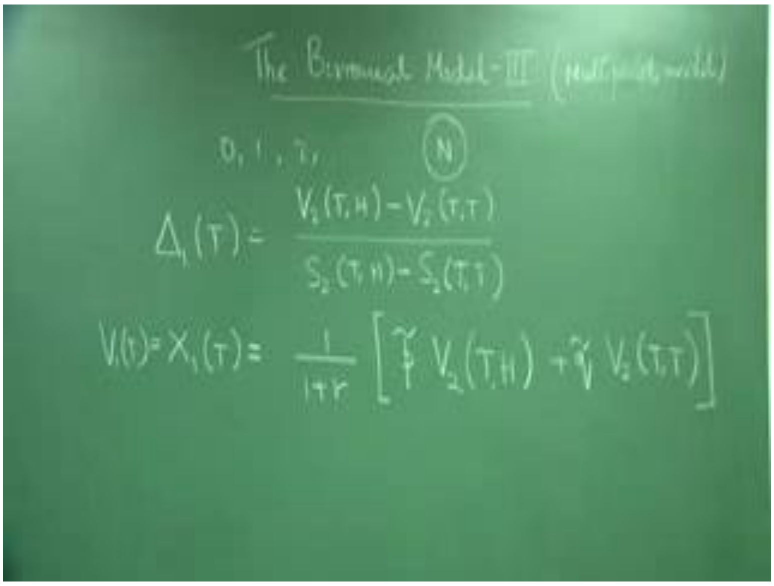

Let me first write down the case of delta 1 T. So, the delta hedging at time 1 how much stock you have to buy? So, delta 1 at time T is V 2 T H minus V 2 T T. So, this is the first the outcomes T in the first and the second outcome would be H and outcome would be T, so V 2 V H and this is the same one S 2 T H minus the same story just writing it in a slightly a (Refer Time: 22:14). Next one T, X 1 T you can write the H form, I am writing the T form. You can write, put H here. If I put H here it will become H T H H minus H T H H H T that is the thing. X 1 T it is I will tell you what is happening? Here it is 1 by 1 plus r it is again the expectation in terms of the risk neutral probability, p tilde V 2 T H plus q tilde V 2 T T because V 2 and X 2 are same and this is exactly equal to your V 1 T, that is the worth of the option at time t equal to 1.

So, V 1 T is exactly equal to this, the worth of the option at time t must be the same as my portfolio then only I am doing a perfect hedging. So, I am not writing the delta 1 H delta 2 H and also those things, so you can just write down just changing the all things. Now I will write down the case for a very for nth. If the capital N, say, I have periods starting with zeroth period 1, 2 these are the F periods. So, expire time of my contract is N. So, you can exercise the option only at time t equal to N, then what is the price of that option is what we are going to calculate now.

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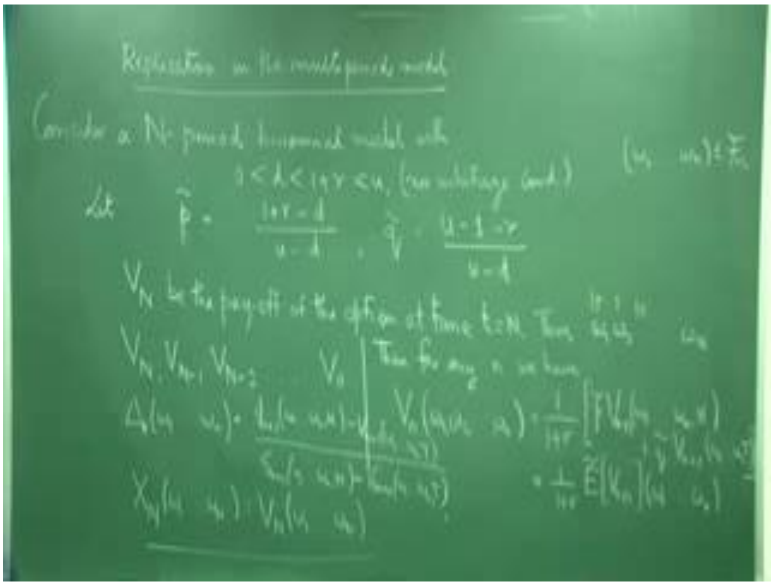

So, now we are going to write the general formula for replication in the multiperiod model. I am not going to prove this because this is the same sort of repetition what you increase the number of periods. So, I am just writing the results.

So, replication in the multiperiod model, there are. So, we have N period binomial model. So, you consider N period binomial model. So, if you want me to write down everything in detail, consider N period binomial model with 0 less than d less than 1 plus r strictly less than u this is an (Refer Time: 25:29) transcondition fine and of course, consider the risk neutral probabilities. Let p tilde and q tilde risk neutral probability is as given as 1 plus r minus d by u minus d and q tilde is given as u minus 1 plus r or u minus 1 minus r does not matter u minus d. So, this is there.

So, let V n be the payoff V n be the payoff of the option at time t is equal to N. So, it has to follow you know t equal to N means, thus there is a path by which we have written V 1 omega 1, omega 2, omega n. So, this could be head, tail, and head like this. It could be a head tail, head right.

So, basically you will start with V N, you know V n would be is equal to X, capital X N. The worth of your portfolio time N go to V N minus 1, V N minus 2 and recursively go down to V 0. This is exactly the way dynamic programming is done. The going recursively back to V 0 right, once you do that what would happen? Then for a general N then for any N, we have that V N that is the worth of the option at time n is the expectation of or expectation of risk neutral probability of it is values at time n plus 1.

So, suppose I want to know V n with the outcome omega 1, omega 2, dot, dot, omega n. These are my outcome. It may not be commas also I am just giving comma space. So, I like it. So, you can just give omega 1, omega 2, omega n, this is nothing, but a discounted price p tilde V n plus 1. So, omega 1, omega 2, omega n, as occurred up to n then at either head can occur or tail can occur. So, this is exactly the price, this is exactly nothing but expectation of V n plus 1 given. So, if I know the information up to n. So, it is a omega 1, omega 2, omega n is in the filtration F n of the sigma algebra. If you want to write it like that, then this is nothing, but the conditional expectation. If you have learned conditional expectation and that is what we will talk about in the next talk. So, what it says, the worth of the option is nothing, but the discounted value of the conditional expectation of V n plus 1 and; obviously, V n is random variable is nothing, but 1 by 1 plus r of the conditional expectation, under the risk neutral probabilities.

So, this is the way you compute V n, delta n is computed in a very similar way. So, if you know this as occurred then this is nothing you will just have to write V n plus 1 omega 1, omega 2, omega n, H minus V n plus 1 omega 1, omega 2, omega n tail divided by S n plus 1 omega 1, omega n H minus V sorry S n plus 1 omega 1, omega n, T. This is what you are going to get. Now of course, at the end you have this fact, that X N omega 1, omega n it must be same as V N omega 1, omega 2, omega n. So, this is the general formula, once you know this fact you just put the fact reversibly and go back and go back to V 0 this looks very easy. You write down the theory, but not easy to compute actually, but of course, you can do it they are waste to do it, but there is no time to really discuss above those things in full semester course that has to be given also discussions and thus nowadays a lot of way of doing finance.

Finance has moved beyond black (Refer Time: 32:03) what are issues in incomplete market that in, for example, you in the different pricing all those things and largely things are down in the incomplete market and you can look in terms of utility functions all those things. There are lot of issue have come. What are the economic issues have come, statistical issues are come.

So, here we are trying to get you into the basis of mathematical finance. For those who for example, working optimization and say finance is an application they should remember, math finance is a much-much bigger (Refer Time: 32:36) than math programming and they should also realize that or we are trying to do here is just trying to light, put a light in the entrance over huge hall, call the hall of mathematical finance. So, with this will go on to the next thing. We will show how this risk neutral all these things, what you just written down is done. What are the natures of stock price? Is it a martingale or not? What are the natures of the wealth?

So, we will talk about wealth equations and also those things. So, at X when you will write X N we will call that the portfolio value at time n is called the wealth equation when you write the equation down. So, all these things will discuss (Refer Time: 33:16) which will be or discussion in the binomial model, which would be the large discussion on the binomial model. Next week we continuous will discuss this whole thing, that we have discuss in continuous time and so, that we will able to finally, compute the black shoes price.

So, black shoes price is going to be just a beginning of your study in finance. Please remember somebody wants to have some knowledge in finance, has a huge pressure on him because he needs to know a lot of important aspects of mathematics. He starts with linear algebra, optimization, probability theory, stochastics process, differentially equations and so and so one can get into the model of Bessel functions and everything and going to classical analysis and classical differential equation, things can (Refer Time: 34:01) special functions things can get bit messy. So, one really has to focus on particular area optimum of mathematical finance, he wants to do anything. I just want to say that mathematical finance in indeed extremely huge subject, trying to understand each and every aspects of it might just back fire. So, one should focus and then just learn one part very well. Portfolio optimizing, focus on portfolio optimization, there is risk management, option pricing, focus on option pricing, statically tools, focus on statically tools and something like that, CAPM focus on CAPM something of this.

Thank you very much.