Probability and Stochastics for finance-II

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Lecture - 13
The Binomial Model-II

Welcome to the third lecture on the third week. So, we are essentially into auction pricing.

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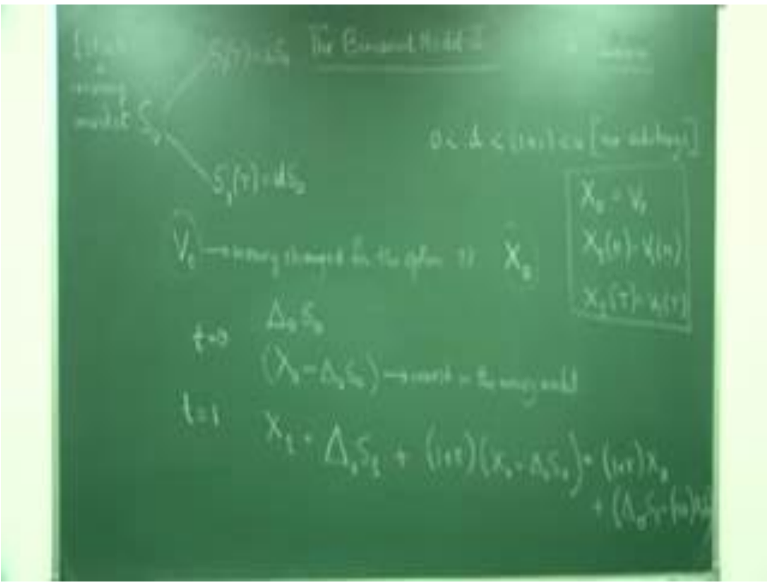

And we are today supposed to look into the single period model where the price here was s 0 and then we had two possibilities of the price at time t equal to 1 and of course, you have border auction with the strike price k and you want to know what is the price of the auction that even charged, so all those things have been completely discussed. I would just remind you that in our analysis here, the market is complete this means we will assume that once you take the auction price; the money and then invest in the market and the worth of your portfolio time one is same as a worth of auction that is exactly main idea behind the completeness of the market where the things can be perfectly hedged, risk that you are exposed to can be perfectly covered.

So, suppose I take somebody, so v 0 is the money charged for the auction, we will have to find it in terms of other data available which is this, what is the data available u d r and s 0. So, let this be the money charged for the auction, so what you do you take that money part of it you buy some delta nought amount of shares of the given stock, if this is the market with one stock and one bound actually basically one stock and one money market, so one stock or the market is a very simplistic market one stock plus one market.

Now we have charge; charge is money, so with this money a part of this money you buy delta naught amount of shares of the stock. So, you buy delta naught amount of shares of stock at the price s 0 and the remaining money which is your; so we will not put v 0 at this form v 0 would be the finally, the worth of the portfolio. So, I assume that I start with some money x naught, so let us not bother about that v 0 has to be charged, v 0 is the money that will be charged this is that we have to find out.

So, suppose I this is just for you to; I can use v 0 also does not matter it come out to this same. So, what I am trying to find is a worth of my portfolio at time 0 or what I am doing, suppose I have got x naught of money. So, in x naught amount of money which is same as your v 0 that is essentially what I am trying to say and instead of writing v 0, let me as just put x 0. So, v 0 is something I do not know but suppose if I add x naught of money then what I would have done.

I would have got, bought some shares and invested the rest of the money in the money market, so what is that. So, if I invest the rest of the money in the money market means this is the amount of. So, delta naught, s naught is amount of money I have spent and then remaining part of this x naught, that is x naught minus delta naught, x naught; these are invest in the money market, does not matter you can write v naught also. So, I am writing x naught because x naught could be any amount of wealth, it does not measure any v naught; v naught is auction price will at the end put v naught equal to x naught, but here I just want to show that if you start with any wealth x naught, at time t equal to 0; this is what you do because as we have already said in the previous discussions no investor should ever put money only in risky asset, but should divide the thing risky asset and the sure be non risky asset.

So, what is the wealth of my portfolio at time t equal to 1; of course, the wealth that I have depending upon whether it is head or tail. So, that is the random variable, so at time t equal to 1, my wealth of the portfolio is this what I will do, I will sell this delta naught shares that I am holding in the market at price s 1, which could be s 1, t s 1 h, x 1 is a function it is x 1 of something t or h, we will explicitly write this. So, you will sell delta naught plus the money that you have invested in the money market with interest rate r, which is given it to you from the last lecture.

Of course, let us please do not forget that we are always using the no arbitrage condition this I do not want to write, but this I am just writing to remind you that you cannot escape the arbitrage condition. So, because you want to get a price of auction which is free of arbitrage if; so at time t equal to 1 my wealth now is so the worth of my portfolio is this. If you are reminded of what we have done in the part one for those who are done part one, we have already done this sort of things in the continuous time, but we will come to them at a later stage also, but here I am doing for those who are not done at course everything can be done in a very simplistic fashion, but still as a huge practical importance.

You could write this thing possibly in a more compact fashion basically separating x naught and s thing. So, what should be the initial wealth I should start with, so if I am in the auction market and ever sold out a auction, my x naught should be equal to v naught and what is the meaning of replication that x 1 of h should be equal to v 1 of h, the worth of my portfolio if h comes should be v 1 of h, worth of the auction when you have head coming and use is the backward thing, first know these and go to this. This is something like dynamic programming; if you have done some dynamic programming, x 1 t should be equal to v 1 t, these are the conditions that should be satisfied. So, I now know these so I can separately write two these conditions from the infra from what is x naught.

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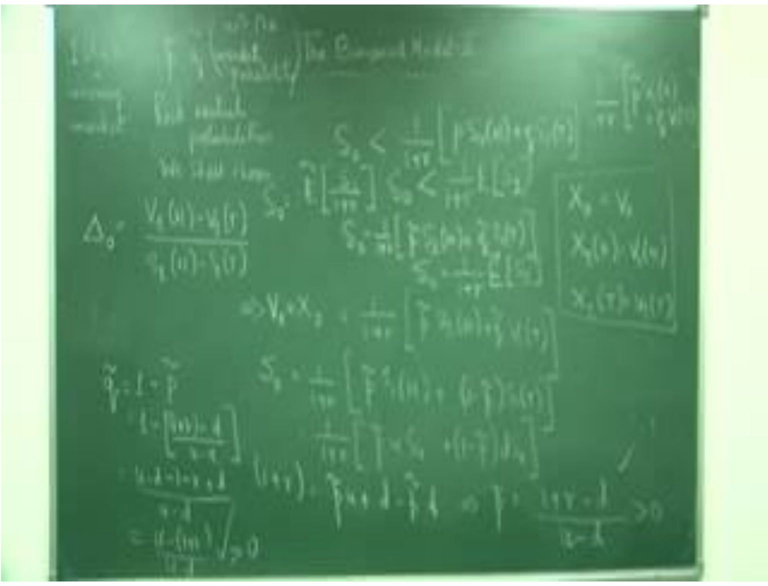

So, I will just rub this part and then I will let you know rather write down for you explicitly this things. So, I can write the first case x naught plus just writing it down from book so then you check the calculation, so this is (Refer Time: 09:44) when h occurs. So, what it gives while dividing by 1 plus r, we have got a discounted price worth of the auction at time t equal to 0, at time t equal to 1, at time t equal to 0 this is the discounted worth of the auction.

Basically dividing by 1 plus r on both sides that so the second equation, it looks the same only h is now replaced with t, so this is exactly these two facts. So, now multiply the upper part, multiply this equation by some number p naught and multiply this equation by some number q naught; sorry q tilde such that p tilde, plus q tilde is 1. So, you multiply this by this, multiply by this by q tilde and see what happens, if we do so then finally let us see what we get.

So, if I multiply them we will show why we are doing so, so these are two quantities such that they sum up to 1, you might think over they look like probabilities; of course, these are two unknown quantities and we have not, you cannot say this probabilities because we are not said whether p tilde and q tilde is greater than or equal to 0, we said just two quantities add up to 1, we will multiply this first term by p tilde, second term by q tilde.

So, finally, we would have and then add these two things, so finally we would have x naught plus delta naught 1 plus r, p tilde, s 1 h plus q tilde, s 1 t minus s naught. We know when you add here p tilde x naught plus q tilde x naught it will become p tilde plus q tilde x naught which is x naught and this becomes 1 by r, p tilde v 1 h plus q tilde v 1 t. So, does not matter you can take any possibly any p tilde, q tilde you think, but no here you will be the unique choice, we will make a very different sort of choice; what is that choice.

We shall choose p tilde and q tilde in such a way, this p tilde and q tilde are greater or different from p and q equal to 1 minus p by which the coins are tossed; p with head and q with the probability of tail, they are different in general different from p tilde and q tilde such that this quantity vanishes, so I am using them to solve this two equations basically and find my v naught. So, because x naught is equal to v naught, if I put this to be 0 then my v naught is nothing, but this thing that is all; it is a convex combination of v 1 h and v 1 t divided by 1 plus r, the discounted convex combination, so that is essentially the black shows price actually.

So, s naught, so we assume p is such that s naught is nothing but 1 by 1 plus r, p tilde s 1 h, plus q tilde s 1 t. We will discuss more about the nature of p tilde and q tilde and they would have a name which we will soon call them. So, this would immediately show from the above equation the following, this will immediately show me that this would imply; at x naught now is equal to 1 plus r, p tilde; v 1 h plus q tilde v 1 t. So, I have altered the term convex combination while speaking about this, I have also altered the term that they should be behave like some probability that is what our inner wish.

So p tilde, q tilde should be equal to 0, but then I should be able to find p tilde and q tilde; I cannot just say that p tilde and q tilde is some number and p tilde and q tilde should be computed through the data of the problem which is u, d and r that is all that is s naught. So, p tilde and q tilde 1 should be able to compute it right, so once you know x naught x naught is equal to v naught so this means that v naught is also equal to this auction price is this. Now I should be able to know what is p tilde and q tilde in terms of this and then there we will see deduce p tilde q tilde are indeed greater than or equal to 0.

So, from this equation I have s 0 equal to 1 by 1 plus r, p tilde s 1 h plus q tilde which is 1 minus p tilde, s 1 t what is my; see here once simplify things are looking better is u of s naught plus 1 minus p tilde d of s naught. So, that will give me s naught will actually get out and cancel, so it will become your 1 plus r is equal to p tilde u, plus d minus p tilde d. So, that would imply at p tilde is equal to 1 plus r minus d, divided by u minus d; u minus d is always greater than 0, strictly greater than 0 and 1 plus r is always strictly greater than 0, so this quantity strictly greater than 0.

Similarly from here you can now compute out q tilde which is 1 minus p tilde which is equal to 1 minus, 1 plus r minus d which gives me u minus d minus 1 minus r plus d divided by u minus d. So, this gives me u minus 1 plus r divided by u minus d, again by the no arbitrage condition. So, it is a no arbitrage condition which is guaranteeing that p tilde is greater than 0, it is a no arbitrage condition which is guaranteeing that, q tilde is also greatest strictly greater than 0 and this sum up to 1. So, we can now view as actually probabilities, but remember they are not the market probabilities, there is something we have not figured out, we have to figured out, how much stock I have to buy that also I have to been known by the problem that is the beauty of this model; at this just three quantities u, d, r is giving all the data. So, they behave like probabilities but they are not the market probabilities, it is not that by that probability that this becomes head or tail, p tilde is not the probability of occurrence a way it is p, we will speak up it on this, but let us compute delta naught, so delta naught can be computed out.

So to compute delta naught, you have to go back to the equation these two equations and subtract them from each other and I leave this computation to you to figure out delta naught is equal to v 1 h minus v 1 t divided by s 1 h minus s 1 t, it is called the delta h of the portfolio, it is called the delta of the portfolio means it tells you how much amount of stocks you should or the shares or the stock you should buy in order to completely replicate the portfolio, p tilde and q tilde has a very important name; p tilde and q tilde they are called the risk neutral probabilities, I will tell why. Why I am calling p tilde and q tilde is risk neutral probabilities, what is the reason they are not the market probability; I would like to tell you remind you these are not the market probability which is p and q; which is the p is the probability of head coming, q is the probability of tail coming which is 1 minus p, this is something have to is not the market probability.

Why do people invest in risky assets because they want to make a quick buck, so they want to very frank with you guys I really never made a investment in the market, but I know friends made investment in the market and have lost. So you put your money in risky assets because you want to make a quick buck, so the rate of growth of the risky asset has to be faster than the rate of growth of the fixed deposit naturally. So, if p and q are real probabilities then we expect that this should happen because if the rate of growth of the risky asset is same as the rate of growth of the risk free asset then under no circumstance I am going to invest in the risk; risky asset, I will always invest in the risk free asset the money market.

So, s naught must strictly be less than this, it cannot be equal to this; the discounted price, so what is this if you take this p and q as your probabilities, what is this; it is a expectation of s 1; random variable s 1 and you have the discounted expected value. So, the discounted expected value of the stock price at time t equal to 0 is this, the value of the stock price at t equal to 1 is discounted value, for the present value rather than discounted value is this and that must definitely be bigger than s 0 so that people would actually invest in this stock, otherwise we will never invest in such a thing right that is the idea.

So, the key issue here is the following that you would rather invest in a stock because your market probability should be such that your expected return or expected money or expected value of the stock should be; obviously, strictly bigger than the expected price of the stock or the worth of the stock; that discounted value at this present moment should be bigger than s 1 then only I am confident and have to invest in the stock. While the risk neutral probability says; risk neutral probability says the following, this is what it says; it does not say anything.

So, what does it say, so it says s 0 if e i e tilde is the expectation under the risk neutral probability and this is nothing, but expectation of s 1; the random variable s 1, stock price at s 1 which has two possible value, s 1 t and s 1 h because here the omega, the sample space is h and t, is e tilde is s 1. So, this is the expectation under this neutral probability, so risk neutral probability of those probabilities so the under which the expectation remains, the discounted expectations remains same.

Essentially if you have remember what is the martingale from my last talk, this is exactly the thing; risk neutral probability makes the discounted value of the stock price that is here what we can write s 0 is equal to 1 expectation tilde, s 1 by 1 plus r, the discounted value of the stock price into martingale, so that is exactly done that is s 0, s 1 this process is actually a martingale.

So, what it says that under this probabilities my expected return is the same as this, so why should I invest in the stock market, I can just keep as my money in the money market. So, it essentially says that rate of growth of the stock market is same as the rate of growth of money market; it does not change my money. This is what I can expect, my net worth, my net expected; worth of my stock; is as same as what is the current price now, so I would not invest in such a stock, but what are these why we use this because these quantities allow us to find v 0, which is the general p naught, q naught p and q help us because they would have inequality and it will be difficult to figure out v naught, it can just allow me to find out the bound on v naught, these things come in incomplete markets, but this is exactly what we do in complete market.

This study we done in complete market, to study incomplete markets essentially would have 20 odd course, which starts with complete market and goes into incomplete market that is really not a course we give in MOOC; MOOC is to essentially give people ideas and people then can start over and talk about it, thank you for listening patiently and now we will go to the next step in our third class the multi period models and if there is more one period of trading, what is it right, what is my; how do I price it. You will see always will go at start from the end; we know what should happen at the end when the trading ends.

So, then you can actually when the expiry time of the auction when the trading end, the last period then you can dynamically come back to v naught. So, this is exactly a process and one uses in dynamic programming, the very important aspect of modern optimization whose founder was Richard Bellman dynamic programming. So, this exactly the same process that we are using backward coming down backward, backward induction sort of thing and that is not the correct word but it is just a backward process. Starting from the very last what should happen and then coming and see what should I do in the beginning.

So, we end it here and we will start over multi period by nominal model.