Probability and Stochastics for finance-II
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Lecture – 11
Capital Asset Pricing Model

So, here from today's lecture onwards in the third week, we are making slight change in the main ideas that we have been discussing till now or main idea that we have discuss in the last week was portfolio optimization which says how should we make our investments. What investment would give me the maximum return with the minimum risk that was the clean idea, but here we will be talking about pricing. So how do you as price an asset? How do you know what should be the relation between return on the asset and it is associated risk what are the relationships?

So, these concepts come under something call capital asset pricing model. Then we are talking about the capital asset pricing model that we are going to talk about now essentially relates the risk associated with the single asset with it is return. Because here we had been lastly we were just looking at not the relationship between the risk and the return, largely we were looking at under minimize the risk and get some return. To do this, we have to introduce something called one fund result one fund theorem.

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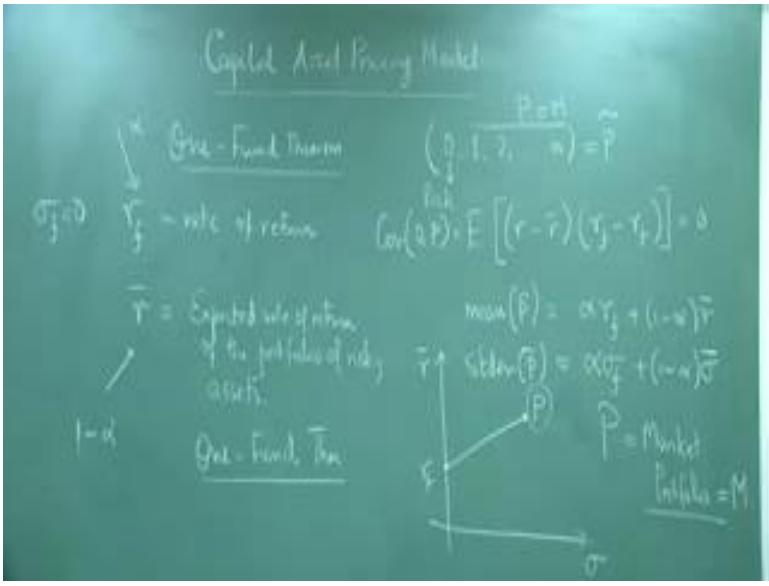

So, what is this one fund theorem all about? See when investors invest, they do not invest all in risky assets. It is not a good idea to invest everything in risky assets. We have to have some risk free assets like a bond or a fixed deposit something like that. So, that some part of your investment has a sure return right. So, what would happen if I add a risk free asset? So, added with your portfolio I have one more asset risk free asset.

So, now, your portfolio new portfolio will have a risk free asset say asset number 0 and then you will have the risky assets. So, now you will have n plus 1 assets right. So, risk free asset has a rate of return r f, see it is something like interest rate. Risk free asset risk free so, rate of return of the risk free asset. So, we this asset is risk free. So, if we invest one rupee in this asset then, in the next trading period you will get 1 plus r f.

Now we want to say that a over money an r bar is the expected money is the expected rate of return. This is the fixing. So, this is the expected rate of return of the portfolio or risky assets. So, what is the co-variance between this risky assets risk free assets and this part. This co-variance is actually 0 because if you observe it very correctly then, this nothing, but rate of return of the portfolio minus this expected return into rate of return minus expected return this is constant. So, this is 0, so co-variance between 0 asset and the remaining portfolio which I write P for example, P is actually 0.

Now if you look at that what is the mean? So, what is this r f? Here sigma of f that is no risk is a zero risk portfolio zero risk asset sigma of f is 0. So, now if I have an optimal portfolio right, if I have a portfolio in which I have made an optimal investment then, if I add a risk free asset and I change my wades a little bit. Can I can that investments remain optimal? That is the question. So, if I now look at the mean of this portfolio say portfolio P tilde. So if I look at the mean of P tilde what would happen? Suppose of the total money I give alpha I have invested in the risk free portfolio and 1 minus alpha I am invested in the risky portfolio. Alpha in the risk less asset and this is in the risky portfolio.

So, my mean is and what would be my standard deviation? Standard deviation P tilde is alpha of course it will be the same sigma f, sigma f is 0 actually. So, if you just look at that. So, sorry there was a mistake. Here it would be sigma, sigma r what that we have done earlier also sigma r what diagram. Here sigma is 0, so is r f is some risk less portfolio and P is the risky portfolio. So, basically now as alpha varies you trace straight lines between P and r. This is nothing, but the convex combination those who have some idea about it, need not get so worried about it, but it traces as a straight line.

So, this becomes now the efficient frontier. So, P is an optimal allocation on the portfolio then, I can change the wades a bit right and I again put some in the risky asset. And then I can change put remaining money in a risk free assets and still get an optimal portfolio which you can prove by yourself which I am not getting into. So, you can always construct efficient portfolio by combine your portfolio risky assets with risk or risk free asset. So, this is called the one fund theorem which says, there exists a portfolio which is also called a fund. So, this called the one fund theorem. So, there exists a portfolio of risky assets with whom you can have the combine a risk free asset, in order to generate an efficient portfolio. So, this if you look at it, it simplifies the efficient frontier very much. It makes it a straight line right.

Now the question is what should be this P? This question is answered by the capital asset pricing model. They say that this P, this set of risky assets must be the market portfolio. Must be the market portfolio means you have to invest in all the market, all the available instruments, all the available risky assets in the market. So, if they you cannot say I invest in the couple of them and do not invest in others. So, if you take the market portfolio that is what the capital asset pricing model says, that if you take the market portfolio and if you take a risk free asset, you can still generate an efficient portfolio rather optimal portfolio right, which will give you the minimum return and there is of course, mathematics can be done in the more detail way using optimization like we have done earlier but we are not going to get in to the details.

So, P is equal to the market portfolio which we write as M. Our job would be now to show that given my market portfolio, what is the expected return? Now I have a risk free asset also. So, risk free asset I do not need to bother about expected return I will know what is my expected return which is fixed and I know that sigma f is 0. But with the remaining part of the portfolio which is a market portfolio now, there is a question of expected return, there is a question of risk. How is that expected return and risk varies that can be beautifully describe once you put in a risk free asset in your combination? So, now, that is a starting of the capital asset pricing theory and that is what we have now going to describe.

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So now, we are going to again look at the r bar sigma bar model, r bar sigma model. When sigma is 0, let r f be the rate of return in the free asset. See the rate of a return of a risk free asset is always lower than the rate of return in a risky asset. That is why people put money in risky asset. If the return was same and I need not put money in risky asset, I can put in risk free asset. So, the market portfolio is a portfolio whose expected return is something like here and it has some standard deviation. Now join these by a straight line segment dot like this a thing. So, this is my sigma M, risk associated with the market portfolio and this is my r bar M, the expected return of the market portfolio. This line is called the capital market line. This line joining r bar and M, this line is often refers to as the capital market line.

So, let me try to find the slope of this line. So, the coordinate here, this coordinate here is 0 r f this coordinate actually. And this coordinate here is sigma M r M. So, what is the slope of this line? So, if I take any point here, any point say here or any point on the line it does not matter I am just checking you. So, I am taking the arbitrary point here whose expected return is some r bar sorry whose exposure to risk is given by sigma an r bar expected return. Now how do I relate this? So, I take any portfolio. We are risky portfolio and I am relating the things. So, if you write down the slope. So, slope of this line can either calculated through this two and also through this two. So, if you do that then, simply that calculation will show you r bar is equal to r f plus. So, r bar minus r f divided by sigma minus sigma f, y axis is y coordinates difference by x coordinates difference ratio that is slope. But here sigma f is always 0 sigma f is 0. So, then that will give me, again you take this one. So, I should write it in this way r bar m minus r f by sigma M, this whole thing into sigma.

So, here given any portfolio which is on the market line. Of course, sigma and all, but this is all nonnegative numbers right, you cannot have a sigma negative. So, there all here right. So, anything that you choose here, any other portfolio that you choose on this line on the ray basically emanating from r bar will satisfy this equation. So, this equation shows how given a portfolio and if you choose a risk asset how for any portfolio the expected return is associated with it is risk. The portfolio optimization that we had studied does not say the relation between r bar and sigma. So, once you induce, once you induct risk free assets, it becomes a very simple equation.

We are now will go to look at what is called the pricing model right. We are now going to talk about something which all investors talk about the beta of stocks. We are going introduce this and that will be the thing that we are going to talk about in this discussion. So, in the pricing model is called the CA model, the CAPM model. Our job is to relate the return of a given stock, given assets i-th asset to it is actual risk. So, how is the as return of the asset i is related to it is risk that is sigma i right. So, that is we are going to do.

So, in this case we assume that the market portfolio M is an efficient portfolio and if you take the whole market portfolio, you can always make, we will always get a solution to that. Of course, technically if you look at remember last discussion where we have taken the variance, co-variance matrix to be for obviously, proper reasons to be strictly positive, I mean a positive definite then, you are getting one efficient solution. So, you can always meet the market portfolio efficient.

So, now let us write down the theorems which we writing down from Luenberger's book Investment science. So, we take an asset i, again I will tell you and I am relating it with it is with the assets expected return with it is risk exposure that is sigma i. Let M be the market portfolio which is efficient. And r i bar is the expected return of i-th asset. Then this (Refer Time: 17:31), then r i bar minus r f this is equal to beta i into r bar r M bar minus r. Then what is this beta i? This beta i is called the beta of the stock which is the very important indicator for practitioners is the co-variance of i M. That is sigma i M. co-variance between the single asset and the market portfolio itself sigma i M divided by sigma square M, the variance of the market portfolio which I can also write as co-variance i M divided by sigma square M.

So, here see in this situation, we are assuming that everybody is the mean variance optimizer when somebody calculates betas in usually in the market, everybody assume that you everybody rather everybody has is a mean variance optimizer and everybody knows that the all the stocks the rate of return is same and everything assume to be same. So, for everybody every parameter is assumed to be same. Only under these ideal conditions you are getting this. So, we are going try to give you a proof why this is so? How do we get this formula? And the beta is a very important indicator. It tells you, it relates the asset here and the risk of the asset i with right. It relates the risk of the associated with asset i with the risk associated with it is own relationship with the market portfolio in relating to the market portfolio.

For example, if you look at this one. This term r bar M r f by sigma M, this is often called the price of risk because it tells you how much and what is the change in your expected return, if you increase your risk by one quantity. So, here suppose I have r bar is r f plus this price of return, this r bar M minus r f by sigma M into sigma. Now I have taken r bar dash which is r f plus r bar M minus r f sigma M into sigma dash where sigma dash is the sigma plus 1. Then r bar minus r bar dash minus r bar is nothing, but r bar M minus r f by sigma m. So, it shows that this actually tells you how much change you are expecting in the expected return right.

So, if you increase your risk because this is positive r m bar is always bigger than r f. So, if you increase your risk right, your expected return increases. If you decrease your risk, your expected return decreases. So, this simple line straight line actually expense a lot of think about a market and that is the beauty of this whole analysis. Now how do you come to this conclusion and that is exactly what we are going to prove now. And with that we will finish this very very basic study of capital asset pricing model. After which we will start our study of option pricing.

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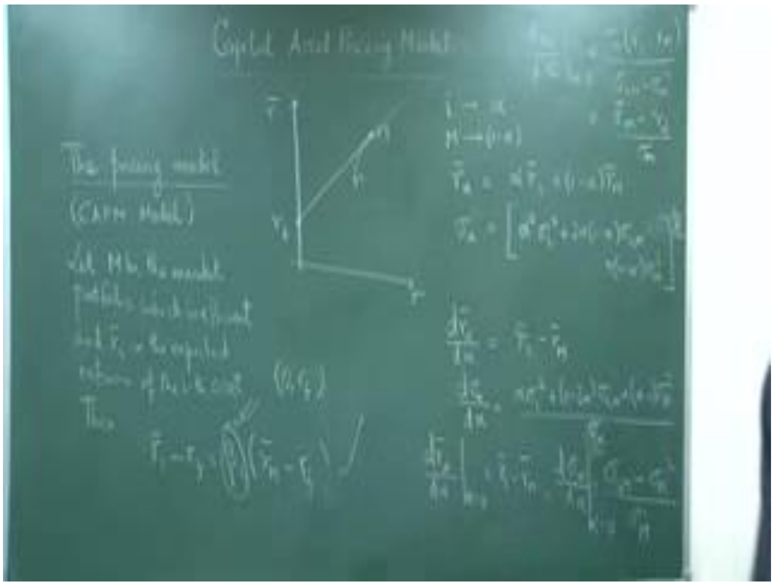

So, we know that the market, assumed that the market portfolio is efficient. So, which is associated with addition of the risk free rate of return which is r f and this is the market portfolio then this is the capital market line. Now I take any asset i. Now let me forget about r f for some risk free asset. Take an asset i, put in alpha amount of money here and in the market portfolio put in 1 minus alpha. So, in this new system we have taken some asset i and the remaining market portfolio, say these are very strange system you might say okay. Why are you doing this? Because your asset i must within the market portfolio know.

Basically then I am additionally putting an alpha on the asset i. I am keeping it separate and looking at the market portfolio as a separate. I might be say looking at market portfolio as if it is a single asset. Then if I do that, so you could change the alpha 1 minus alpha that is up to you. So, r bar alpha now with these i and M would be alpha time r bar i plus 1 minus alpha time r bar M. And sigma alpha, just I am writing the standard deviation would be alpha square sigma i square plus 1 alpha 1 minus alpha sigma i, M co-variance between i and M plus 1 minus alpha square sigma M square. So, I am just writing in the next line so that it makes clearer 1 minus alpha square sigma square M. So, this whole thing to the power half or you have taken a square root. So, that is what you have.

So, now as alpha varies you can vary the alpha. So, you can have the portfolio i must be having originally this r i and sigma i. And as you vary the alpha, this has to put in the capital market line this M. So, this r bar alpha sigma alpha would actually trace out a curve. When alpha is 0, this is nothing, but the market portfolio. So, at alpha equal to 0, the curve should touch the market portfolio and then this is the asset i. So, alpha is equal to 1, it is asset i right. So, you can vary alpha accordingly. It could be negative also for short selling.

Now you see this curve cannot go over this capital market line. If it goes over this capital market line then, there is a big problem. The problem is the following. Problem is that then basically they could be inefficient portfolios which crosses the efficient portfolio because none of the inefficient portfolio paths can cross the efficient portfolio when an efficient frontier because this is efficient frontier. Then it violates the fact that M and i are both are on the efficient frontier, it violets the fact. So, what happens is that, that at M this curve is tangent to the capital market line. That is what should happen.

So, once that happens, we can start thinking about that. Now we can do the derivatives and so, which means the slope of this curve, slope of the tangent, slope of the capital market line and the slope of the tangent rather and slope of the tangent to this particular curve given by this parametric equation must be the same. So once you can do that you will get the answer. So, let us just do it. Just rub this part for a while I am sure have copied it down. So let us go and do the step by step proof.

So, d r bar alpha d alpha. So, that will give you r i bar minus r M bar. This is what you will get. Now d of sigma alpha by d alpha is, I will just write down the expression because is this expression, I do not want to keep on computing the expression. So, but we are essentially looking at the slopes at M. So, a view everything has calculated at alpha is equal to 0, that alpha is equal to 0, this seem as same does not matter. So, d r bar alpha and alpha equal to 0 is r i bar minus r M bar, but d sigma alpha by d alpha is equal to, put alpha is equal to 0, right. Once you put alpha equal to 0, it will give me sigma i M minus sigma m square divided by because if you got alpha equal to 0, it will be 1 and the sigma i M, alpha is equal to 0 it will be minus, here it will be 0 divided by sigma.

Now, this is the formula when you put d alpha, alpha equal to 0 because sigma alpha when you put alpha is equal to 0 is just give you sigma M because this is the calculation of the square root right. This is sigma alpha. I am writing it much more, may be this is because 1 by 2 root alpha into the chain rule basically. Now once you have this, you want to compute, but you want to compute know the relation between r alpha and d alpha. Basically you want to compute d r bar by d sigma alpha that has to be computed at alpha is equal 0, which now is a ratio of this by this. And if you take the ratio of this by this it will so that that is exactly the slope of the tangent to the curve defined by r alpha third r alpha bar and sigma alpha. And once you do that, what you get is sigma M into r i bar minus r M divided by sigma i M minus sigma M square, this is the comma i M.

So, this is what is happening, but this must be same as the slope of the capital market line, the tangent line. And you know what the slope of the capital market line is. The slope of the capital market line, so then this must be equal to r bar M minus r f by sigma M. So, these two quantities are equal. Once you know these two quantities are equal, you can immediately find out this formula. This is beta i of the stock is very important because these, if you go to stock market they will always list down the beta of the stocks for everyone, every asset because that tells you that what is the relation between the expected return. So, what is the relation between the expected return and the risk exposure to the stock in viz a viz the market portfolio. So, or what is the relation between this particular return on this particular asset and the return on the market portfolio. So, that will tell you that whether should I give more money to this asset or whether I should give less money to this asset? That would decides the beta of a stock is very very important parameter.

So, thank you and we start option pricing from the next class.