Probability and Stochastics for finance-II

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Lecture -10
Last Lecture on Portfolio Optimization

I have no clue how many of you have pointed over it; we have just kept the board as it is as I left in the last class. So, I am written that this is the last lecture on portfolio optimization which it is. Now if you have not pointed over it let us see how to really draw such diagrams the first good thing hear the parameterization. I think I met a little mistake I just go ahead and do the parameterization again I just go by Romans books. If you anybody sees the books it will be much easier for them just.

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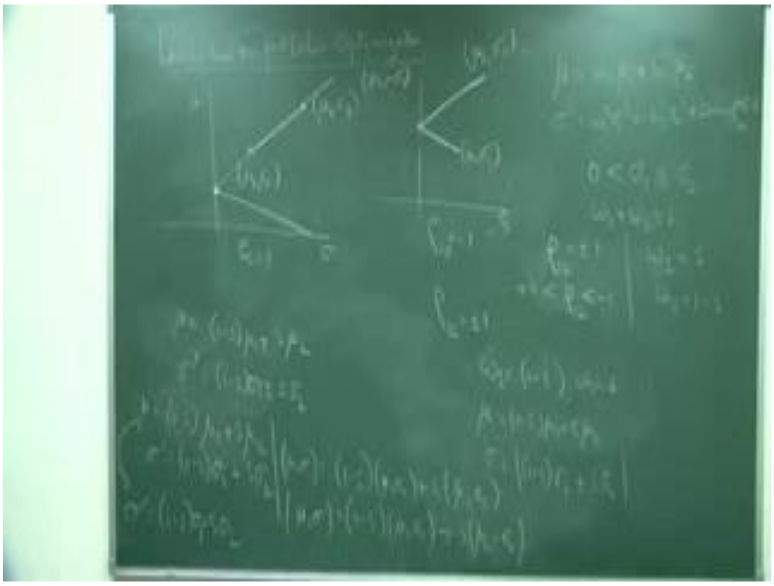

I have taken W 1 to be 1 minus S and they have taken W2 to be S. So, in that sense mu has become 1 minus S mu 1 plus S mu 2 and this has become this just a shift switch is nothing only the then, things would be slightly different you have to change because basically we want mu 2 to do be bigger than mu 1 that that is that. Then we draw the diagram that that is the reason for all these change. So, will become 1 minus S sigma 1 plus minus S sigma, how we are going to draw the diagram the first trick is just consider this parametric equation 1 minus S mu 1 plus S mu 2 just consider this do not bother about 1 minus S plus minus S sigma 2.

So, now take this diagram sigma dash and mu. So, suppose you have these 2 points mu 1 sigma 1 and mu 2 sigma 2 first consider the plus case first consider this set mu equal to 1 minus S mu 1 plus S mu 2 that is why. So, if you consider the plus case in sigma dash. So, it is sigma 1 plus minus, if you consider the plus case what does this represent actually what is mu sigma. If you look at it very carefully mu sigma is nothing, but 1 minus S mu 1 sigma 1 plus S mu 2 sigma 2, it is the line segment or the line passing through mu 1 sigma 1 and mu 2 sigma 2. So, basically this is showing 2 different port assets with them this is just a line passing through mu 2 sigma 2 and sorry in this case we are always keeping mu 2 as higher.

So, here is the line segment joining mu 2 and sigma 2. Now if I have minus 1 that is sigma dash as the same mu, but sigma dash is 1 minus S minus S sigma 2 1 minus, then it is nothing, but same thing I should draw it in a better position. So, it is a same thing, but this time it is a line passing through mu 2 minus sigma 2 and mu 1 sigma 1, that is what is happening. So, it is passing through mu 2 minus sigma 2 and mu 1 minus Sigma 1 right. So, that is exactly what is happening? Now, it is just the reverse, here if I have the diagram, here if I had the same mu 1 sigma 1.

Now, it is the lines upon a line passing through mu 2, but minus sigma 2. So, whatever was sigma 2 there here, I have to have minus sigma 2 here. So, it will be same mu 2, but minus sigma 2. So, it will. So, also it is mu 2 minus sigma 2. So, so it will come here. So, here I am drawing this. So, this point will be same mu 2, but minus sigma 2, but here you basically then basically then in this particular case we have mu sigma represented as 1 minus S mu 1 sigma 1. That is if you what is happening, if you put s equal to 0 it is giving me mu 1 sigma 1. If s equal to 1 it is giving me mu 2 sigma 2. So, when you move from 0 to 1 you go from here to here. So, it will become plus s into mu 2 minus sigma that is what is happening.

Here I have this new joining this line, but when you take the mode everything is positive. So, what will you do if you reflect this line along the x axis in this case, when it is plus what will you do you will have to reflect this part along the mu axis, if you reflect it will come this side. So, basically then you forget this part and you really concentrate on this part here also you will have reflection, but reflection come here.

So, now this case it is done like this 1, when this is 1, when you have rho 1, 2 is 1 here rho 1 2 is minus 1. So, this loaded part here this part the line segment joining this 2 points and the part of the curve or the broken straight line that is between these 2 points or call the efficient frontier of the portfolio efficient frontier. So, basically you see in both these cases the minimum value. Now I can remove sigma dash and write sigma. So, in both these cases the minimum value of sigma is 0 that is there would be a value of mu or there, you can have your expected returns in such a way that you have your minimum risk to be 0, but that is really not found in the real world and so, we really have to move towards the situation when, you have everything between plus 1 and minus 1. So, we will have now, the case when rho is lying between plus 1 minus 1.

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Now we come to the case where rho is between plus 1 and minus 1 this is the most important case to deal with, now you must be asking why I need to do a rho sigma study the rho sigma study is important because, you can visualize the points that are lying on what is call the efficient frontier.

So, basically those are the portfolios rather any portfolio that you can choose you can choose. Now you can choose any mu sigma combination that you choose there or actually a good choice as a portfolio right that is either you choose mu 1 sigma 1 mu 2 sigma 2 or anything in between them. For example, you once you saw line segment. So, any combination of those 2, if you choose that as your optimal portfolio, that is your distribution that will be enough that will be a good one, that is the meaning of this. So, that is why this particular study is done here the thing will gradually become more clear when you have rho is between plus 1 and minus 1. So, what you have observe what even if you have drawn the last part I will just go back to the last part of it we had 2 types of graphs 1 for s is going to plus 1 and minus 1, 1 was a line segment like this. So, anything this was the only case you have here, it was another case which was like this. So, 1 this is the point where the minimum variance is reached at anything above the minimum variance point is called the efficient frontier. So, this is the efficient font. So, some people call this as the efficient frontier some of point of view of Pareto minimization which I am not discussing and this whole thing can be efficient frontier which I had explained to you all dear.

So, this point is sometimes if just this point is called an efficient frontier because, these are essentially for risk says that, if you choose any convex combination with no short selling between these 2. They are the best minimum variance portfolio for you guys. So, so we will try to look into the case, when rho is between plus 1 and minus 1. So, here again ill re recall that will put mu as mu 1 sorry I should write it as W 1 mu 1 plus W 2 mu 2 and sigma square is W 1 square sigma 1 square W 2 square sigma 2 square plus 2 W 1 W 2 rho 1 2 sigma 1 sigma 2. So, this is exactly what is we had all ready written down. Now again I will put W 2 is equal to s and W 1 equal to 1 minus S. So, W 1 is 1 minus Sand W 2 is equal to s where this is some real number of course.

Now, I can write down mu what is mu in this case mu here is a 1 minus S mu 1 plus S mu 2 which can be written as mu 1 plus S into mu 2 minus mu 1 this is the standard way of handling convex combinations. Once you have done that you also write now sigma square and here is something we really need to worry about the nature of the curve the mu sigma curve. So, sigma square in this case would be the following. So, here is the sigma square yes stop the camera and wrote down this bigger expression this is the sigma square expression. So, you have to observe the sigma square of the function of a is truly quadratic because this term sigma sorry sigma 1 square plus sigma 2 square minus 2 rho sigma 1 sigma 2 can be expressed as sigma 1 square plus sigma 2 square plus 2 into 1 minus rho sigma 1 sigma 2. Now rho is strictly less than 1. So, this is strictly bigger than 0.

So, hence this is truly a contradict expression. So, let us look at the nature of this sigma square mu curve right we will be interested in drawing sigma square mu curve, what is the link between sigma square and mu here writing the sigma would be a very difficult thing to understand how the nature would be let us write sigma square and mu. So, if it there what is the relationship. So, suppose for the time being mu 2 is bigger than mu 1. So, if I think for the time being mu 2 is bigger than mu 1. Then for any s which is positive mu is positive right not mu is positive means of course, when you draw diagram mu sigma it is very important to understand the sigma square increases in this direction mu increases in direction actually it should be sigma mu there would have been a better understanding of the stuff, but that is the way finance changes to do it in a reverse way. So, what happens is that when mu increases.

If s is positive let us consider mu 1 and mu 2 be both positive for the time being right mu 1 and mu 2 to be both positive just for the timing being, it need not be mu 1 and mu 2 as assume that expected return is positive of course, you can say that expected return is never negative that is theoretically it could be negative, but in practice. Of course, now would you expect and expected return is going to be negative I am expecting a loss that is a very bad thing now investor would do. So, form an optimistic point of view we will always have mu 1 to be positive. So, when s is increasing the mu is increasing right. So, mu is increasing mu increases as s increases here also as sigma square increases this will increases faster than this part sigma square keeps on increasing. So, as mu increases sigma square would keep on increasing.

Now, you think of s to be negative if s is negative then, this is positive than mu is decreasing you keep on going towards the negative side mu will keep on decreasing right. So, this whole mu will keep on decreasing, but the here value of the s square as I go make toward minus infinity, this will become larger and larger. So, as mu keeps on decreasing when s is negative sigma square keeps on increasing. So, there is a when mu keeps whenever mu keeps on decreasing. So, mu keeps on decreasing sigma. So, from here mu from 1 point mu is increasing sigma square is decreasing and for so, this is the part where s is non negative this is the part. So, suppose this is the point from where s is actually non negative and then when mu increases, s increases mu, increases s is also sigma square is also increasing and there is a part when s is negative in that mu decreases, but s increases is some curve like this. So, so mu decreases s increases a sigma square increase, s is this is the point this is the turning point that is s goes from positive to negative. So, till when s is positive and mu 2 is bigger than mu 1. So, suppose here mu 2 sigma 2 mu 1 sigma 1 anyway sigma 2 is bigger than sigma 1 5 and I have assumed that mu 2 is bigger than mu 1. So, this is a nice explanation. So, this is a turning point this is the point where s has become negative.

This is the point from where the curve takes a different term. So, when s mu 2 is bigger than mu 1, as s I am keeping on increasing as s is a positive quantity s is 0 it is still mu 1 it is positive. So, if s s is 0 then sigma square if this s value is 0 then you this part has to be positive. So, sigma square would always be positive, but mu 1 would be some positive quantity also right. So, where s 0, we can think like this even this is the point when s becomes 0. So, between 0 it is going and when s is 1 here I have mu 2. So, when s is 1 I come here. So, as I move from 0 to 1 s from 0 to 1 I am going along this curve from mu 1 sigma 1 to mu 2 sigma 2 and what is happening is the following. So, how do I got the curve. So, for s it is 0 to 1 right s between 0 to 1. Now s starts from s is 0, when I keep on s is s is increasing now, when I make s positive, but less than 0 right half of this. So, the whole thing keeps on decreasing sorry even not just s positive I have made a mistake when s strictly bigger than 0. So, here it starts from 0 to 1 from here s starts to be 0 when s is strictly bigger than 0 it is increasing, as mu increases s on a between s equal to s is some for some value of s which we do not know. So, for 1 part u see between 0 to some part as s keeps on increasing mu value keeps on increasing mu goes up.

Now, what happens to the s value because s is now between 0 and 1 this thing does not keep on dominating because s is bigger than s square? So, though it is finally, positive this value actually keeps on decreasing the value of s keeps on decreasing then after some value of s this thing dominates this part and then it goes on increasing that is that is the clear clean idea why the curve looks like this. So, as you move from here to here you are changing s from 0 to 1 and that is the way this curve is drawn this is call the mean variance curve and that is the way it look. So, here let me just write down for you what would be a minimum s in this particular case minimum s is sigma 1, sigma 1 minus rho sigma 2 sigma 1 square plus sigma 2 square minus 2 rhos to rho, rho 1 2 sorry rho 1 2 sigma 1 sigma 2. So, this is the optimal portfolio and the risk that you are exposed to it cannot be 0 it is a positive risk that you are exposed to sigma 1 sigma 2 1 minus rho square sigma 1 square I would ask you to go home and do the calculation 2 rho sigma 1 sigma 2.

So, this is even if sigma 1 sigma 2 is equal does not matter it will see you will see get something. So, from here lot of more conclusions can be drawn, but this is very, very important this is a very important drawing. So, for the risk investor this part is called the efficient frontier this 1 not the lower part. So, I would like you to really compute out this part and check it for yourself, you see from this mean mu diagram you have now found the weight and found. Of course, you now have to see if you know do not short selling you have to put a this is rho equal to 0 also you have to 1 minus s mean to be greater than equal to 0 and then you will see what conditions you require on sigma 1 and sigma 2 and then relation between sigma 1, sigma 1 2 sigma 1 and sigma 2 and rho 1 2. So, that there is no short selling, if there is short selling what should be the condition. So, that I would leave it to you and I would ask my t s to put them as an exercise in the coming future.

So, with these we finish the second week and in the next week we start with capital asset pricing modal may be 1 plus maxima 2 plus and then we go to pricing European call options and which we lead us finally, to the black shows formula.

Thank you very much.