Basic Concepts in Modal Logic
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Lecture - 04
Semantic Tableaux Method for Propositional Logic: General Examples

Welcome back, in the last lecture we discussed syntax and then, the in the next lecture. We talked about semantics of propositional logic and basically, we are trying to cover a kind of crash course on propositional logic because it is consider to be, the most important thing for doing modal logics because modal logic is considered to be an extension of classical logic.

So, in this lecture I will be talking about some of the important definition that we have left it out or we need to discussing, somewhat more detail they are logical consequence consistency satisfiability etcetera. With some examples and then I will quickly move on to one of the important methods that we will be using it in our course that is it is also considered to be a decision procedure method, which is called semantic tableaux method.

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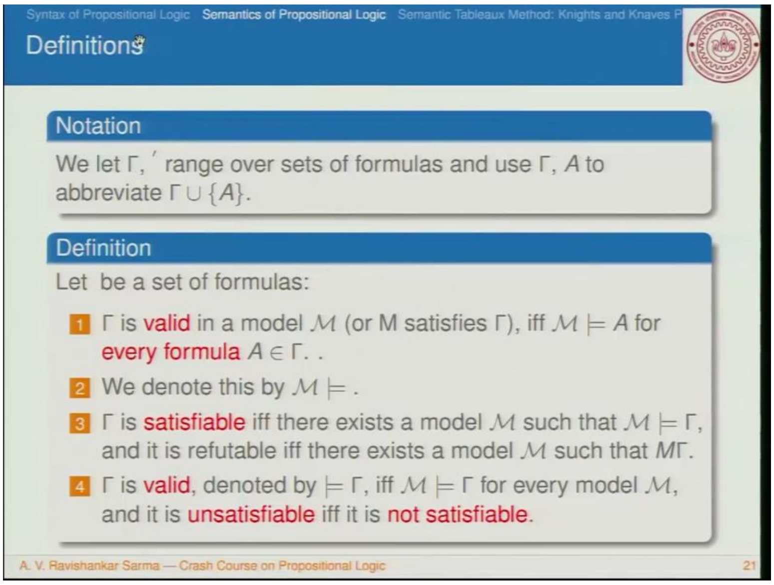

So, before that let me begin with the definition of logical consequence. So, the notation goes like this, if you take gamma as some set of formulas and we use gamma for that particular kind of thing which includes all the formulas that exist more propositional logic and a is any kind of propositional formula and if you write gamma and union a, it is set of all formulas together with a given formula. If that is the case then gamma is set to be valid in a any given model m; that means, n satisfies gamma if and only if a is considered to be logical consequence in that model m it happens for every formula that is a n which exist in a given to the formulas set of formulas gamma and in all interpretations given formula is true, then it is consider to be a tautology and all tautologies are considered to be valid formulas.

So, we define we denote this thing as n any other letter a as something like that and when we say that gamma is considered to be done set of formulas, that we have that is considered satisfiable. If and only if they exist in model m such that gamma is considered to be logical consequence in that particular kind of model m and it is refutable, if and only if their exist in model m, such that m gamma, gamma does not follow from m a gamma is considered to be valid which is denoted by gamma follows from something. Gamma follows in a model m it happens for every model m and it is considering unsatisfiable, if it is not satisfiable at so, basically talking about three important things logical consequence satisfiability tautology etcetera.

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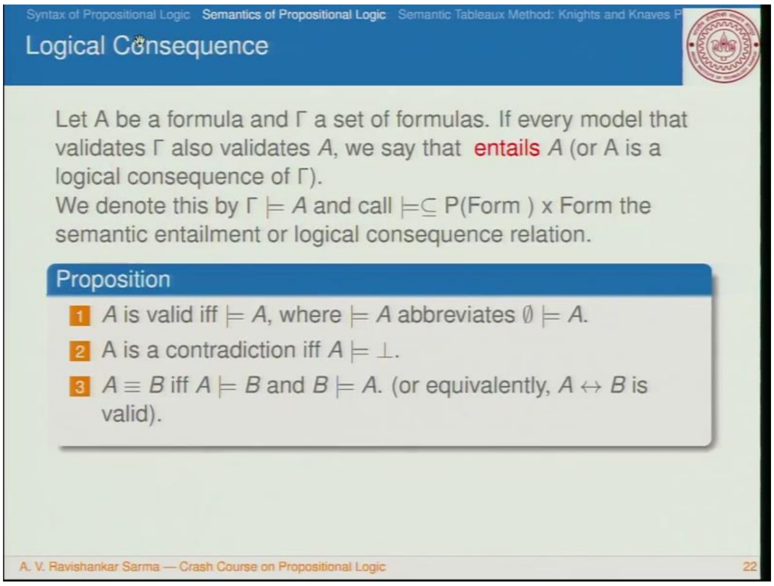

Now why we stress little bit on logical consequence because logical consequence occupies central position in any logical system. See in classical logical we defined logical consequence like this. If a b formula and gamma is some set of formulas and it is it happens in this way that in very if, very modal that validates gamma also validates whatever is there on right hand side that is a then, we say that gamma entails a or a is a logical consequence of gamma since that the formula is gamma.

So, a is considered to be a prepositional needs this is the set of formulas or formula is considered to be valid as it is written in the sense a is logical consequence especially where a abbreviates less thing we do not try to anything on the left hand side means a has to be obtained from empty side. So; that means, it has to be tautology; that means, a does not require any kind of do for anything values. So, it is considered to be tautology. So, in the same way, a is considered to be contradiction especially, when both is consequence of a given formulas if and two formulas are set to be logically equivalent each other especially when b is logical consequence of a and the same way, a is also logical consequence of b or rather way fond of saying this is that a if and only if b has to be valid or a if an only b as to be a tautology.

In classical logical we mean tautologies are considered to be valid formulas and valid formulas are also considered to be tautologies.

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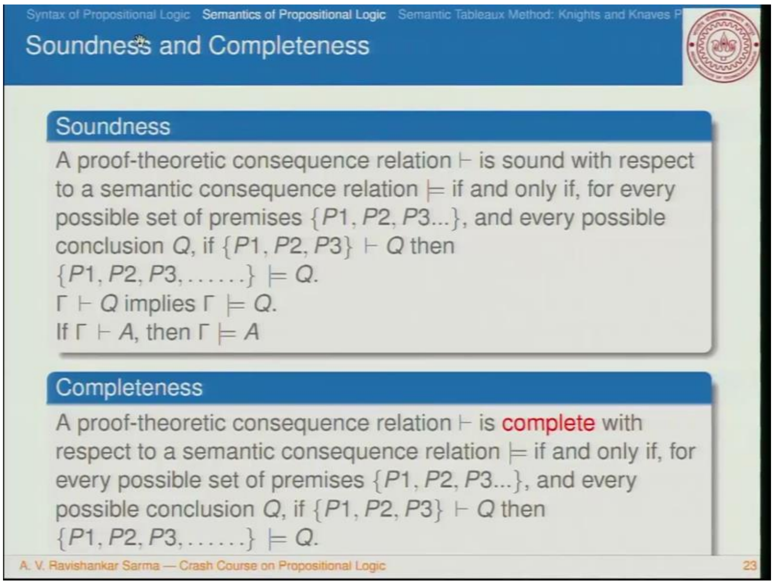

There are a two other important things which you come across a while doing any logical, these are like this soundness and completeness usually an argument can be valid any given argument which considering to be valid and it also has two premises, then it is considered to be sound argument if any given argument is valid, but it has I mean one of the premises is considered to be false then it is considered to be unsound argument as for as the argumentation is concerned.

So, in propositional logic soundness is defined this sense particularly when, whatever is proved is also consider to then the system is considered to be sound and the system is have to be consistent when all the usually you required this thing for this reason that in we want all the prove all the truths to be provable and all the provable things are considered to be true at tautologise.

So, proof theoretic consequence relation single constant have used two symbols single constant double constant which is considered to be sound with respect to a semantic consequence of relation double constant. If and only if every possible set of premises let say P1, P2, P3 etcetera, and every possible conclusion q; P1, P2, P3 from that q follows then q is the has to be logical consequence of P1, P2, P3. So, in a nut shell what it says this is if q is reduced from gamma it implies that q has to be true then q has to be logical consequence of gamma.

So, if a is the ways from gamma then a is the semantic consequence of gamma a is considered to be logical consequence of gamma, it means all provable things are considered to be true and the completeness is like this a proof theoretic consequence relation single constant is considering to be complete with respect to a semantic consequence relation, double constant. If and only if for every possible set of premises P1, P2, P3 etcetera, and every possible conclusion q and this q has to be semantic consequence of all those thing P1, P2, P3; that means, all this premises should make the conclusion true it is like has been as saying that a given a argument is valid.

Especially when it is impossible for premises to be true and the conclusion to be false if you it happens in such; it happens in a way that your premises are true and conclusion is false if the given argument is invalid.

So, this is the this is also it can be viewed as it can also viewed has relationship between syntax and semantics in the syntax, we talked about single constant and semantics we talked about double constant the relationship between these two. If you want to have a relationship between these 2, we need to have soundness and completeness property, but the basic ideas is that whatever, you have proved at the end of the day, you proves have to be true and what all considered to be proves in mathematics etcetera to have a proofs, but this is objectionable kind of thing that in then as come up within an interesting. is incompleteness theorems you came up with a view that this cannot be the case, they are obvious truths, but they are not provable any given formal system which is following it is own rules etcetera, but he cannot sure that it is consider to be complete it means all truths cannot be provable.

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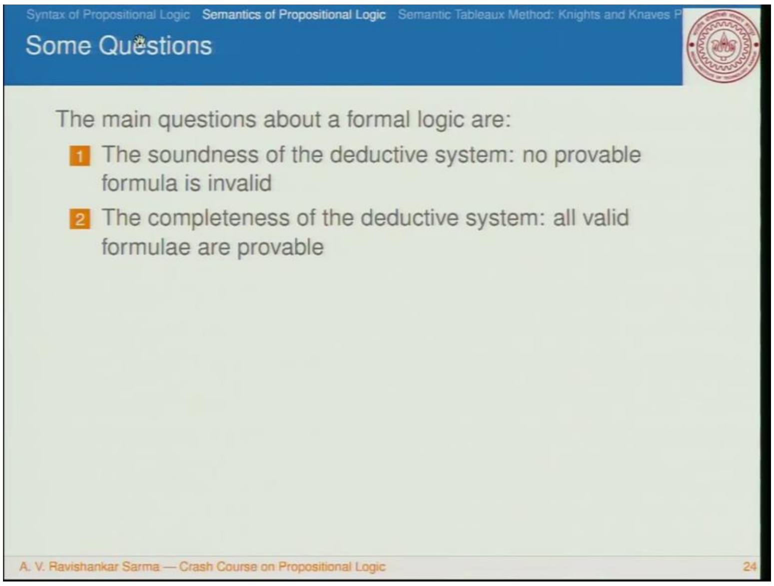

So, a basic questions we asked in the formal logic or this things the soundness of the deductive system any given formal logical system. We will be asking these questions how sound it is; that means no provable formula has to be invalid; that means it as to be false if something is proved it has to be true and the completeness of the deductive system all valid formulas have to be provable.

So, we ask this important question that is the system is consistence is the system is complete or sound etcetera. There are two important things for the logicians some other things are considered to be very important let us consistency completeness etcetera these 2 are considered to be the most important questions at logicians we would be asking or working in any kind of logic.

So, there is another important thing which we need to know all this things which will be using it in the modal logic while discussing about the modal logic.

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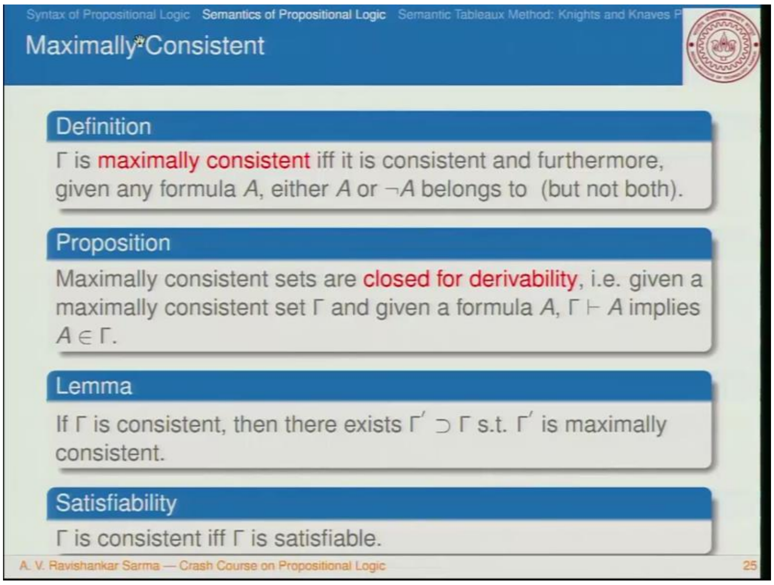

So, these are the terms that you come across even there also what do you when, do you say that system is considered to be formal logical system is considered to be maximally consistent a set of formulas. Again gamma is set to be maximally consistent if and only if it as to be consistent and there is one more requirement that is further more given any formula this is important requirement that is gives back either a has to be provable or it has to be provable not a is the case.

But definitely not both the things there are some logical systems for example, in the case logics etcetera suppose if you cannot prove a it cannot even true not a then we have to withdraw your fundamental principles of logic one of the fundamental laws of logic that is law of means tell us that either p is the case or not p is the case one, excuse the other one if you cannot prove p and it cannot prove even not p there is no way in which we can validate this fundamental law of logic that is the law is excluded middle.

So, for the maximal consistency, either a as to be derivable or not a as to be the case, but it should not be a case that both are derivable from a given system in maximally consistent sets are considered to be closed for derivability that is given a maximally consistent set gamma and given a formula a, a is reduced from gamma implies that a has to be in that particular kind of set of formulas gamma.

So, while we talked about maximally consistent set because when we discuss about some other concepts like possible words etcetera in the case of model logic you required this maximally consistent sets the possible there is defined in the sense that it is considered to be maximally consistent set of sentences. They are considered to be possible worlds in that contexts we need to know little bit about maximal consistency there is supporting theorem that is lambda that is that, if gamma is considered to be consistent then there exists gamma prime which is super set of gamma. So, is that gamma prime is considered to be maximally consistent.

I am just for superficially I am talking about some of the important theorems it is possible to discuss all this things in 10, 20 minutes like that all this things require a rigorous proves etcetera. There are very interesting books and rechecked to introduction to logic there you will find this integrate one best text book is that introduction to mathematical logical by Mendel's this is a good starting point for understanding all this the theorems and Laminars.

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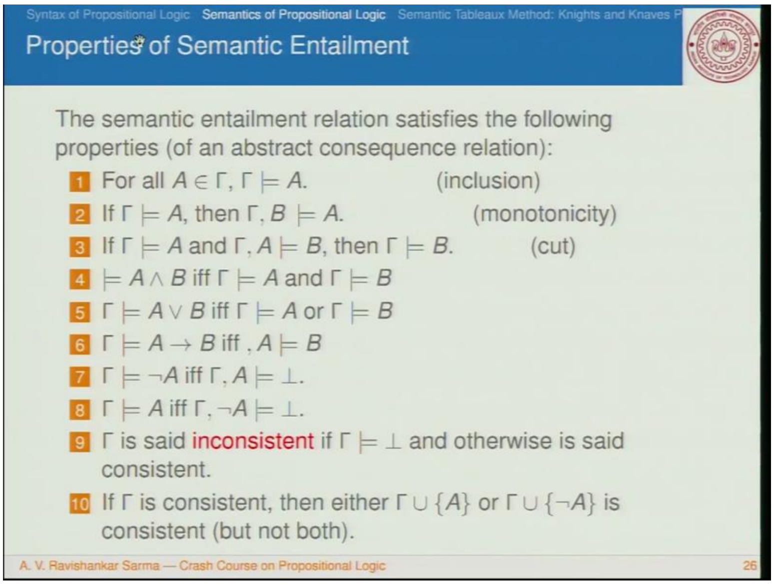

So, another important thing we need to notice is that gamma is set to be consistent if and only if it is satisfiable that is also we need to note there are some important properties of this entailment relation and it is like this I do not want to going to the retails of all this things, but I will quickly switch on to the ninth one that is gamma is set to be inconsistent if and only if, you derive contradiction form in a given formula gamma otherwise it is said to be consistent.

This is the one which will be using it in our method decision procedure method is semantic tableaux method gamma is to be consistent then, either a as to be gamma union a as to be the case or gamma union is not a as to be case, but not definitely not both if any system in which you derived the both things a and all b also in that particular kind of system is considered to be inconsistent and these are some of the important properties that any classical logic follow base in particular.

So, particularly there is one interesting property that we need to highlight here. So, that is monotonicity property, that is second one, if a is logical consequence of gamma. When you can you can add some further statements gamma and see and from that also a follows, but in most of the cases may not happen. For example, if is if you represent it like this in the case of conditional statement that we are going talk about little bit later.

Suppose if, you say that if there is a sugar in the coffee and would be tasty; obviously, it is case that sugar makes a coffee tasty some of us and then, if you say that in a addition of new information such as if there is a sugar in coffee and then in coffee then it will be tasty. So, from a implies b 1 of the logical consequence of a implies b is that a and c implies b where c is considered to be the new information there. So, that need not have to be the case in non monotonic logics where in you need to drop this particular kind of thing is follow in classical logic that is a monotonicity property in the whole lot of logics that are emerged from this one they come under the category of non monotonic logics.

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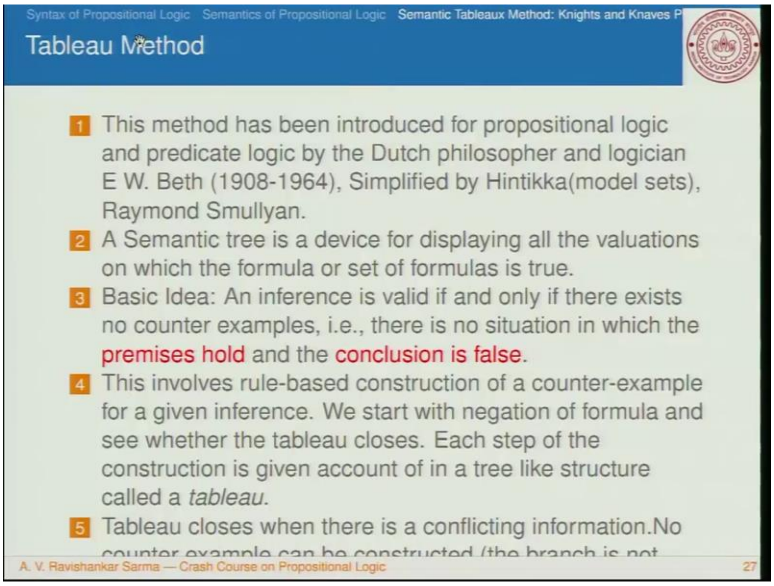

Now, let us get back to this important method that is semantic tableaux method I will be highlighting this method because, you will be using this method and we will be extending this method for modal Laplace etcetera. It will be later this is consider to be one of the important methods and easy to use and this method is used to it is like a decision procedure method with which you will be able to tell when a given proposition formula is valid formula or a tautology. And when two formulas are said to be consistent to each other or when two formulas are said to be logically equivalent each other many things you can talk about when, two formulas are said to be a formula to be satisfiable etcetera.

So, they are independently it worked out by many the three important logicians any one claims that method is due to them only. It is started with Beth's works and it is further simplified by Hintikka in the in his model sets and Raymond Smullyan also used is considered to be the founder of this particular kind of method it does not matter of who as come of this method, but this method is very interesting.

Here what we do is in what we essentially do is that given any formula which are in the propositional logic. We draw it is corresponding tree diagram a semantic tree which is called as semantic tree semantic tree is considered to be a device for displaying all the valuations and given set of formulas are considered to be true the basic idea is that one of the important things which you will need to notice is that truth table method is also considered to be decisions procedure method. But it is going to be difficult for us to track all the rows are truth table especially when the number of variables increases, if they are three variables we have 8 entries in the truth table it is easy to manage and the number is increases n is equal to 7 or 8 then, we have 124 increase very difficult to monetised those things it is difficult for us, but the computer can do it easily, but it is difficult for us.

So, semantic tableaux method simplifies those things. So, the basic idea here is that any inference is considered to be valid if and only if, their exits no counter example in argument is considered to be invalid if I have counter example like a premises are true in conclusion is false then, it is set to be invalid kind of argument. So, the same way suppose if you have well formed formula something x. If you denied the formula and it leads to contradiction then that not x is considered to be unsatisfied unsatisfiable; that means, x as to be satisfiable x as to be valid argue unsatisfiability of not x guarantees that x is considerable to the valid argument.

So, we need to ensure that if you want to show that x is valid you have to show that not x is unsatisfiable. So, these are some of the basic ideas that we will be will be using it here the basic ideas is that any invalid arguments as true premises and a false conclusion. So, this method involves a rule based construction of a counter example, for a given inference. So, what we do start with negation of the formula in a given formula and then we will construct tree diagram for the given formula and then, we will see if all the paths of the particular kind of tree closes or not this whole thing is called as tableau and this is also called tableau method.

So, when the tableau method closes after denying the formula; that means, not x is considered to be unsatisfiable and that guarantees such that x is considered to be valid it is kind of reduction add up certain kind of method suppose if you are asked to show that x is the case then you start with not x and if not x leads to contradiction. Then; obviously, that is not x is considered to be the unsatisfiable; that means, x as to be valid truth these are considering one of the most efficiency base are checking semantic properties of propositional formulas like truth tautology etcetera validity all those things.

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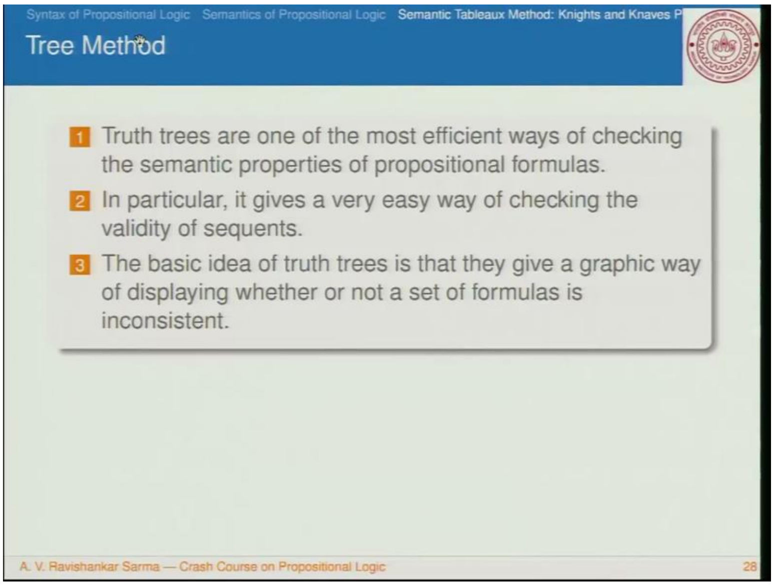

And in particular it gives very easy way of checking the validity of sequence especially when the member increases four five etcetera propositional variables increase then it definitely help us the basic idea of truth is that, we give graphic way of usually say the picture say, inverse it if this is the effects whether or not set of formulas is set to be consistent or inconsistent etcetera.

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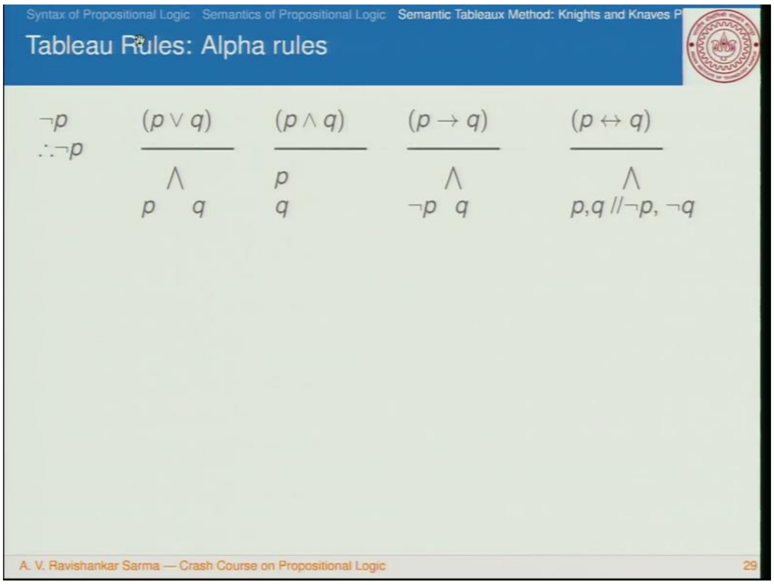

So, this is some of the rules that we used for constructing tableau as you seen in this tableau rules this three tableau ends with atomic formulas, it is most important which we need to note for not p it is same as not p when, you have a compound formula p r q it is a branch living to p q and if we are appealing to both p and q have to be true p implies q has this particular kind of structure not p or q and p is a negative q it is written as p and q and as a not p and not q these are considered to be alpha rules.

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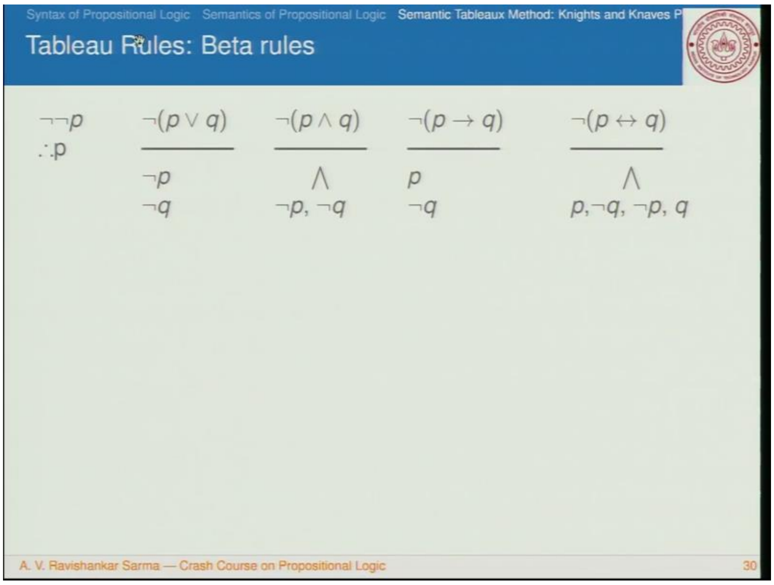

Now, beta rules exactly negotiate of that one are like that not p is same as p not of p and q that use Demorgan's laws and not p and not q. So, that is why you have a trunk of the tree and not of p and q is not p not q.

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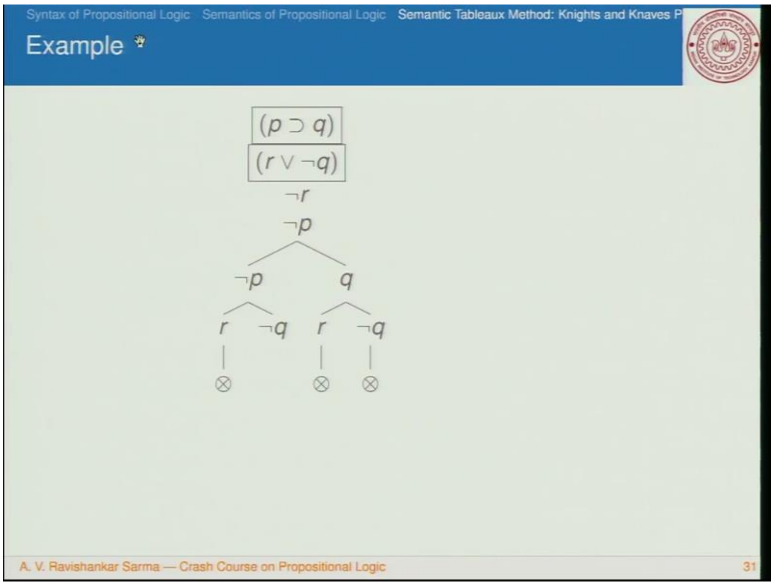

So, we just considered one example and then we will see whether from p plus q and r are not q and not of p forgets or not. So, here p is considered to be the conclusion.

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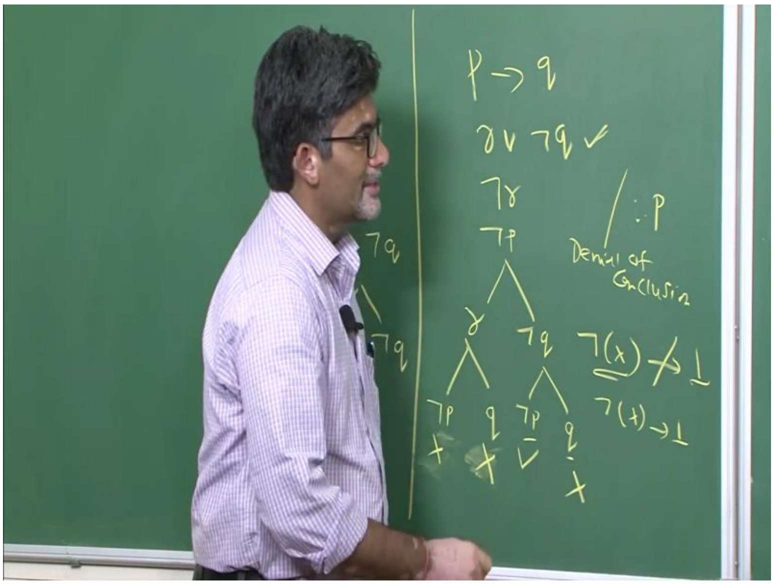

So, it is like this then p r q r r not q and then not r and then whether p follows from this r not. So, we want to check whether the p is at consequence of this r not. So, in the semantic tableaux method what you do the first initial step is that you deny the conclusion. So, this is what is denial of conclusion; that means, you are trying to construct a counter example here.

Now what you do is you simplify this formulas with alpha semantic trees that we use the. So, now, if you apply on this one the tree diagram for this one r r not q it is to be like this r not q. So, it is a branch that is why it looks like this. Now, you write the same bit of information here r not q now this is over that is why we checked it. So, that is why we put a tick mark right here over here.

Now, the formula left first is this one. So, this can be written like this not p r q. So, this particular piece of information needs to be written under all open branches. So, this branch not yet closed. So, branch will be closed only when a literal hence negation after symbolic branch then we have to close the branch. Now, same bit of information you need to write here.

So, now we need to see whether this branch closes r not. So, we have p plus q r r not and we exhausted all the things and ultimately we have ended up with only atomic propositions. Now, this branch remains open because you know in which has this branch gets closed here because r and not r is here. So, this gets closed here this remains open this sides also closed because r and not r is there.

Now, here we have not p not q and all. This branch remains open where as this gets closed. Now, what kind of situation is that denial of the conclusion does not lead to contradiction so; that means, we cannot surely say that x is considered to be a valid argument. So, what essentially we are trying to do here is this the if I want to know that p is logical consequence of the given formulas we are starting with negation formula and then we construct a tree diagram for these things and then ultimately we are trying to see whether not x leads to contradiction.

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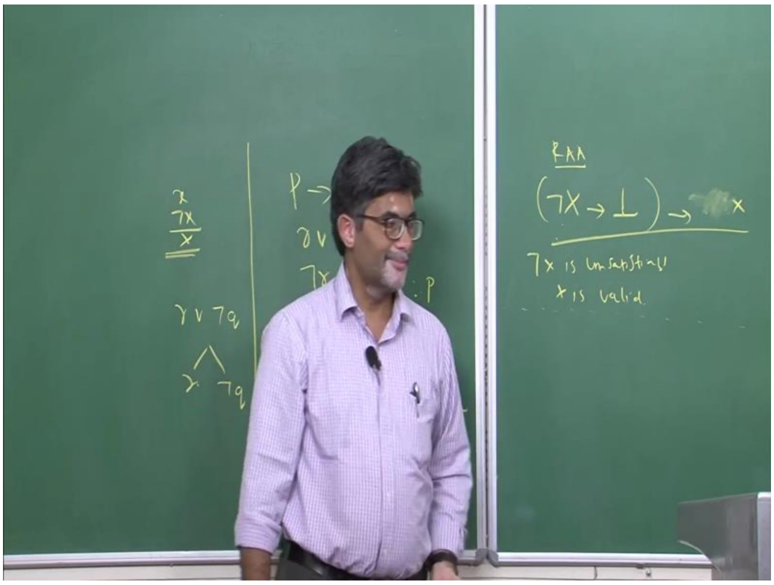

So, there is a principle in logic which is considered to be reduction and up 7 and this is like this, If from not x leads to some kind of contradiction. So, then it should not be not x, but it should be not of not of x. So, this is same as x 1. So, that ensures such that x is construct to be valid, but in this case it appears to be the case that after exhausting the tree rules here I mean you under go with the automatic formulas you did not come up with the branch closer.

So, branch closer suggests that not of x is unsatisfiable, if not x is unsatisfiable then x is considered to be valid so; that means, your counter example did not work there. So, in the next lecture we will be talking about this semantic tableaux method in greater detail and we will be using this semantic tableaux method for solving solve the important puzzles these are the some other which are considered to be very interesting and exciting logic as a subject matter with particularly interesting especially when you study this puzzles try to solve this puzzles by using the basic concepts of logic and another thing which exercises is this is the paradoxes etcetera.

So, they are all valid arguments, but let us something wrong with those things you can only show that they are considering unsound there is only be a resolution part this paradoxes or otherwise you are considered to be valid kind of arguments.

So, in the next lecture we will be talking about semantic tableaux method and how we use it in solving two important puzzles one is and navy’s puzzles and the second one is tiger kind of problems. So, these two problems are very interesting exiting we are up by famous logician Rehman and we will be taking few problems from that particular kind of these two books then, we will be trying to solve those puzzles by using semantic tableaux method.

Thank you.