Basic Concepts in Modal Logic

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Lecture – 20
Semantics Tableaux Method – 1

Welcome back. In the last lecture we have seen how by imposing various kinds of considering some accessible relation is leading us to different axiomatic systems. Like t requires reflexive relations, s4 requires transitivity, whereas B requires semantic properties etcetera. So, in this class we will be talking about one of the important methods decision procedure methods that we are using for this course. So, that is semantic tableaux method. So, little bit of history about semantic tableaux method, it was introduced by many people around the same time the credit should go to Raymond Smullyan Intica, in his modal sets they are proposing the same kind of idea and it was latter used by (Refer Time: 01:01) and all others in US in particular.

So, and Beth, is another important logician whom to we need to credit for this semantic tableaux method. So, the idea of semantic tableaux method is simple and straight forward. Just in the case of propositional logic, if you want to show that given formula is valid or tautology, all tautology is valid formulas. What you do is that you negate formula and start constructing tree diagram. It is upside down kind of tree and then you see whether all the branches are closes or not. If all the branches close; that means, not of x is un satisfiable, if not of x is un satisfiable guaranties you that x is considered to be valid. Similar kind of thing we do here in this case as well. So, what we are going to do is the. So, we have different axiomatic systems K T D S4 S5 etcetera.

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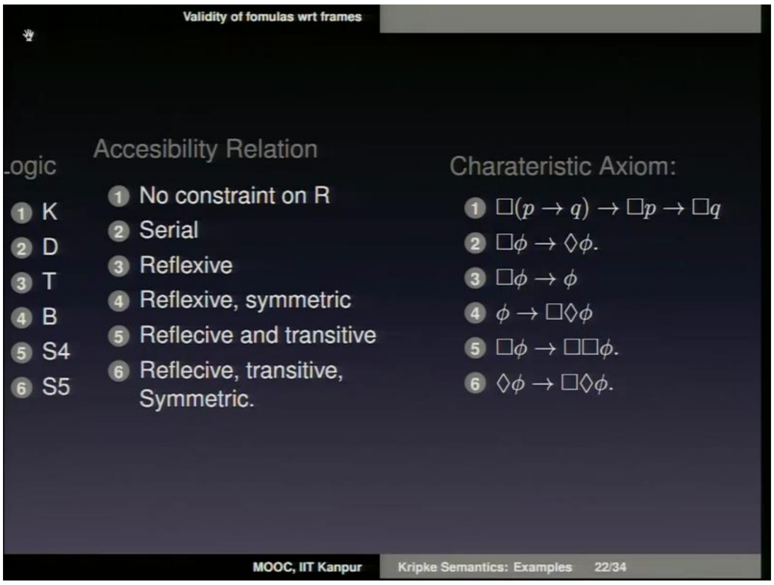

Now while constructing the tree diagram for this things, while negative in these formulas, we will see what kind of constraint we need to impose. So, that the branch closes.

So, before that I need to talk about some of the important rule tree rules. So, that all the tree rules with respect to the propositional logics are intact, but we will we need some more rules with respect to necessity and possibility. So, before I begin idea here is this thing.

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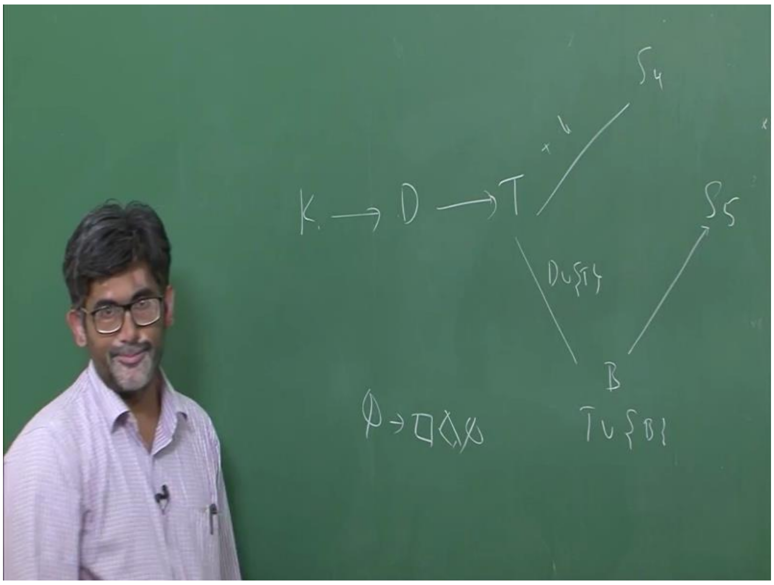

So, K is the one which we began with, it has a characteristic axiom it is necessary p implies q, means it is necessary p, implies it is necessary that q. It does not require any constraint on without I have to impose any constraint on the accessibility relation, whereas, in case of D, it has to be c D m. I mean the constraint you need to impose and accessible relation as to be serial, serial necessary for any given world it as to at least some one world y and where this x is accessible to y. Which are always exists some kind of world y, such that x is accessible to y. For any given world x, you always some kind of world y and which is axis to y or y is accessible from x, there are same things. So, then that is considered to be D.

So, interestingly this this axiomatic system was introduced before D sometimes re flexibility and serial is sometimes leads to some property called serial. So, anything which is valid in K has to be in valid in D and it has to be in valid in T also. So, anything which is valid in D has to be valid in T. Then it goes like this S4 we have S4 here. And then we have something called S5. And then we have some other axiomatic systems like B which requires semantic property and then it leads to S5. So, this together with 4, S4 or 4 leads to S4 and then D together with T leads to B. B is nothing, but T is to B as his axiom corresponding axiom it is something which is K S implies that is possible phi as to be this is, similar what we are going to do in this class is that, you list out all this formulas hence we will see while negating this formulas what are the things what kind of constraints you need to impose on the accessible relation is one of which are trying to see here.

So, now before that we need to talk about some rules. So, we are all the tree rules for proposition logic is same here and they are going to be like this. So, we have something called alpha rules and we have beta rules.

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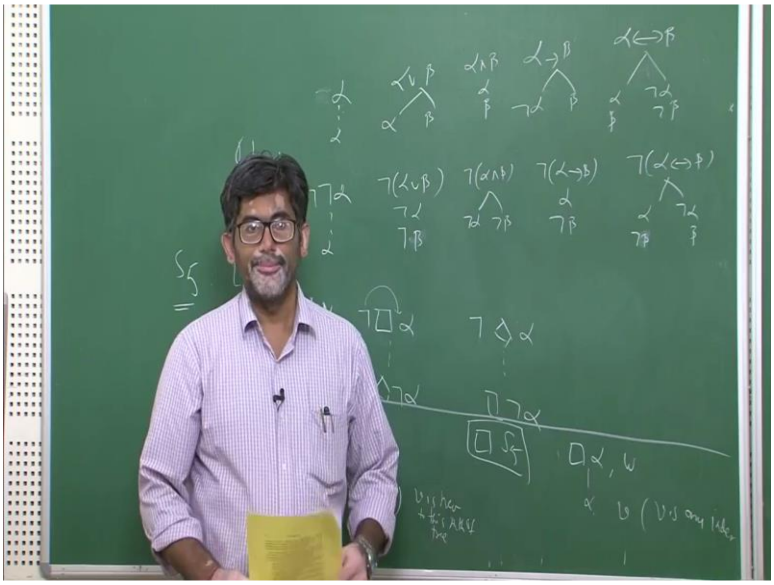

So, if a formula is there like this alpha, and then it is same as alpha only, then alpha or beta it is constructed be a branch and alpha and beta it is a truck of the tree it will be like this. The tree diagram, for the basic formulas that existing in your moral logical system there like this. Now alpha implies beta is not alpha. And beta and then alpha if only if beta is like this. Both alpha and beta has to be true n not alpha, not beta. So, said to be some kind of alpha rules is that I will talk about positive things here. So, exactly the negation of this one is like this. So, the negation of negation of alpha is substituted as just alpha. And negation of alpha or beta, it is a conjunction negation of negation is conjunction and you need to write this.

Negation of alpha followed by negation of beta what it means is it both had to be true. So, that this is going to be true. And negation of alpha and beta is the branch which insists of these formulas. So, negation of alpha implies beta is not alpha and beta and the negation of alpha implies beta, is alpha not beta or you can take even not alpha beta also not alpha beta.

So, these are things that we have in the case of classical logic. So, now, we have this 2 things, necessity and possibility. So, now, we are talking about S5 system particularly we are talking about let us we talking about S5 and this are considered to be rules all the rules are propositional logics are intact together that we have this rules, and something called as modal negation suppose we have formula like this, and the modal negation of is this one is like this negation of necessity of alpha. So, if any formula such kind of formula exits, then we need to simply replace, this negation goes inside it will be if negation goes inside, and the negation of necessity is going to be possibility, and you have put this negation inside. So, it is going to be like this and necessity of it is not possible that alpha. So, this is this can be written as necessity not alpha. So, this is with respect to modal negation.

And there are some rules which are considered to be this thing possibility S5. So, this is going to be like this. If any formula alpha possibility of alpha exists, then you replace it with alpha, but you need to ensure that this suppose if this is a world w, this says that possibility alpha true in the world w, you replace it then it will become alpha is true in a world v, but this as this is v is nu to the nu to the, v is considered to be nu to this path, this path of the. So, the idea here is that suppose if you have 2 formulas like this, possibility of alpha and possibility of beta for example. Now first thing when you remove this thing you choose one letter u and you are not supposed to use the same kind of u again when you remove beta now it as to be beta and the any other letter other than u is one which you take into the consideration D as to be true in any other world.

So, no 2 possible worlds are alike. So, possibility of alpha is there you can replace it with alpha, but v as to be nu this w is accessible to v. Whereas necessity of S5, that draw a line. So, that will be clear necessity of S5 rule. So, this tells us that, if a formula necessity of alpha as to be true in a world w, as by the definition you can say that necessity of alpha is going to be true in a world w when even alpha is true in all possible worlds so; that means, this alpha has to be true in your world v. So, this v in any index indexical letter that exist in that given path of your tree. So, v can be any index. So, let me all such kind of v Sit can be v1, v2, v3 anything all those things your alpha has to be true. And these are the things which related to necessity S5 true.

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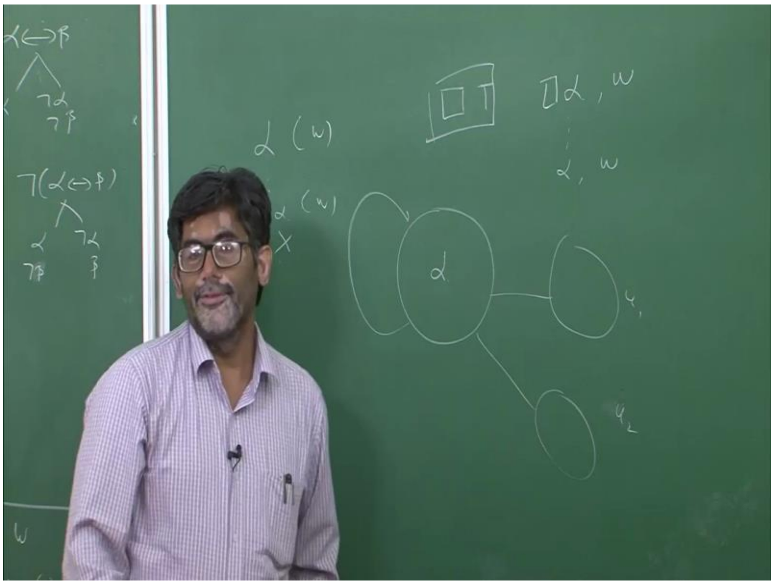

So, there is a one more formula which we use it is what is called as closure a formula. It comes across a formula alpha, and your alpha come across a formula not alpha. This is also true in a world w, then this is called as closure this is reading to in consist kind of situation.

Now, the only formula which is nu here is this thing, which is what is called as necessity t. So, this is an interesting one which, so, just say this thing necessity of alpha is true in a world w. When we say that a necessity of alpha is to in a world w if alpha is doing all the wells that w as axis to, if w as axis to u it as to be true their w, as axis v. In v also this alpha has to be true. So, now, if you come across a formula necessity of alpha is to be world w, then t rule tells us that this alpha has to be true in that world itself. So, it is like necessity of alpha means, it is true in this one this one etcetera, let us say this is u 2 u one etcetera.

So, in addition to that thing if you have necessity of alpha which is true in a world alpha means and this alpha is true, in this world itself then you are following some reflexivity relation that is nicely mentioned by this rule. Whenever you come across a formula necessity of alpha to in a world w, we replace it with alpha and put the same index, so that we will solve our purpose. So, now, using this rules let us talk about the formulas that we have see here.

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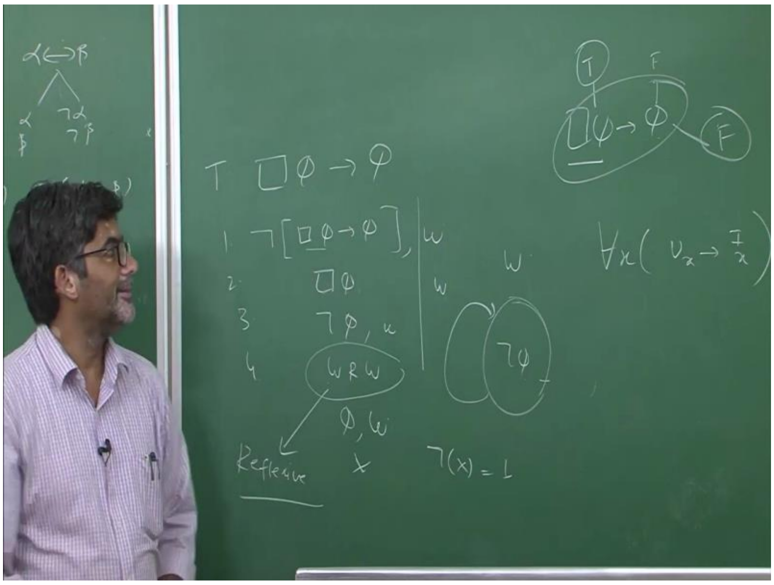

Why it is case that simple formulas like necessity will begin with the simplistic kind of formulas like this thing. So, this is an axiom T. So, we are trying to show that, this is going to be invalid in K and we will construct a counter example and then we will nicely draw the relational structure please do it quickly.

So, now in the semantic tableaux method the first step is to get the formula and start constructing the tree based on these rules. So, this is like x implies y. So, this is psi not phi. Sorry, this symbol is this one necessity of phi implies phi. So, the simplification of this one is this. Because negation of x implies y is nothing, but x and not y. So, I have used the same here and we have written like that. And we need to note that in modal logic unless and until you mention the world that you are in that make any sense. So, modal logic you only talk about truth with respect to the worlds. We are we are constructive to be possible worlds.

Till now we did not exactly we mean by possible worlds. We will have another special kind of lecture now possible worlds. And then we will introducing the concept of there is a view which is dominant in philosophy in particular, that is what is called modal realism, do you treat possible worlds are real as well as the concrete things that you consider it as concrete, or you have to treat it as in abstract kind of things. So, there are some other questions that we will answer and them we will deal with such kind of questions in a separate lecture and that is the lecture is based on possible worlds and modal realism.

So, now coming back to this one; this is true in world w. So, this is case now necessity of phi is true in world w. So, now, we need to put likes. So, now, there is no way in which you can close the branch. There are 2 things which you require. So, whenever you are require necessity of phi it does not implies that it is always some kind of world which exists out there and in that world phi is true. That cannot be a case. When have possibility of phi, it should be the case you know if this is not the way you cannot say that there is no world such kind of thing. So, there is always some kind of world and in that world phi has to be true, but in this case you cannot say anything, but when you say possibility of phi, is true in a world 0 means which should always be some kind of world one, where in that world one, the phi as to be true that is only difference between these 2 thing is like the difference between universal quantifier and the existential quantifier.

We do not talk about assume with things to be this all unicorns are intelligent to be the case, but the movement when we talk about there exist some unicorn and that x is considered to be intelligent for example, if you write it like this, unicorns are represented like this, if x is a unicorn x is intelligent because u and I are considered to be predicates and the one hand we have another statement which is like this. X is unicorn and x is intelligent. The movement you say this there exists some x; that means, it presupposes that your unicorns has to exist in a world. So, this is a big kind of para-see, which is called existential import existential Pharisee which we are not going to detail of it.

But the idea is same here also. Whenever there is a possibility of any formula it already presumed that you can assume that, there is already a world which accessible to that one. So, answer may be possible that is raining outside. It means there is already a world in which it is raining is considered to be true. In the actual world it is not the case. So, if it so, happen that you know this formula remains open because there is no we are not able to say that you know you cannot reduce this thing necessity of phi to phi. So, now, the counter example in K system could be like this. From the open branch you can construct a counter example.

Now, the counter example is going to be like this. Not phi of course, this is not accessible to anything we can leave it just like this. We can say this thing w. Simple example is this thing, now, why this serves as a counter example. So, now, substitute this thing again into this one this is the only one world and in that world not phi is true. Now coming back to this formula, necessity of phi implies phi. So, as I said in the last class that if there is no world which is accessible to this one, then all the formula that begins with necessity are going to be true, and all the formulas that begins with possibility are going to be false. So, in that case it is consider to be T; now, n phi where phi is considered to be false here. Phi is already false here. So, this this is true and this is false this makes the whole condition false. So, the counter example the relation structure that we can draw is only this one.

So, now what happens under what conditions this branch closes? This branch closes only when if there is a world w and that w is accessible to itself. And it is accessible to itself then necessity of phi the mean phi as to be true in that world w itself. So, the moment you draw something like this, w is accessible to itself then your phi as to be true in that world w. So, now, you have not phi is true in a world w and phi is true in a world w and this closes. So, under what conditions we close this thing when an accessible to relation is reflexive, when the branch closes. And not x is going to be un satisfiable, only when you know you impose reflexive, you impose a constraint in the form of reflexivity on the accessibility relation so; that means, this formula holds only in reflexive frames.

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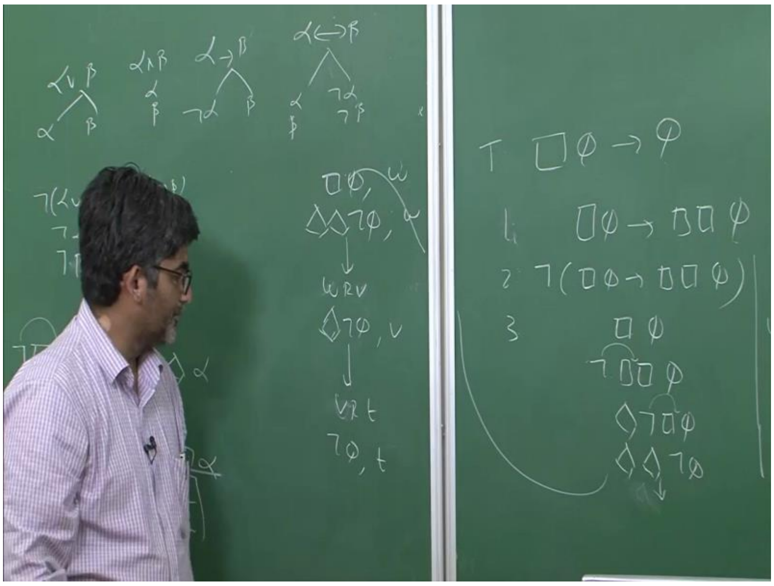

So, now let us consider another example. There we will see why it is the case that this formula which is characteristic of 4 axioms. So, that is necessity of phi, implies necessity of necessity of phi. That is necessity true, as to be necessary true. So, what it means. So, now, again you begin with the negation of this this formula where it leads to this this is the second step. 1, 2 in semantic tableaux you start with an inner of the formula and then ultimately what you show is, not x is un satisfiable; that means, all the branch close it become sun satisfiable. If not x is un satisfiable what is satisfiable is not of not x. Only if you remove all the possible of not x you will come to x. So, that is the idea here. So, now, expand it little bit further. Then it will become this not of 2 necessities and then phi. So, now, we have to simplify all these things are true in your assuming that these are true in a world w.

The first thing when you use a negation thing you have to push negation inside. The negation of necessity will become possibility and you are pushing this negation between this things then it will be like this. First time you apply like this. Now second time, when you do it negation of necessity of phi, you have to you have another possibility because of negation of necessity is possibility and you are pushing this negation further inside, then it will become like this. 2 possibilities and then phi not phi, now, there is a formula with which you have this thing necessity of phi and then you have this thing. So, now, let us remove this possibility operator by using the rules that we have here. So, possibility of alpha when you come across the formula like this, it is true in a world w then you can simply replace it with alpha, but that v world v where the alpha is true as to be nu. So, now, first time when you remove this formula, then it will become you have to write like this. So, I will mention it here because space problem I will consider this here. So, this will become like this, I am just writing here. So, we have possibility of possibility of not phi and then we have necessity of phi which is true in a world w.

Now, first time when you remove this formula then we have w it is accessible to v. Some you take a letter v and then in that world v this possibility of not phi is true. So, now, next time when you remove the same possibility operator demand operator then you are moving from v to another world so; that means, v this v has to be v is accessible to something. Some world randomly I have taken in to consider t. And in that world t not phi has to be true in a world t. So, now, what we have is this thing if some formula possibility of phi is true in a world w. So, now, we apply then the t rule. The t rule says that whenever you have a formula necessity of phi is true in a world w, it already taken into a consideration right it is already reflexive kind of relation. And then where you do not you do not achieve anything if you take this thing to consideration. Suppose if h necessity of phi is true in a world w and then what you get is phi is true in a world w. There is no way in which you can close the branch. So, the idea here is that negation of the formula and then we are questing under what kind of conditions this branch closes.

So, now w is already accessible to v; that means, necessity of phi is phi is true in a world v. And necessity of phi is true in a world w means wherever this w is accessible to in that world it as to be true. Now this branch will remain it closes only even the accessible relation is considered to be transitive. So, this goes like this.

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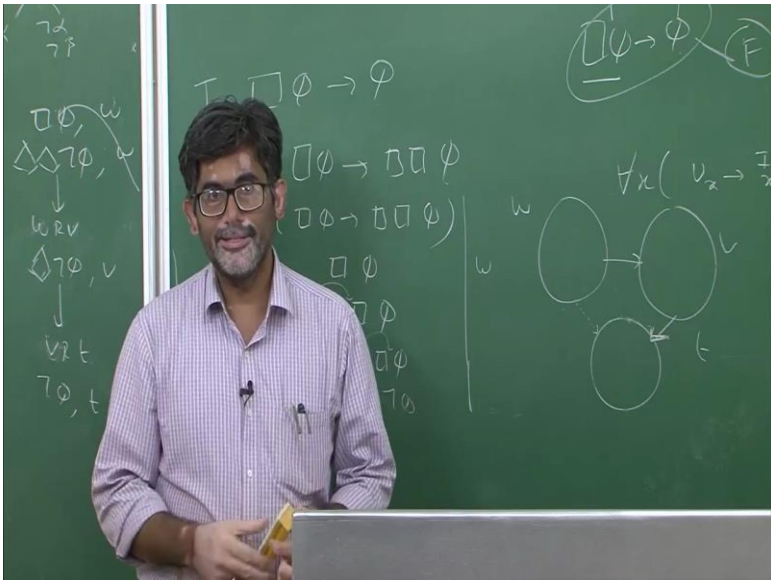

We have taken 3 worlds w v and t. Let us take t is the world which we taken to the consideration is w and v and t. So, w has to be accessible to v. So, this is one which we have w is accessible to v and we will see here v is accessible to t, but in this case there is no way in which we can move from w to t. Unless and until you bring in the transitivity relation that transitivity relation is this one. Which I am drawing it like this unless and until this happens, a w is accessible to v and v is accessible to t, w has to be accessible to t then that is considered transitivity problem. If there is no transitive property this branch will remain open. So, now, the idea here is that we have w r v and v r t and that should imply w r t. So, this is what is transitivity property.

Only when you invoke this thing the branch closes; if that is the case when phi has to be true in that world t also. Only in that case your phi is going to be true in that case it closes. So, what did we get out from this one? So, if we delay the formula with this thing that it is necessary the t phi means implies that whatever is necessary has to be necessary, it closes only when you impose transitivity relation on the access going to relation or if the accessible relation is transitive in this sense then it only close otherwise it is going to remain open.

Suppose if it remains open here itself then it is going to be invalid in t. So, why because you know there is no way in which you can close this branches, and from the open branch you can study the counter example and this counter example can be represented in terms of K structures like this, the K structures would have possible worlds first of all, and then you need to know how this world are related to each other that is what we have, and then we need to know where your atomic formulas are true. In which world a given atomic sentence that exists in a modal logic formula are construct to be true. So, let us consider one final example and then we will see why if it a case that this particular axiom requires symmetric property or any other property.

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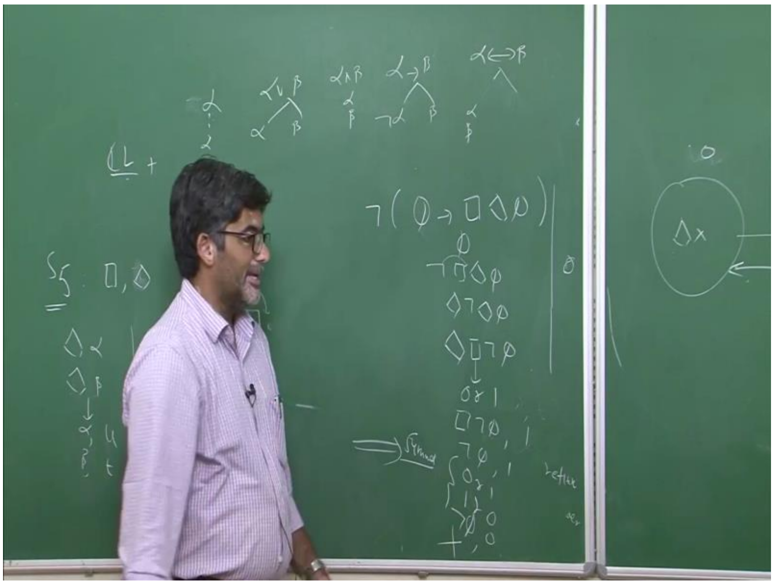

So, that we have going to see, suppose now if we have a formula like this; if phi is actual truth with this, we will end this lecture. Phi is true implies that it is necessary that possible phi. So, now, under what conditions in which axiomatic system it is going to be true, that is what we are going to see. It is true t K B C B D etcetera. So, now, again you deny the formula and start constructing a tree then it will become like this. So, all this thing that true in a world 0. You take w you can take alpha gamma it does not matter. So, now, negation of necessity possibility of phi this goes inside and then negation of necessity, it become possibility and this one possibility, negation possibility phi.

So, now if you simply it further this will become negation of possibility will become necessity, and then this one. So, now, you have to reduce this thing remove this possibility necessity operator etcetera. So, till now we are in world 0. So, now, we have a possibility operator; that means, you can move go to other world 0 r 1, and in that world one your necessity of not phi has to be true. So, this can be defecated like this. So, the moment you have possibility of some formula x, you have to get into another world w x, is considered to be true. So, these are different worlds w one and w, 2 necessity of negation of alpha. Now there is a rule t rule, which tells us that whenever you have formula necessity of not phi is the case, we can simply substitute not phi I can say this is true in a world one, and this has to be true in a world one already it is reflexive relation. So, then only this hold here.

Now, but that is not going to serve our purpose. Because you have phi is true in a world 0. Phi is true in a world 0 and not phi is true in a world 1. So, when this branch closes are the one which we are trying to study. So, this branch closes only when we have a relation like 0, r1 and if it. So, happen in that one the world 1, necessity of not phi is 0 if 1 is also accessible to 0 it. So, happen that let us assume that this is 0 and this is 1, and if 1 is also accessible to 0 in that case necessity of phi is true in a world 1 it has to be true in a world 0 also because 1 is also accessible to 0. So, this is what is symmetric property only when you invoke this symmetric property, then there is away in which you can move from world 1 to back to 0.

So, in that case your phi has to be true in a world 0 not phi has to be true in a world 0, in that sense this branches closes. So, this we will stop in this lecture. We have just considered semantic tableaux method. With which you are just which trying to see various axiomatic systems that we already studied, in some more detail when we are taking about, when we are when we considered axiomatic system for this moral logics. So, what happened now is this that things have become simplified in sense that depending upon you, if you if you just know if you have some set of worlds and how these worlds are related to each other, and if you have an accessibility relation etcetera then you can say when a given formula holds in which kind of frame t holds in reflective frames where as B holds in semantic frames etcetera. So, I stop here.

Thank you.