Basic Concepts in Modal Logic
Prof. A.V. Ravishankar Sarma
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

Lecture - 19
Kripke Semantics for Modal Logic: Examples

Welcome back, in the last lecture we considered various examples some examples related to Kripke model, and the Kripke model consists of a set of non empty possible worlds w and you have a accessibility relation r and you have a valuation function which you assign some kind of values to the propositional variables that existing your given modal logical formula. So, Kripke seems to be he is trying to talk about four levels of truth; first truth is with respect to a given possible world and truth with respect to a frame a frame consists of set of a possible worlds plus accessibility relation it is an order by which consists of set of non empty set of possible worlds and accessibility relation and then modal consist of W R and V.

So, we have seen, far we have seen various modal logical systems till 1960s, modal logicians were working in that tradition that synthetic tradition, there they were trying to come up with new freedoms by taking the this characteristics axioms in to consideration what are the characteristics axioms? Every logical systems has it is own characteristic axiom and these modal logical systems are named after some logician are there.

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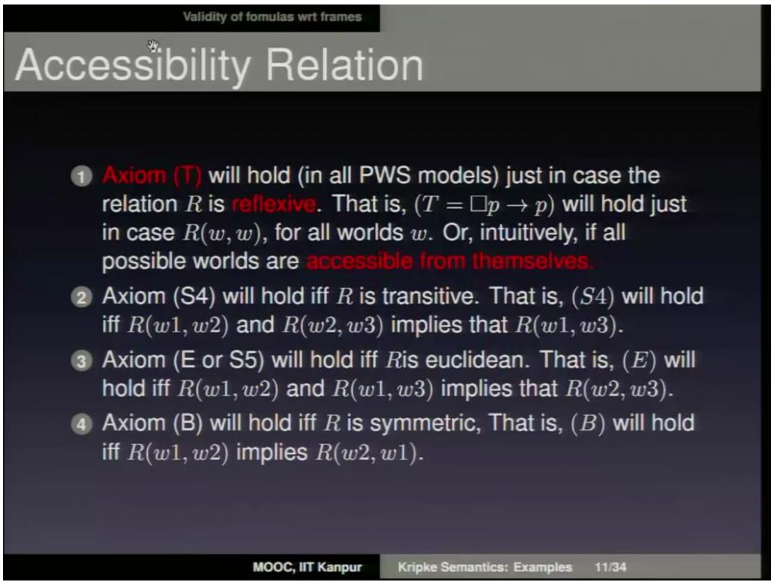

For example, in the case of T. T has a particular characteristic axiom which says that it is necessary that p implies that it is p and for example, if you have another axiomatic system which is which has introduced before t that was that already occur in the branch of deontic logic name of the system modal logical system is d d has is characteristics axiom that you know something is necessary in it has it is possible that p is the case.

So, you have to note that some other some of the axiom some of the theorems, which are consider to be part of some axiomatic system mean at have to be theorem in other thing. For example, if he says it is necessary that p is implies that it is actually in the case at p that does not applying in the case of deontic logic. For example, I have been telling you the same example you are to follow the traffic rules that does not mean that you actually follow the traffic rules. So, Kripke adjust with the help of some relational structures, W R V it is like a dieted kind of graphs with which he could talk about many things such as validity satisfiability and there are many things which is trying to talk about using this relational structures. So, Kripke main observations are that, when you considering the accessibility relation then it results in on this different model logical systems.

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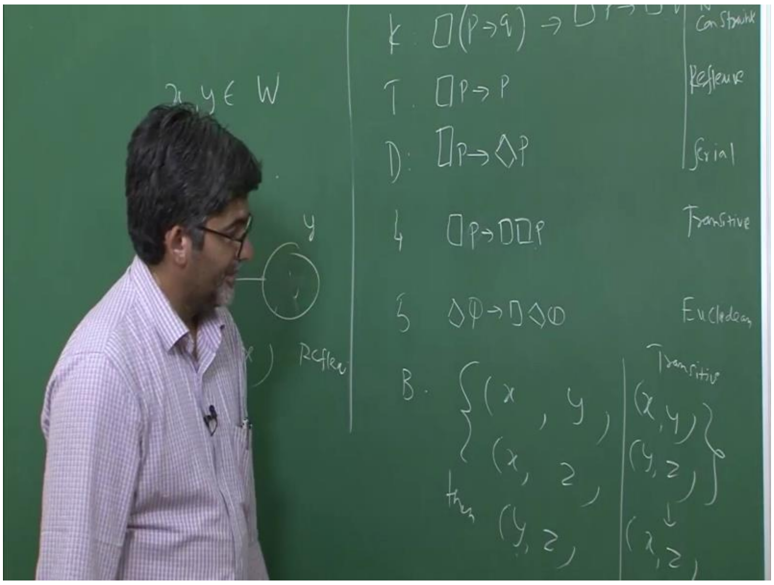

Suppose if you see here in k, k has this characteristic axiom p implies q it is necessary that p implies q means it is necessary that p in necessary that q. You don t have to impose any constant on the accessibility relation. So, that is why it is considered to be the minimal modal logical system.

So, the moment you impose on the constraints on the accessibility relation R, we have we require we need to impose a specific kind of accessibility relation on the specific kind of constraint on the accessibility relation he observes that. When t that moral logical system T has his characteristics axiom it is necessary that p implies, p this requires that accessibility relation has to be reflexive. So, when it is consider to be reflexive, especially when there are two worlds let us say you have x and y belongs to your set of worlds in a Kripke model, if x is accessibility y then, y is also accessible to x if that is the case then, accessibility relation is consider to be reflexive.

So, in diagram it is like this. So, sorry this is considered to be symmetric kind of property reflexivity is like this. So, if x is accessible to itself then it is consider to be reflexive property if there is a world x and if this x is accessible to itself then, it is consider to be reflexivity property. So, if there are two worlds x and y belongs to set of worlds w if, x is accessible to y is a way you write the things x forward by y; that means, this x is accessible to this another world y, this is x and this is y then, y is also accessible to x if that is the case then your this accessibility relation to be symmetric and there are some systems which follow this symmetric kind of if you need to impose symmetric symmetry on the accessibility relation.

So, then there is a resulting system could be B or something like this. So, like this Kripke could study all the axiomatic systems while imposing some kind of constraint on the accessibility relation the only thing you need to note is that, k does not require any constraint. So, all these things we will talk about the same thing when we deal with particular technique we are using in this course that is symmetric tablets method. So, what we do is we negate the formula we will construct a tree for this all this formulas and we see when the branch is closed.

So, now observe this one it is necessary that p implies it is necessary that it is necessary that p requires that, your axiom your accessibility relation has to be transitive and particular kind of axiom 5. It is that something is possible that 5 that has to be necessary has to you need to impose the constraint of Euclidean kind of property, Suppose if x is related to y and x is also related to z then y is accessible to z.

If these two things happen then, y is accessible to z if that is the case then it is a Euclidean property transiting the property is like this, if x is accessible to y and y is accessible to z these two taking together these to axis accessible to z this is what is transitivity property. The Euclidean property is little bit different that is it looks like a more transitivity property. But it is not x is accessible to y the same x is also accessible to z. So, that is why y is also accessible to z. So, x and y z are all these things are possible worlds. So, now for example, let us take a simple Kripke structure and see why it does not this necessity of p implies p does not hold in k.

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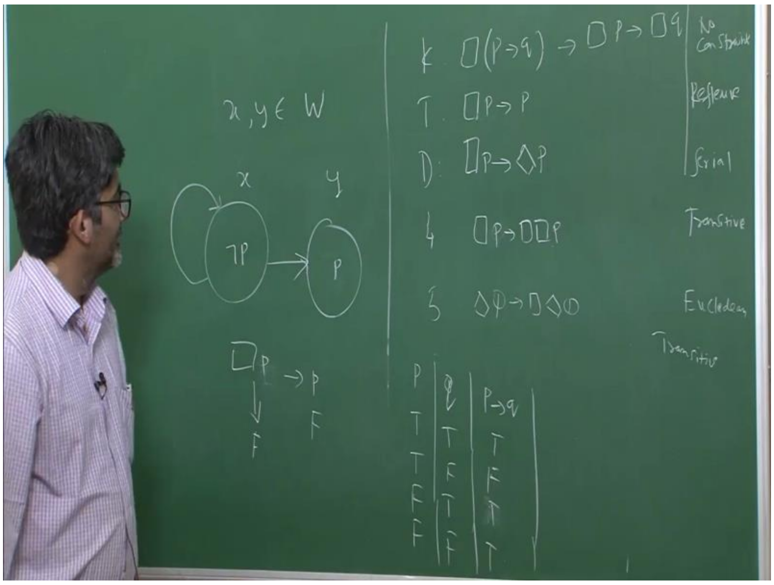

You can have a counter example where necessity of p is going to be a true, but this p is false.

So, now let us consider a relational structure where they have two worlds x and y. So, now, situation is like this p and here not p is true. Now this x is accessible to y. So, now, this will serve as a counter example for this one y because now if you evaluate this thing necessity of p in all the possible worlds which are accessible to x in that world p has to be true. So, this is true, but p is considered to be false here why because not p is true here that is a p is false here. So, this is true and this is false this makes the whole condition false.

So, now Kripke could easily talk about these things that now the same kind of axiom if you do not impose any restriction on this one, it is going to be invalid in k and the counter example is going to be like this. So, we can draw a relational structure and this relational structure will indicate us that there is a counter example for this one. So, necessity of p is true, but necessity of as a p is considered to be false here that makes a whole condition false. So, like that in you can see.

For example; if you draw some kind of if you if you make this thing little bit different then that is going to be valid here. For example, if x is accessible to itself then what will happen in the nearest worlds where p is true that is the case, this is the nearest world to this one where p is true in this case in world it is the case. So, now, there is another world which is accessible to itself x is accessible to itself in that world necessity of p is consider to be is false; why because p is false here p is not true here. So, that makes this one. So, I will go in a simple method. So, so what we have done is that we have drowned one extra arrow to this one and then we are trying to see whether it holds or not.

So, this tells us that x is consider to be we are using some kind of reflexive relation we are imposing the additional constraint that we are drawing it like this; that means, x is accessible to itself. So, in that case necessity of p necessity of p means in the entire world which is accessibility to this one your p as to be true. So, the world which is accessible from this one is p. In the first case it is true, but the world which is accessible to itself this is the one in which p is false; that means, are case in one world where p is false the necessity of p is also going to be false.

So, for that reason this necessity of p is going to be false. So, now p, p is already false there. So, now, we have a truth table for p, p q and p implies q. So, this is going to be false only in this case when antecedent is true the consequent is false is going to be false in all other cases it is going to be true. F and F is referred to going be true. So, what it tells us is that the same kind of axiom. If you do not impose the accessibility relation has the constraint on this accessibility relation this is going to be you can have a counter example for this one. So, this is going to be theorem only when you impose some kind of constraint on the accessibility relation. So, there is the beautiful innovation of a Kripke he could simply see all the validities etcetera of these formulas by means of some kind of relational structures.

What all we need is that we need to have some kind of non empty set of worlds, we need to know how this accessibility relation is constraint as the reflexive symmetric all this things, if reflexive symmetric transitive needs to be there all are required then it is consider to be an equivalence relation. And most of the model logicians were talking about an axiomatic system S5, S5 is consider to be diagrammatic system which is widely studied why because all this relations are already presenting there you required reflexivity transitivity symmetry etcetera that is why S5 is no other system was studied in on the sense that x y was studied that is why I was studied widely by many logicians and in S 5 also later will see that this S5 will serve as modal of knowledge.

So that means, logics in particular we begin with S5, where but the priority thing is that this necessity possibility will be have in a slightly different way, necessity can be treated as knowledge operator and I thing that something is possible can be treated as this diamond kind p, but in our course we are course, we are just concentrating our attention on only model it is that is we are taking about logical necessity met a physical necessity. So, we are not trying to talk about necessity with respect to the individual agent etcetera.

So, in this way you can have a counter example for in K for T, t this axiom is not valid cased. So, in the same way d requires serial kind of property for any a given world it should always be some kind of world which has to be accessible to that one that is what is serial kind of property and then transitivity property is a x is accessible to y, y is accessible to z x also as to be accessible to z.

So, now based on how you constrain the accessibility relation Kripke has come a with various modal logical systems, you could clearly say that axiom T I mean T was defined in terms of characteristic axiom till 1960, there was no way to say that you know they were not talking about validity semantic other consequence etcetera a many logicians were trying to two theorems by using this axiomatic systems. So, axiomatic T will hold just in case the relation is consider to be reflexive and it will hold just in the case that w is accessible to w, in the same way axiom is four it is necessary that p implies it is necessary that it is necessary that p holds only when, you constrain the accessibility relation to be transitive we are S5 holds, when r is considered to be Euclidean the one which mentioned it here and B will hold when accessibility relation is symmetric x is accessible to y y is also y also should be accessible to x.

So, these are the ways in which depending upon the constraint on the you apply on the access, but relation Kripke could find out easily that you know this is t this is k this is k this is S4 S5 etcetera.

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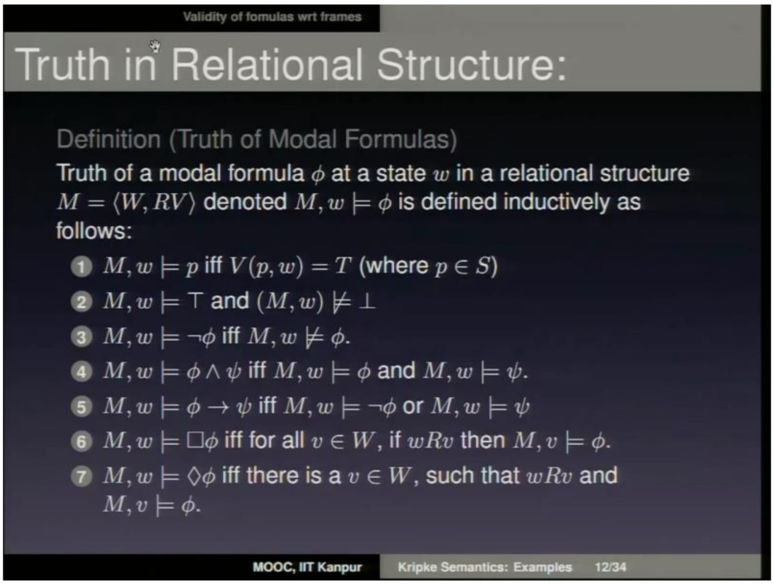

So, now what do you mean by saying that some formula is true with respect to the relation structure what is the relation structure here? W R and V non empty set of world accessibility relation and the valuation function. So, you write M w double phi; that means, phi is a phi is true in a world w with respect to a modal M. So, that is defined in this sense first any atomic formula. If it belongs to any given world is true is represented in this sense p is true in a world w with respective your model M; all total this as to be true in a given world w and all contradictions does not mean follow; that means, it does not does not hold with respect to the world w and not phi is true in a world w means phi should not be true in a world w not phi true in a world w; obviously, it is it is taken phi is false there as phi is to not phi does not phi fallows then phi is not the case the phi does not follow only one of this thing should follow either p are not p should be follow.

So, now phi and psi is true in a world w only when both are considered to be true in the same way is usual disjunction rules at the a conjunction is true only when both conjunctions are true is true at least one disjunction is consider to be true. So, these are all already there in the case of propositional logic, what extra that we are trying to talk about is, something is consider to be necessary when it is true in all possible world this is idea which is carried forward, I mean idea which is taken from this work.

So, is of the view that our world is considered to be the one of the best world best possible world. So, there many world, but our worlds that we are inhabiting is consider to be the best possible world and also consider to be another possible world. So, now in this context necessity of phi is true in a world w, if and only for all such kind of v's some kind of worlds belonging to some set of worlds w and in all such kind of world v this w has to be accessible to such kind of world v and in that v your phi as to be true. If that is the case than necessity of phi false the same way possiblity of phi is true in the world w. If at least there is 1 v some v some possible world v and in and that your actual world is accessible to that particular kind of world v and in that world v and in that world v your phi as to be true; simple thing is that is that true for in the given world your phi as to be true that w is accessible to v and that v for it is for any v, but in the case of four possibility of phi even if it is happens for one possible world where you are diamond phi is true and it is also our purpose.

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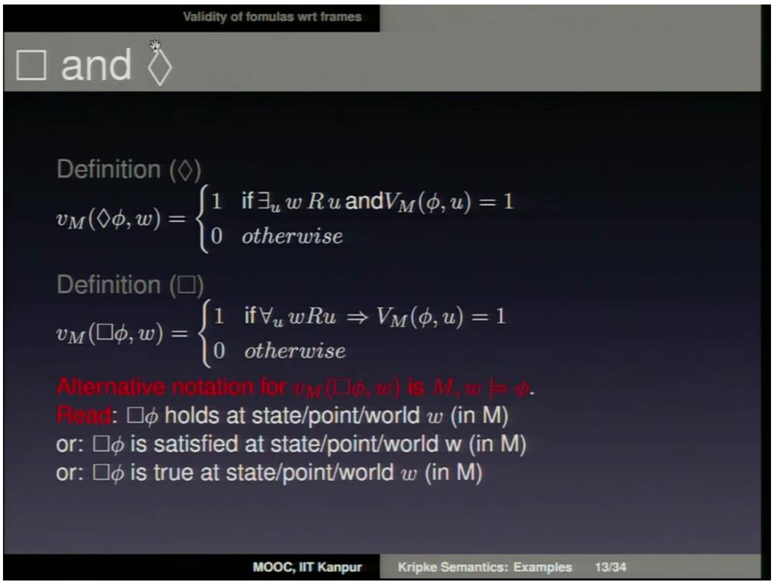

So, the another way of defining the same kind of thing suppose, if I write v m possibility of phi w you have edit in the sense possibility of phi is true in a world w with respect to a model m and that is going to take a value t or one usually mention t by one false as 0. If and only if there exist some u where w is accessible to such kind of u and in that u to your phi has to be true where as in the case of otherwise it is going to be zero so; that means, if u w is accessible to u, but p is false there then it is going to be zero necessity of possibility of phi is going to be zero because p is not going to be true there p is false there.

So, in the same way the necessity of phi is true in the world w with respect your model m there is going to be a true or takes value one and if it happens for all such kind of use where this w s are accessible to those use such that your u as to be true v m phi u as to be true there. That means, in those entire worlds phi as to be true. If w is accessible to u one or u two etcetera in u one phi as to be true u two also it as to be true otherwise it is going to be in any other case that you can think of whereas going to be true. So, this is one important thing which we need to not suppose there is there may be a situation where no world is accessible to anything.

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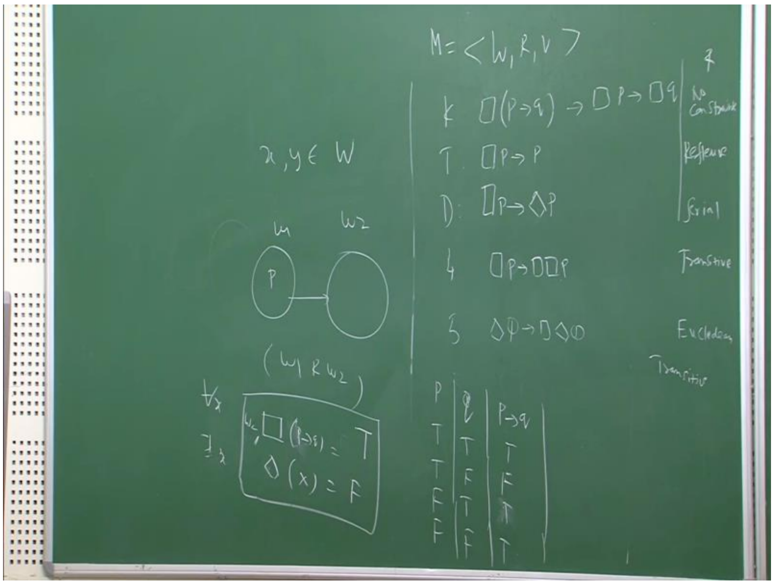

For example in this case let us consider some examples where example if it is like this is W1 and W2 and W2 does not consist of anything. So, now, here let us consider p is true in W1 now we need to see that here W1 is accessible to W2, but we do not have any information about W2. So, now, if that is the case if there is no world which is accessible to accessible from this world W2 then, we need to know about this important thing that is in formula that begins with necessity it can be anything it can be contradiction or it can be any formula p implies q or anything that as to be true where as any formula that begins with possibility has to be false.

So, it is like saying that distinction between there exist some x and for all x. So, the idea here is if there is no world which is accessible from a this W2 suppose if you are evaluating this formula which respect to W2; that means, necessity of implies q which true in a world W2. If there is no world accessible from W2 all the formulas which begin with necessity are going to be true and all the formula which begins with possibility of x is going to be false in some of the examples, you will come across these kinds of situations.

So, when the second worlds are consider to be empty; that means, there is no world which is accessible to that world w suppose, If you are evaluating the formula necessity of phi with respect to such kind of world then; obviously, it as to be true I mean it as to be true and in all other cases, if you take the possibility of phi it as to be false.

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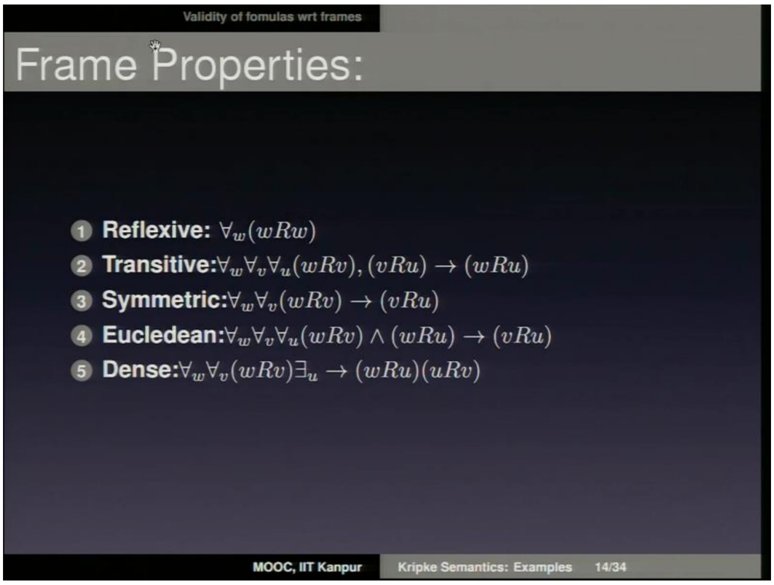

So, depending upon what kind of constraint you impose on frames you will have different kinds of frames these frames can be reflexive transitive symmetric Euclidean dense etcetera depending upon the constraint that you are imposing on the accessibility relation.

So, you need to know that again I was point out the repeating the something that in our Kripke he was talking about four levels of truth is with respect to a world if it is something is true in all possible worlds. It is true it is consider to be a valid kind of statement and valid formula and truth with respect to the frames there are some formulas which holds only in reflexive frames like t necessity of t implies p where is necessity of p needs to be taken something like something is freedom all the necessary true's are have to be true.

So, there is a nothing wrong in, but in if see impose a same kind of thing in other kind of thing then it may not hold when, the case of you have to follow the traffic rules implies that you actually for the traffic rules that may not work there.

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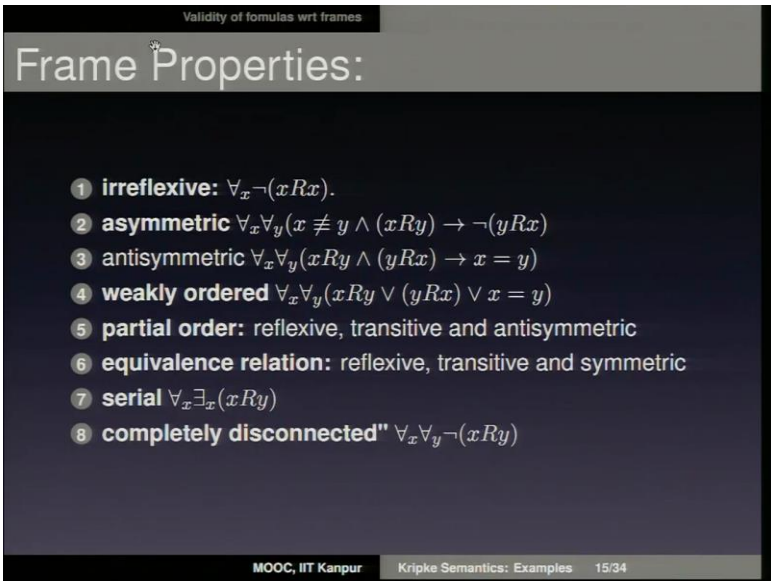

So, this frame properties can be extended further and then you can talk about the reflexivity asymmetric anti symmetric all kinds of other things depending upon, you can impose more and more constraints and then it will result in more and more logical systems for example, in the case of seventh case serial property is required for deontic logics in particular it is necessary that phi implies that it is possible that phi.

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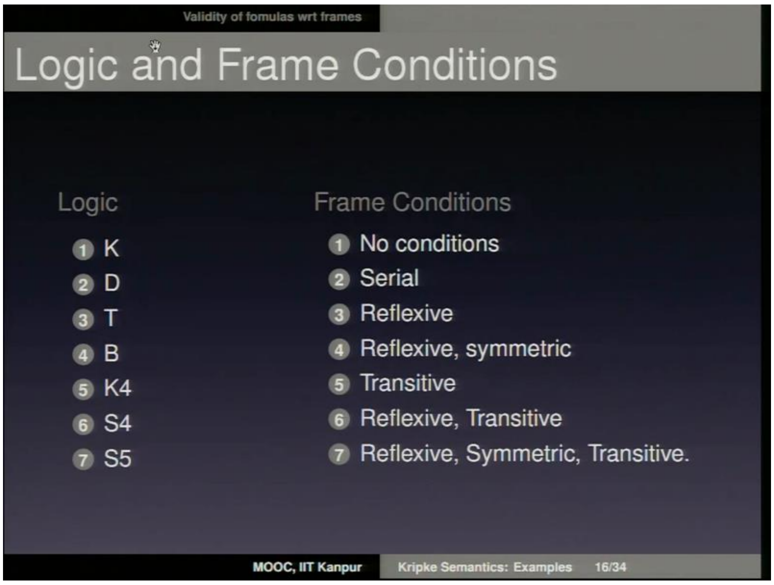

So, here is the picture that we have as of now these are the mostly widely studied logical system K, D, T, B, K4, S4, S5 etcetera and the only thing you need to note is that k, k does not require any condition you don t have to impose any constraints on the accessibility relation where as p requires reflexivity relation and b required to reflex symmetric k for required transitive property and. So, on forth S5 is the one which has all this constraints which is to be imposed.

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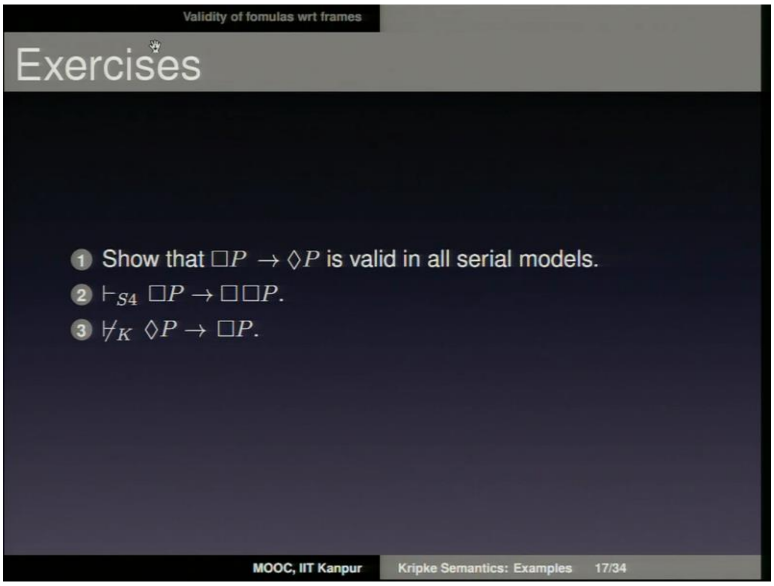

For example, is take these thing that necessity of p implies possibility of p you have to follow the traffic rules means you have only permitted to follow the traffic rules you have to note that each and every modal operating comes in joules. If you represent something as a necessity you are always something which is represented by possibility and necessity and possibility can be defined one in terms of another necessity of p is defined as it is not possible that not p and possibility of p is defined as it is not necessity that not.

For example, you see here it is necessary that p implies it is necessary that it is necessary that p follows in S4, but it may not follow in k and even p because it requires more than reflexivity he required transitivity S4 requires reflectivity together with that we require transitivity for that reason this possibility of p implies necessity of p it is involved in all for instance including the minimal modal logical system k ah.

Once a given formula is invalid you can construct you can easily show it with the help of Kripke modals Kripke diagrams very easily. So, these things will carry it forward and then when we talk about semantic tableaux method we will make use of these things in a better way.

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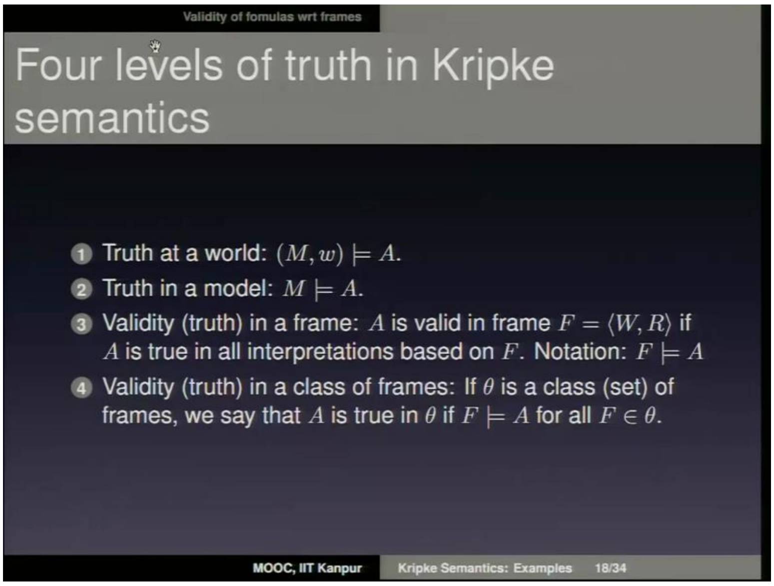

So, in a natural if this lecture is all about he is talking about we are talking about four levels of truths that is mentioned in Kripke s semantics one is truth with respect to the world w in that context we have seen some problems the only thing which we need to know to here there is no world which is accessible to that given world then, we need to view all the formulas that begins with necessity as to be true yeah they are true and all the formulas that begins with possibility is consider to be false and Kripke is also talking about truth is expect to your modal m that is what is represented in the sense a follows followed by m and validity in the frame is defined in this sense. a is valid with respect to only this frame consist of set of non empty set of worlds set of non empty non empty set of worlds and then you have an accessibility relation r and if a is true in all interpretation is based on that particular kind of frame f then, you represent it is in with the a is the logical consequence f or a is true in f is the is frame that makes a in formula a true rather same thing.

So, valid in class of friends is like this if theta is considered to be a set of frames all kind of frame reflexivity transitivity all kinds of etcetera we say that a is true in theta the set of all frames which includes reflexive symmetric serial all kinds of it happens, for all such kind of theta. If f belongs to the theta consists of set of all frames with this we will stop here and then, we will continue with semantic tableaux method with which in you know we can talk about a same axiomatic systems and then, while constructing the while neglecting the formula and constructing the tree diagram for these things we will come to know what kind of accessibility relation, we need to impose on the given formula. So, that the branch closes the branch closing is means not exceeds unsatisfiable then next has to be valid.

Thank you.