Basic Concepts in Modal Logic

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Lecture – 18
Kripke Semantics for Modal Logic: Some Examples

Welcome back. In continuation to the last lecture where we discussed in general about the Kripke semantics and then we have seen that Kripke structure or a relational structure consists of set of possible worlds, an accessible to relation which tells us how these possible worlds are connected to each other and then other thing, which you need to have is the valuation function, which assigns some kind of values to the atomic propositions that exists in your given modal logical formula. So, let us consider some simple examples in this lecture, to see when a given modal logical formula is considered to be true. So, let us start with some simple relational structures and see how given modal logical formula is true or false.

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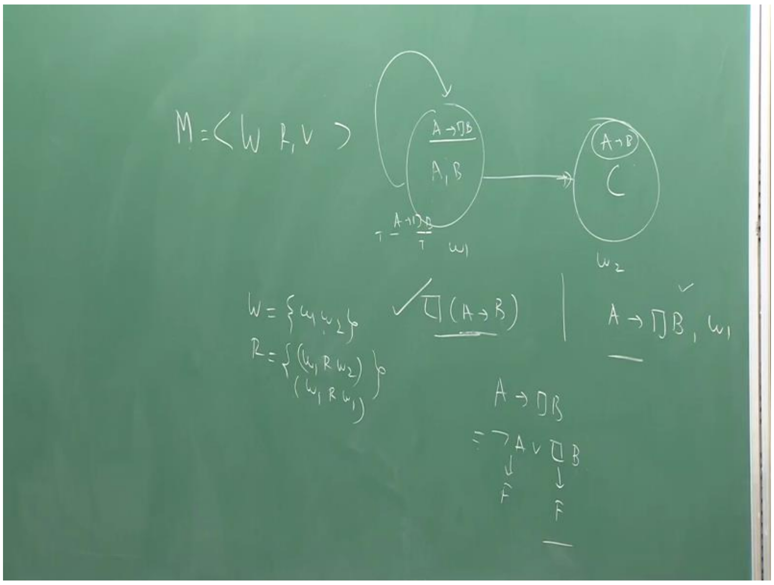

Let us say you have Kripke structure, we have we need we require this thing Kripke modal consists of set of possible worlds which has to be non empty and R is an accessible to relation and V is a valuation function. So, here is a kind of relational structure where we have like this A and B are true in a given world w 1and in case of w 2 let us assume that C is true.

So, now we are trying to find out whether necessity of A implies B and A implies, we want to know which one is going to true in this particular kind of relational structure. So, now, we need to write like this have a set of worlds are like this w 1and w 2 and then accessible to relation there is only one word which is accessible to the other one w 1 accessible to w 2. So, this is the only thing that should we have and then valuation function is that A and B are true in with respect to w 1and with respect to w 2 C is true. So, now, A implies B, when this implies B is considered to be necessarily true. So, there is one more thing which needs to note. So, in this accessibility relation w 1is also accessible to itself, this is another kind of accessibility in the accessibility relation we have this also.

So, now A implies B in all the worlds in which assuming that this is the case. So, in all the worlds in which this word is accessible to in that world you are A implies B has to be true. So, what is this A implies B. So, A implies B is nothing, but not A r B. So, the worlds which are accessible from this one are this and even this also. So, now, you have to have not A r B has to be true here and it has to be true here also. So, with respect to this word if at least one disjunct is true it does not matter. Whether A is true or false, you are A implies B is true; that means, A implies B is true here. Now we need to check whether A implies B is true here or not since both A B are true here so; that means, this A implies B is true here also. So, what we have seen what we are seeing it here is that necessity of A implies B; that means, A implies B is true here and A implies B is true in this world also; that means, it is true in all possible worlds that we have.

So, that is in this necessity of a holds c now coming back to this one A implies necessity of B. So, now, when it comes to this one we are evaluating A implies necessity of B here. So, now, with respect to this in this world for examples A is already true and then B is true here and B is true here also; that means, even necessity of B is also true. So, that is why this formula A implies necessity of B is true here.

So, in both cases it holds. So, now coming back to this one, with respect to this world A implies B, A implies necessity of B holds, but now what about this one. So, there is another world which accessible to this w 1. So, what essentially you are trying to do is you are evaluating this formula A implies necessity of B. Now this means in all the possible worlds in which A is true your necessity of B has to be true. So, this is going to be like this. So, the idea of doing it is that whether a necessity of A implies B that is, what we have seen in the last few lectures, that what is going to be the scope of your necessity operator is it this one this is considered to be correct kind of conclusion correct kind of translation or suppose if we use it in a narrow sense it is same as this one.

So, when the confusion arises we need to work with both the translations. Now A implies necessity of B. So, this is same as not A r necessity of B. So, now, not A has to be true here, but A is true here now this becomes false here now necessity of B necessity of B when those worlds which this B is true. So, B is true here and even B is true here as well. So, in those cases in this particular kind of example, of course, these two both the formulas are going to be true because A is the antecedent of this conditional is already false there. So, A is true, but you have you need to have sorry in not A r necessity of B not a is already false and then necessity of b since necessity of B is true. Why? Because if you evaluate necessity of B here B is true here and B is true here as well suppose if you are if you have this particular kind of thing instead of b you have c here. So, what is going to happen to this one not A and necessity of B.

So, now in this case A is going to be false because not a is going to be false because a is true here; that means, not A is going to be false and necessity of B there is a possible world which is accessible from this one, but B is not true there. So, in those cases this will also become false. So, in with respect to this relational structure now consider necessity of A implies B. So, now, A implies B definitely it is true here whereas, again in this case in all the possible worlds. So, necessity of A implies B means A implies B has to be true here. So, again this is not the case. So, it might be false there also.

So, the idea here is that sometimes I mean it is not always the case that necessity of A implies B is same as A implies necessity of B. So, you have to use this necessity operator over the whole conditional rather than you use it in the narrow sense. So, let us consider some more examples. So, that you know you will understand when a given modal logical formula is true with respect to a given structure relational structure. So, let us draw another relational structure which is going to be like this. So, we start with some simple examples and then we will see how we can do it.

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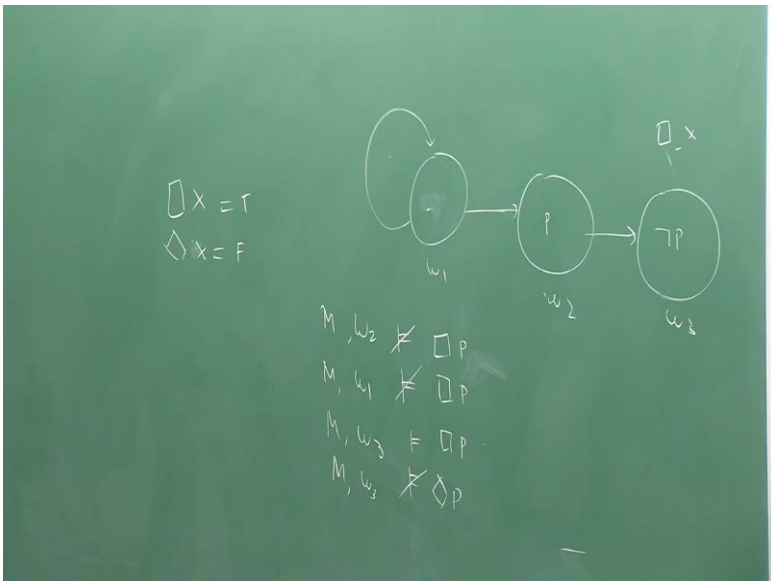

So, let us say you have a world w 1and you have a w 2 and there is another world which is considered to be w 3. So, now, here p is true and here not p is true for example. So, now, with respect to w 2, we are trying to evaluate this particular kind of formula necessity of p is true in a world w 2 we are checking whether necessity of p is true in a world w 2 or not. So, necessity of p is true in a world w 2 means in all the possible worlds this has this is accessible too in those worlds p has to be true, but p is false here; that means, this does not hold with respect to w 2 and with respect to Kripke model necessity of p does not hold, but with respect to w 1. For example, you are evaluating this formula necessity of p with respect to w one. So, now, the world which is accessible to w 1is w 2 only the next nearest kind of world which is accessible to w 1is w 2 only nearest kind of world which is accessible to this one is w 2 in that world p is true; that means, this holds and there is no other world which is accessible to w 1.

So, in that case the necessity of p is going to be true there. So, now what about w 3, now, you are evaluating necessity of some formula with respect to w 3, here is a interesting situation which arises here. So, this w 3 is not accessible to any other world. So, if that particular kind of thing happens, then whatever formula that begins with necessity operator is going to be true and any formula which starts with possibility operator is going to be false there. So, in that context necessity of p is going to be true there whereas, possibility of p it does not hold in w 3. So, this is the rule that we which we use it in modal logic if there is no world which is accessible to a given world w 3 that is empty kind of thing then whatever formula starts with the necessity kind of operator is going to be true there. Whereas, all the formulas that begins with the possibility operator is going to be false there.

So, in that context possibility of p does not hold in w 3. So, there are other things like suppose if this is accessible to this one. So, now, again the things changes here now if you are evaluating for example, coming back to this formula necessity of p with respect to w 1. So, in a world which w 1has accessed to your p is true, but now here w 1is accessible to w 1, but we do not know whether it is true or false. So, you this need not have to follow from this one w necessity of p do not follow in a case w 1. If this is accessible to itself, let us consider a simple straight forward example with which you will see whether.

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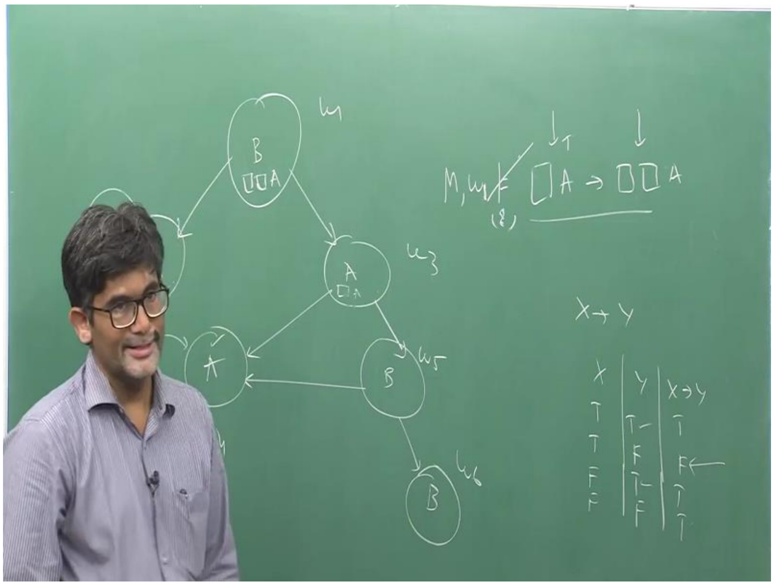

How to evaluate formulas given modal logical formula with respect to this relational structure, here are the worlds that we have w 2 and here is w 1. Then this is accessible to let us say w 3 and w 3 is accessible to w 5 in that w 5 we will see which formula is going to be true there.

So, now w 5 this is w 6. So, we have different set of worlds from w 1 to w 6. So, now, this is accessible to this 1 w 1 is accessible to w 1 and then we have some other world which is considered to be w 4 and this is accessible to itself and then w 3 is accessible to this one that is, why we are putting arrow towards this one, and w 5 is accessible to this. So, this is the relational structure that, we have it is not complete, but we need to have only drawn how these possible worlds are related to each other. But we need to say what are the atomic sentences that are true in these worlds, we have some formulas B is true here A is true here and it is.

So, happened that A is accessible to this one and A is true here and then A is true here B is true and then B is true here. So, now, we have some kind of relational structure. So, now, we are trying to talk about some formulas like I will talk about only one formula. So, that is this one necessity of necessity of A with respect to any world that you have taken into consideration if this formula holds in all possible worlds then this is you have to write it like this is considered to be a tautology if it is not the case if at least in one world it is false then this formula does not hold in all world.

So, now we need to check in which, as I said in last class Kripke is talking about various levels of truth, one is the level of possible worlds which world it is true and then, you talk about truth of a given formula with respect to a frame and then a modal W R and V. So, now, let us assume that you know this is a Kripke modal and then we are trying to see this is question mark, whether with respect to w 1this is a w 1. Now you are evaluating that particular kind of formula whether it is true or not. So, now, this is considered to be a conditional statement and the semantics same for the it is same as the material implication that is like this x and y x implies y; that means, T T F and F T F and T F, if you understand one example in a better way you understand the other examples automatically.

So, this formula is going to be false only in this case in all other cases it is true so; that means, the whole formula necessity of a that it is necessary implies that it is necessarily necessary, this is going to be true it is going to be false only in this case; that means, your antecedent is true and the consequent is false then this formula is going to be false in that particular kind of world otherwise it is going true so; that means, if you can make the consequent, is this thing y consequent true or false consequent is true then; obviously, you x implies y is equal to be true only.

So, now let us see what necessity of A with respect to this is. So, all the worlds which are accessible to this one in those worlds you are A has to be true. Of course, that is indeed the case so; that means, necessity of A is true there. So, now, in this world the world which is accessible to this one is this and even this also. So, that is why in those worlds A is true so; that means, necessity of A is going to be true there. So, now, we need to check whether necessity of necessity of A is true or not. Now what are the worlds which are accessible to w1, w2, w3. So, now, if you if you are trying to talk about necessity of necessity of A here.

So, then you need to move to this world where it will become necessity of A one necessity you can remove it and then necessity of A has to be true here A necessity of a has to be true here also, now we need to evaluate whether necessity of A is true with respect to this of course, this is the case necessity of A is this is a only world which is accessible to this one and in that world a is already true; that means, necessity of A holds here.

Now, we need to check whether necessity of A also holds here also in w 3. So, now, what are the worlds which are accessible to w 3 w five and w four and in those worlds a has to be true because necessity of A is true in a world w 3 means in all the worlds which are accessible to w 3 your a has to be true. So, in this world it is true, but in this case it is not the case that A is true. So, for that reason this formula does not hold because in one particular kind of although this antecedent is true in one particular occasion particularly necessity of necessity of A is going to be false here because necessity of A although it is true here, but when it comes to this one it is going to be false. So, that is why with respect to M 1 it does not follow if it does not hold in one possible world of course, that is good enough to say that it does not hold in the Kripke modal. Kripke modal consists of in this particular kind of frame is that a given set of worlds and then how these worlds are related to each other that constitute say particular kind of frame.

So, many such kind of examples can be given and then the only thing you need to notice is that if there is no world which is accessible to the given world, then all the formulas which starts with necessity are going to be true and all the formulas which are starting with the possibility are that are not to be false. Let us consider some examples, where you know how when, this now Kripke talks about all the modal logical systems K T D S 4 S 5 etcetera and then now it is time to talk about under what conditions the given theorem holds for example, in this case let us talk about this one.

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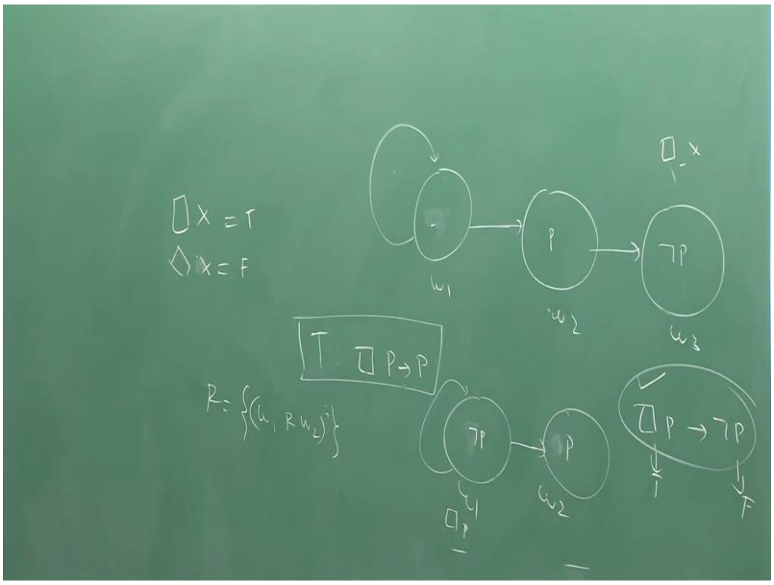

So, let us take this simple example that is a t. So, which states that necessity of p implies p, how do we check whether this is definitely not valid in k you can easily come off with a counter example where you can here you can construct a counter example where necessity of p is true, but p is considerably false there.

So, now here is a counter example let us say you have w 1and w 2. Let us say you have accessibility relation like this is w 1 r w 2 it is a set like this and then valuation function is like this that p is false here and p is true in w 2. So, now, this does not hold in k, but it holds in it holds only when you view reflexive relation in a, if you view accessibility relation as reflexive otherwise it is going it is not going to hold for example, in this case this is this is serving as a counter example why because with suppose, if you are evaluating a conditional like this necessity of p is going to be true there the antecedent necessity of p means in all the worlds which are accessible to this one p is true and now whereas, one second. So, this has to be like this p and not p in all the worlds this will do in all the worlds which is accessible to it your p is true that is necessity of p is true, but here not p is false; that means, necessity of p is true and not p is false that is why it makes the whole conditional false. So, now, one final remark is that, now Kripke was trying to understand I mean of course, using Kripke semantics we can what we can say about this particular kind of formula.

In k definitely this formula does not hold because of this particular kind example. So, now, what happens if you take an accessibility relation to be some kind of reflexive relation a world w 1 is accessible to itself, then this is going to be viewed in a totally different way. So, in those cases if your accessibility relation is considered to be reflexive then the same kind of formula necessity of p implies p might hold. So, now, Kripke was trying to understand these various modal logical systems K T D S 4 and S5 and now he is trying to come up with some kind of constraints on the accessibility relation and then he says that T. For example, holds in the reflexive frames, whereas, d holds in serial kind of frames and then S 4 holds in transitive frames and S 5 holds only when the accessibility relation is reflexive symmetric and transitive.

So, in the next class will be talking about this examples in much more greater detail then I will be talking about what exactly we mean by possible worlds and what you mean by saying that is there any deference between logical necessity physical necessity something, which is necessarily true with respect to necessarily holds with respect to some kind of law statement that is nomic necessity what kind of necessity that modal logics particularly the once which are trying to talk about talks about, we talk about only Alethic modalities it looks like that we are now we are metaphysical necessity or logical necessity.

So, will continue with this discussion when I talk about the when, I talk about possible worlds and it is connection with different kind of view which treats this possible worlds and this position is called modal realism, which is due to David Lewis we will continue with this thing in the next class.

Thank you.