Basic Concepts in Modal Logic
Prof. A.V. Ravishankar Sarma
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

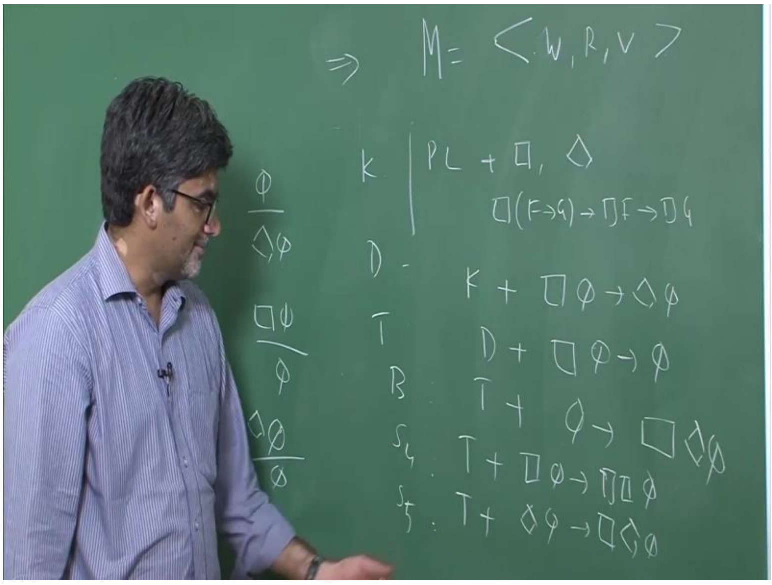
Lecture – 17
Kripke Semantics for Modal Logic Systems

Welcome back in the last lecture we discussed something about the syntax of the modal logic, where we introduce various modal logical systems like KTD S4, S5 etcetera; based on some of the characteristic axioms.

So, in this lecture we will be doing semantics of given semantics of modal logical systems, KTD S4 and S5 and you have to note that, till 1960s logicians were continuously trying to the proving theorems, within these 5 axiomatic system of course, there are many other modal logical systems, which existed in during that period of time. Their interest was only to prove theorems based on some kind of set off axioms, and the actual credit goes to Kripke and just based on some kind of relational structures, he could he could come up with his idea that we can talk about the validity of a given formula, and he also introduced 4 levels of truth, that is truth with respective to a world, truth with respect to a frame, truth with respect to modal and truth with respect to all kinds of Kripke modals.

So, in this lecture, we will be talking about Kripke semantics or it you can call it possible worlds semantics for the modal logic. So, before that what we have seen so far is this thing.

(Refer Slide Time: 01:45)



So, we have we talked about with modal logic is all about extensions of classical logic with these two operators, this stands for necessity and this stands for possibility. And in that context all the total ways of prepositional logic are already there, and if you add these two things, possibility and necessity plus this particular kind of axiom like F implies, G implies. Necessity of F implies necessity of G, then this will become K axiomatic system k and which is due to Kripke himself and this is considered to be the minimum modal logic.

So, now this is constructing to be the characteristic axiom for K. So, for d is nothing, but it is k everywhere you will you will find this k in all the modal logical systems. So, it is not all the mean in d you have K together with that it has own characteristic axiom, so that is this thing, something is necessary implies that it is possible that 5, it does not imply that what necessary truth does not imply something which is actually true. So, that is a case one T. So, in the case of T it is D plus we have this characteristic axiom.

So, this tells us that all the necessary truths are actually true as well, but this same kind of axiom may not apply here; you have to follow the traffic rules implies that you actually follow the traffic rules that need not have to be case, you can always come up with an example where you have to follow the traffic rules is true, but you may not actually follow the traffic rules. So, this may not apply in the case of deontic logic. Some of there are other things which most of the things are studied in the context of understanding the strict implication the Lewis say, Lewis has come up with 5 different axiomatic systems, in order to understand the strict implication unfortunately the attention on strict implication has been diverted and it has been diverted into proving some kind of theorems etcetera and then from 1960 onwards. 1962, 1972 that is what we have discussed in the historical origins of moral logic, this is considered to be the classical era.

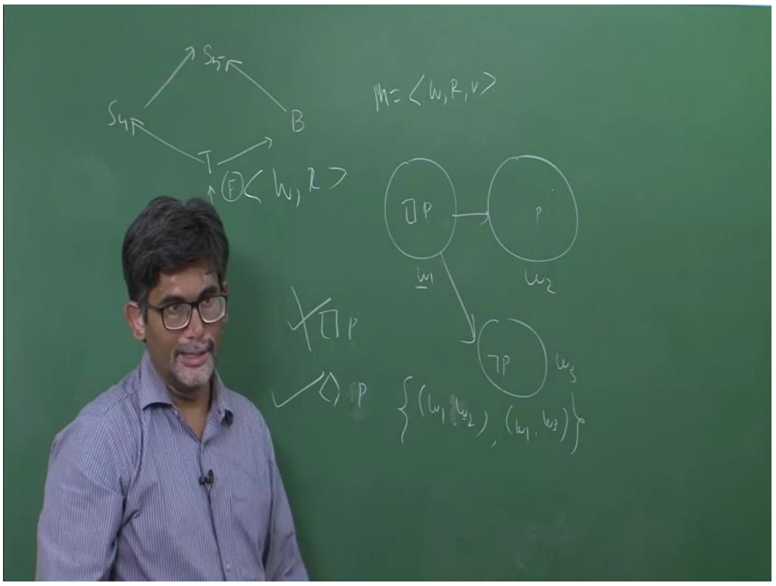
So, in that it was dominated by some kind of relational structures that is what we are going to do in this course. In this class the relation structure is defined in this sense m is equal to set of possible words and accessibility relation and the valuation function. So, now, B is another axiomatic system, which results in when you add this particular kind of characteristic axiom to T whatever is T is already there in B and you have to add this particular kind of phi. So, that is if something is actually true and this implies that it is possible that phi has to be necessarily true. So, usually we have some phi, you can derive possibility of phi. So, I am philosophy teacher means it implies that I could have been a philosophy teacher.

So, in the same way from necessity of phi you can derive phi, but you should note that you cannot this possibility of phi does not imply. This phi something which is actually the case of phi suppose if I say that it is possible it is raining outside then, it may not be actually it is not actually true it may be possible that it is not raining also. So, B has this particular characteristic axiom and the other things are S4 and S5, S4 and S5 has this T you already having this axiomatic system T plus this it is necessary that it is necessary that phi. So, why these other things are introduced because modal logics have been extended to dealing with arguments which involves knowledge claims etcetera.

So, this can be viewed as listen I know that something is a case implies that whatever is known to you is also known to you this necessity is viewed as knowledge operator then, you have to read it in that particular kind of formula in the same way k, T rule can be write like this. Whatever is known to you have to be actually true you cannot say that I know something and that is actually false, you cannot come across with the case where you know something, but it is false. S5 has missing again T, T means D plus this phi what is D? D is k plus this one. So, this is t possible t of phi implies that it is possible that phi has to be necessary.

So, these are some of the most widely studied axiomatic systems in modal logics, each and every logic resulted in to explain. For example, in the case of deontic logic this takes care of the deontic logic D and then, say some other logical systems such as S5 for example, which takes care of epistemic logic.

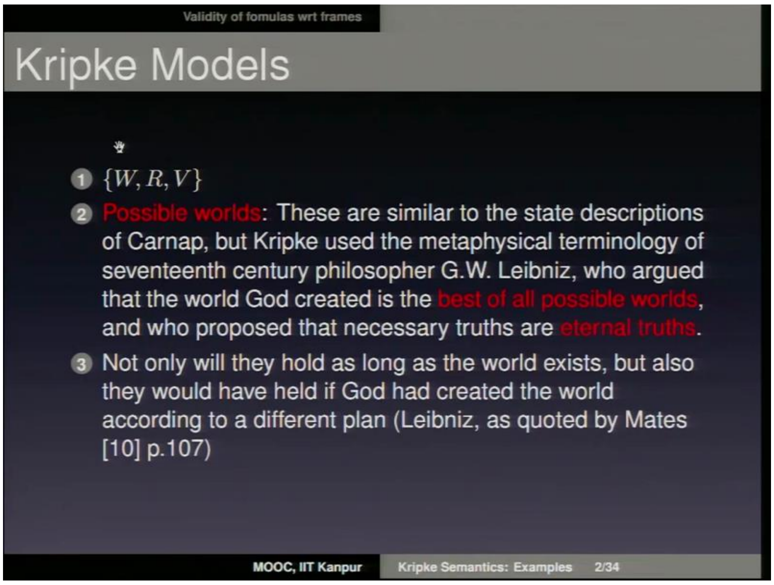
(Refer Slide Time: 07:34)



So, if you draw a diagram or diagram will appear appears to be like this, you start with the minimal modal logic k and then it goes all the way up and you have D because this is an extension of k. So, you have D and then followed by that you have T here arrows are important here. So, now, from T you add some more things you have B here and the other direction you have S4, now this arrows are important.

Now, from S4 all the way up you end up with S5, this is the arrow that goes going like this now goes like this from the adding something you move to S5. So, this is the syntactical kind of things that we already have. So, now, what Kripke has to say about these things? What did Kripke, what are the innovative ideas of Kripke? Why it is Kripke semantic is constructed to be a revolution in the history of logic particularly in the case of modal logics? So, now, let us talk little bit about what exactly we mean by a Kripke modal? Before considering the Kripke, modal Kripke modal is considered to be a triplet which consists of set of possible words W

(Refer Slide Time: 08:52)

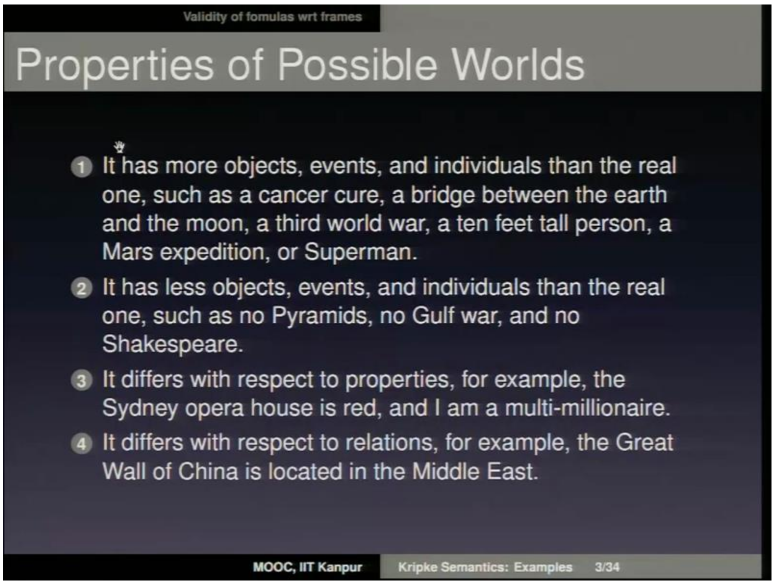


We have an accessibility relation with which you can you will be able to know what in what way is possible worlds are related to each other and then we need to know the access that is the accessibility relation and we have a valuation function which assigns some kind of values to the preposition variables that exist in your modal logical formula.

So, now before we begin we need to start with the set of possible worlds. So, this W R and V is viewed as the relational structure. So, later it is viewed as some kind of directed graphs etcetera, to start with we need to begin with the possible worlds, I will be talking about the motion possible worlds in a separate lecture there I will be linking it with modal realism, but for time being possible worlds there are things that could have been different in things could have been different in different ways. So, these are similar to the straight description that are having mentioned by Rudolf Carnap, but Kripke used the metaphysical terminology of the seventeenth century philosopher Leibniz. Leibniz is Leibniz viewed that and argued that the world the god has created is considered to be one of the best possible worlds.

So, there are several worlds and this world is considered the world which we are inhabiting is considered to be the best possible world, that is last time viewed as possible world, but it is considered to be the best possible world and in that context he propose that the necessary truths are consider to be eternal truths. For example, necessary truths are like two plus two is equal to four, these are the truths which that are considered to be true in all possible worlds including the best possible world that we inhabit. So, this idea has come from like this motion of possible world, where he defined in sense necessity of p means something is necessarily true means it is true in all possible worlds and something is possibly true it means, it is true in only some possible worlds not only these possible worlds will as long as the world exist, but also they would have held if god had created the world according to a different plan.

(Refer Slide Time: 11:28)

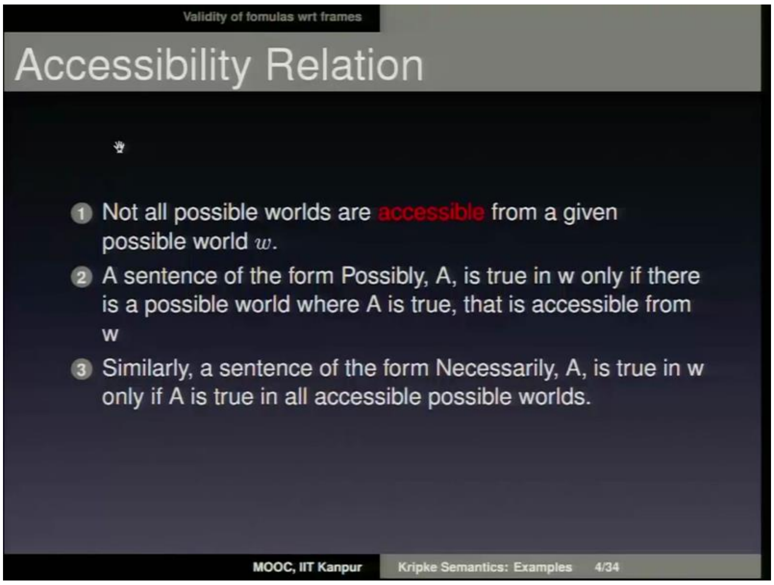


So, this is what is quoted from is what is mentioned in Leibniz which is quoted by the logician mates; so some of the properties of this possible worlds. So, it has more objects events and individuals than the real ones. So, we have to view this possible world. For example, let us take simple example suppose if we take a snap shot of this room that is considered to be the real world assuming that is the real world and then you alter it a little bit from that snap shot you move one object from that thing and the resulting one is going to be the possible world. It is not exactly same as a real world, but this is something different from the real one, but it is closer to the real world.

So, the possible world has more objects events and individual than the real one such as for example, cancer cure or a bridge between the earth and the moon or a third world war etcetera. They are different possibilities which it could it can happen or a 10 feet's tall person are is considered to be a mars expedition, it is possible that there will be mars expedition which is already the case and superman etcetera, where all these things are related to some objects.

It can even have less objects also like in this case events and individuals the less, less objects events and individuals than the real one such as pyramids no gulf war and no Shakespeare etcetera, it will be a very limited set of worlds it can go in either way the given anything you will have more objects than the actual thing and lesser objects than the actual thing. So, it differs with respect your properties for example, in this case Sydney opera house is red and I am a multi millionaire etcetera, or it differs with respect to relations in this example the great wall of china is located in the middle east. So, what have been talking about this is the relational structure consists of possible worlds.

(Refer Slide Time: 13:33)



Accessible relation are in the valuation function say let us talk little bit I will be talking more about this possible worlds little bit later may be in the next after few lectures. So, I will be talking about possible worlds and its relationship with modal realism.

So, accessible relation this is the not all possible worlds are accessible from a given possible world w in a sentence of the form let us say, possibility is considered to be true in a world w only. If there is at least one possible world which is accessible from the actual world to that particular kind of world if that is not accessible to that one then, it is considered to be false similarly a sentence of the form necessarily an is going to be true in a world w only if an is true in all accessible possible worlds suppose if you in your diagram it is like this. So, these are some of the worlds that we have W1 and W2 etcetera and W3.

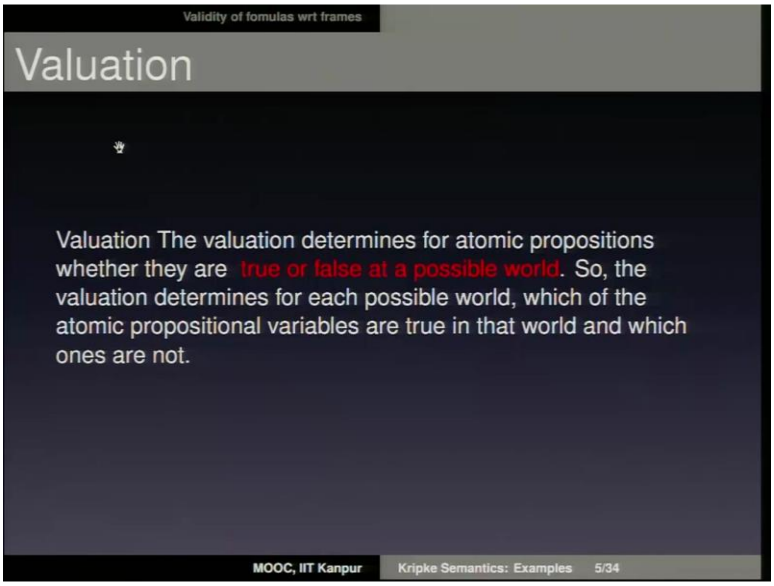
So, let us say you are talking about the truth value of some kind of modal sentences like this with respect to some kind of relational structure that is what is we are calling it as Kripke modal w r and v. So, without some set of worlds and accessible relation in valuation function you cannot talk about Kripke modal. So, now, necessity of p for example, p here and p here and then let us say possible p of p. So, let us say when necessity of p is going to be true in this world W1 any world W1. So, this you write it here like this necessity of p which is true in a world W1 necessity of p is true in a world W1 especially when whatever worlds this has access to. So, this is a world W1 is accessible to W2 this is what you write this is a way you write it and here W1 is also accessible to W3.

So, now in all the world this W1 has is accessible to in those worlds your p has to be true if that is the case the necessity of p holds. For example, if you have a situation where you have like this then in this world the world which is accessible from this is this one and this one in this world p is true, but here it is false in that case necessity of p does not hold whatever possibility of p possibility of p is defining in the sense that something is possible means that is one possible world it is true. So, in this case against in this one W1 is accessible to W2 and in W2 your p is true, but if you go to the other direction in this way W1 is accessible to W3, but p is false there although it is false there, but for possibility of p if at least one world exist where in that world your p is true then this false.

So, this is the way you need to view the truth value of a given modal logical formula with respect to a world you need to note that Kripke is talking about truth value of a given modal logical formula in different levels first he talks about truth with respect to the worlds and the second he talks about a frame a frame consist of suppose if you in this w r and v. If you take only w and r you take only set of possible worlds and accessibility relation and this is what is called as frame and then, if you take all the three things it is consider to be m modal and if it is it occurs in all kinds of modals which happens for a case of total list then something is to in all Kripke modals.

So, now what is accessibility relation a sentence of the form possibility of A is true in a world w only if there is one possible world that is one possible world where your A is true similarly a sentence of the form necessity of A is going to be true only, if A is true in all accessible possible worlds.

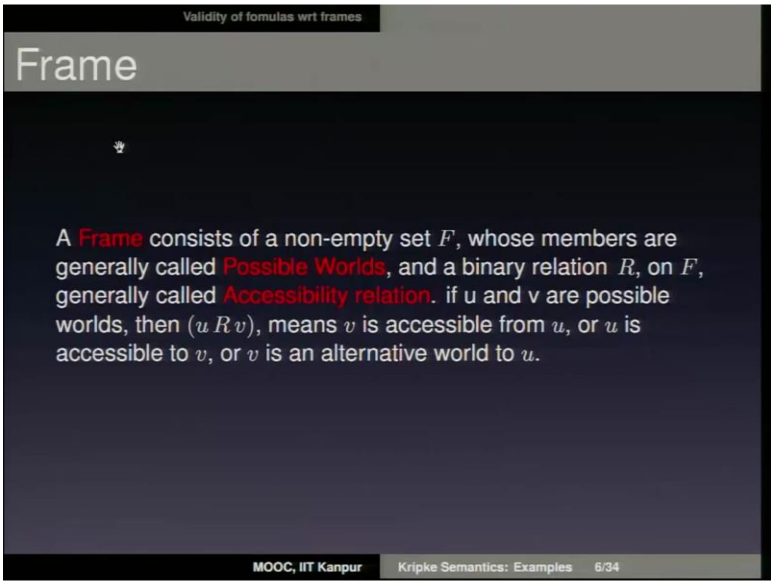
(Refer Slide Time: 18:03)



Now, what about the valuation function? So, valuation determines for atomic proposition that exist in a given modal logical formula, that they are either true or false at a given possible world. So, this is a valuation that you are given p is true with respect to world W2 and here we do not have anything. So, this may cannot be the case that this world may accessible to itself in that case your things will change the value of this things will change.

So, the valuation of p here in the world W3 is false. So, what all you require is that in order to talk about ultimately what we are trying to do is that in the case of synthetic tradition that is the second era of modal logic, people were continuously logicians were continuously trying to prove theorems and of course, all the theorems are considered to be valid formulas. So, Kripke talked about validity of a given modal logical formula in totally innovative way, that the innovative is this thing a truth with respect to possible world that is where the possible world semantics has come into existence. So, we will be talking about more examples little bit later.

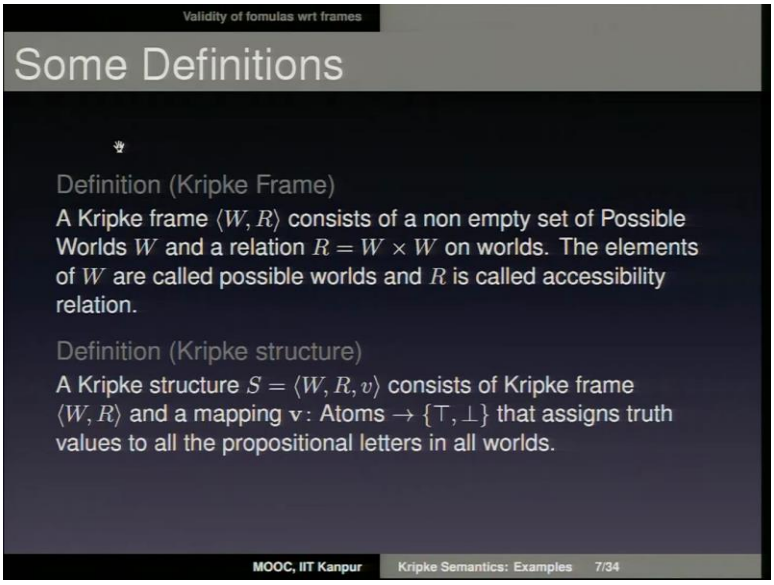
(Refer Slide Time: 19:32)



But there we let us talk more let us talk in some detail about some more details about Kripke modal. So, what is considered to be a frame with respect to Kripke modal a frame consists of set of possible worlds and accessibility relation whose members are generally called possible worlds w and a. Binary relation r which tells us which possible world is linked to what if u and v are said to be possible worlds u R v that is what is accessibility relation which tells us that u is accessible to v say if you right u first and then followed by at r and v this means this means u is accessible to v if for the case that v is accessible to u.

So; that means depending upon what logics we are talking about this accessibility relation changes. So, in the case of epistemic logic accessibility relation is viewed as some sort of indistinguishability relation, that we will not talk about in this class, but we have to view this accessibility relation in a slightly different way.

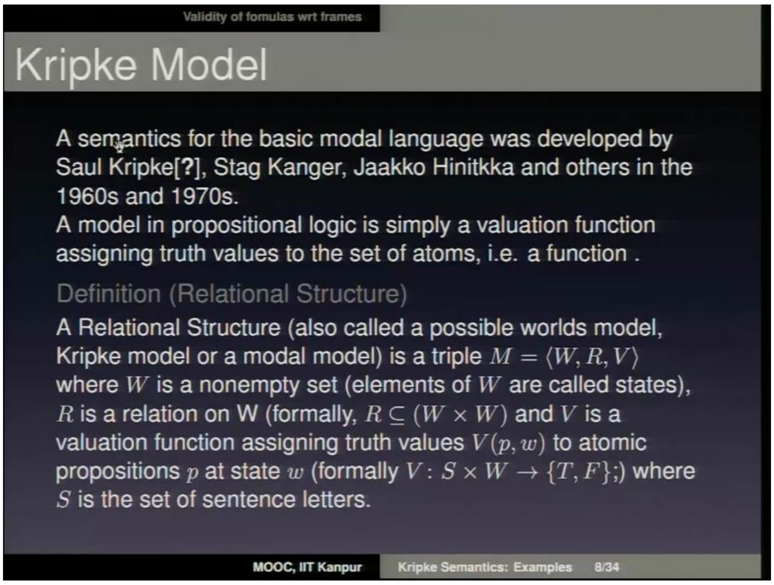
(Refer Slide Time: 20:29)



So, these are some of the standard definitions a Kripke frame, which is considered to be an order frame W R which consist of non empty set of worlds you can have empty set of worlds also, but in ideally speaking you should have at least one world which should be accessible to the actual world it happens only in the case of some kind of impossible worlds. Suppose if you say two plus two is equal to 5, you cannot come up with any possible world which is closer to 2 plus 2 is equal to 5, you cannot even imagine that world that world has to be either impossible or empty world.

So, always we take into consideration that a Kripke frame should consist of non empty set of possible worlds and you have an accessibility relation, which is considered to be trans product of various kinds of worlds that exist in our case and the elements of w are considerably possible worlds and r is consider to be accessibility relation that is what is Kripke frame Kripke structure or modal is W R and v, it is a triplet which consist of set of possible worlds accessibility relation and the valuation function. So, there is a mapping from atoms atomic sentences like p q s r s etcetera and it generates some kind of values either it is true or false; that means, you have to assign some kind of truth values to all propositional letters in all kinds of worlds.

(Refer Slide Time: 22:00)

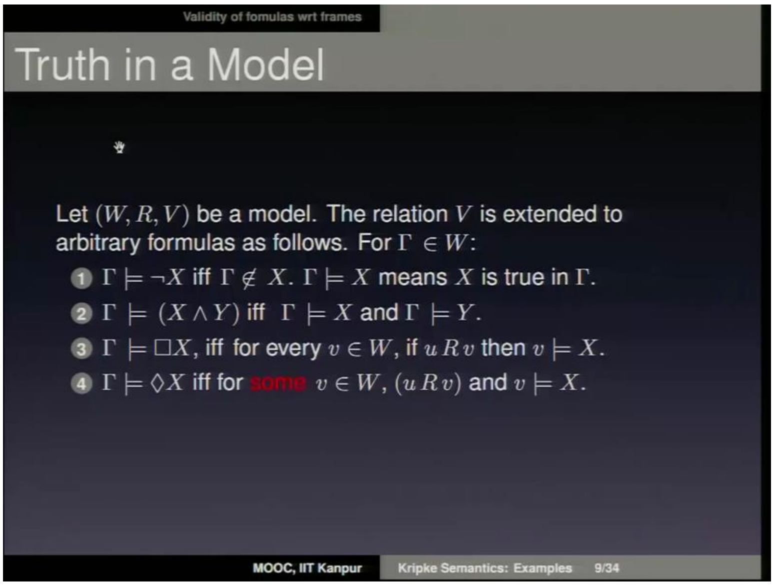


Kripke modal the semantics for the basic modal logic was developed initially developed by Kripke and there are several other logicians Kanger and Jaakko Hinitkka and all others during the time that particular period of time 1960's to 1970's who has come up with this idea first we do not know where even says that this is their own idea.

So, it is debatable kind of thing whether Kripke is come up with this first or Kanger or Hinitkka the Parllely all of them are working on similar kind of idea. So, if you see the history of logic into consideration it appears to be the case. So, a modal Kripke modal is considered to be in propositional logic is simply a valuation function assigning truth values to the set of atoms. So, now, you can talk about Kripke modals even in the context of propositional logic also. So, a relational structure that is what is it was given utmost importance till 1972 etcetera. Excessive emphasis on this relational structure, the relational structure again is triplet it consist of non empty set of world’s W accessibility relation R and V. V is considered to be valuation function assigning truth values to the atomic sentences that exist in your given formula.

Suppose if I write V P W; that means, p is true in a world w that is considered to be true and formally valuation function is defined as prosperity of s and w s into w, it has to take some kind of value T r F. So, in that atomic sentence p it assigns some kind of values to the atomic sentence that exists in your modal logical formula, but thing has to be either true or false. So, what all you need to know the validity of a given formula balls down to a simple Kripke has come up with the simple diagrammatic representation with which you can express the validity of a given modal, logical formula what all you need is that non empty set of worlds and you have accessibility relation it tells us how this possible worlds are connected to each other and then, you should you should know when where a given atomic formula that exist in your modal logical formula is true in which world that is true.

(Refer Slide Time: 24:32)



So, truth in a modal truth in a modal it is like this, if you take W R V to be a Kripke structure, relation structure modal this relation V is extended to some arbitrary kind of formulas and this is like this suppose if we take set of worlds gamma set of worlds w which implies W1 W2 three etcetera. The set which consists they are in this set gamma then suppose we say X is not X is not a logical not X is logical consequence of gamma which is written as gamma does not belong to X means X is true in gamma X and Y is a logical consequence of gamma only when, both are considered to be logical consequences then they are taken individually. X has to be logical consequence of gamma, even Y is also logical consequence of gamma these are all things which are already there in the case of propositional logic and in our context, we need to come up with the meaning of this modal logical formula here we need to note that meaning of a formula means giving truth conditions. So, this is a truth condition that is why we have calling it semantics.

So, necessity of X is considered to be logical consequence in your formal system or logical system gamma which it includes set of formulas, if and only if for every v belongs to a set of worlds w and you have an accessibility relation u is accessible to v and in that world v your X has to be true say this should happen for a v any indexical letter v. So, any world which is accessible from the actual world in all those worlds your formula has to be that X has to be true and possibility of X is defined in a sense a possibility of X is going to be true if and only if for some v belongs to w and this u has to be accessible to v and in that possible world v you are X has to be true.

So, in the next class we will be talking about some more we will be talking about this Kripke modal with some more examples. So, the example will simplify our understanding of or the usage of Kripke modals. So, the idea here is that this proving theorems etcetera are showing that a given formula is valid, etcetera are all simplified I mean much better in the Kripke semantics and then we can simply talk you can draw the diagram and then by seeing the diagram, itself you can make out whether a given formula holds in the given world or it holds in a given frame or it holds in the Kripke modal. So, we will continue with more examples on Kripke semantics in the next class thanks.

Thank you very much.