Mathematics for Economics – I
Difference Equations
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Lecture 32: Multiplier-acceleration

Hello and welcome to another lecture of this course mathematics for economics part one. So, over the last few lectures we have been discussing this particular module of difference equations and how we can use difference equations in the analysis of economics. Now at present we are talking about second order difference equations.

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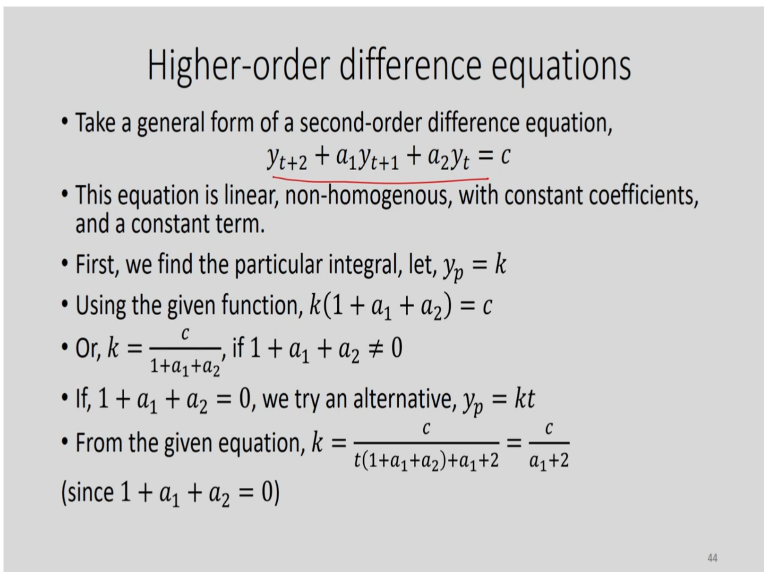

So, as you can see on your screen, this is the general form of a second order difference equation which is linear, non-homogeneous, with constant coefficients and on the right-hand side you have a constant term. So, this is $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$.

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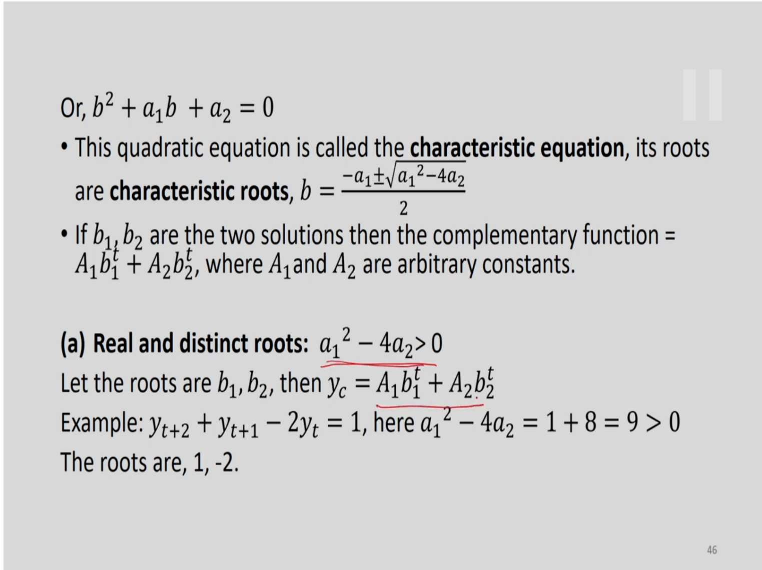

And we have seen how to solve this. We have seen that there are three subcases if we look at the complementary function then if we want to find the complementary function then we will get a characteristic equation and from that characteristic equation we will get characteristic roots. Now, these roots can be of three natures. And the first one is real and distinct roots. Right under this condition, here the complementary function will take this form.

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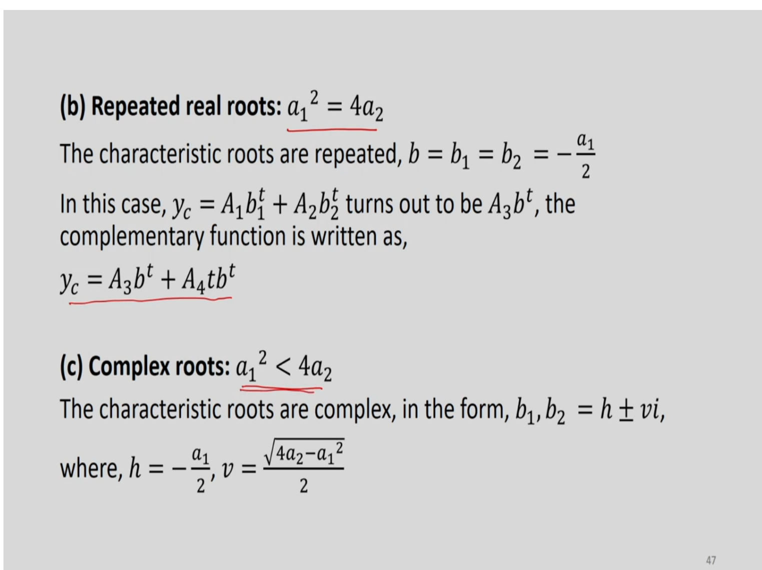

Secondly, it is possible that you have this condition which has been satisfied. In that case the complementary function will have this form. And finally, you can have complex roots where this condition is satisfied $a_1^2 < 4a_2$.

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- Thus, $y_c = A_1 b_1^t + A_2 b_2^t = y_c = A_1(h + vi)^t + A_2(h - vi)^t$
- By De Moivre's theorem, $(h \pm vi)^t = R^t(cos\theta t \pm isin\theta t)$

where $R = \sqrt{h^2 + v^2} = \sqrt{a_2}$, &

$cos\theta = \frac{h}{R} = \frac{-a_1}{2\sqrt{a_2}}$, $sin\theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}}$,

- Thus, $y_c = A_1 R^t(cos\theta t + isin\theta t) + A_2 R^t(cos\theta t - isin\theta t)$

$= R^t(A_3 cos\theta t + A_4 sin\theta t)$,

where $A_3 = A_1 + A_2$, $A_4 = (A_1 - A_2)i$

48

In this case the complementary function will be given by this; $y_c = R^t(A_3 cos\theta t + A_4 sin\theta t)$, where θ and R are defined by these two relations.

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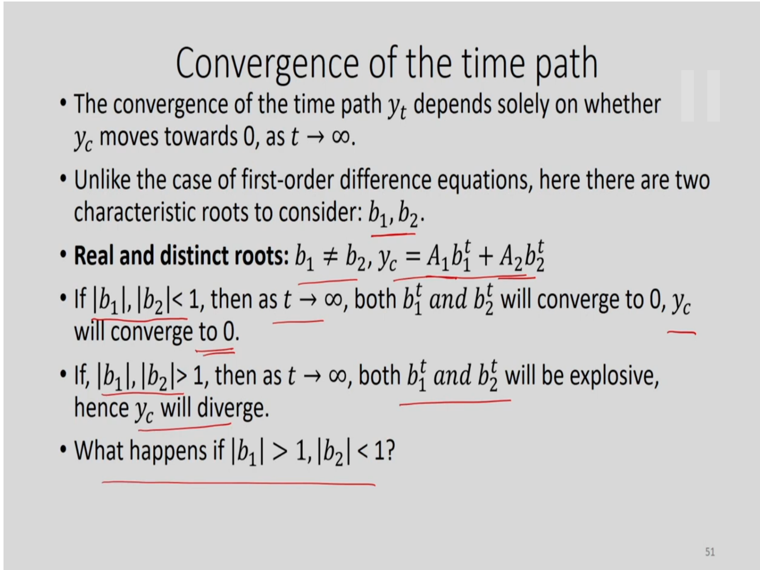

Alright now what about the conversions, under what conditions can we say that this y_t is going to follow a stable path and it will move towards y_t as t goes to infinity. Now, here we need to know that we have two characteristic roots here, b_1 *and* b_2. If in the case of the first **order** difference equation we had just one root and convergence was attained when the absolute value of the root is less than one. But here there are two roots.

So, how to deal with that? So first let us take the case of real and distinct roots, where $b_1 \neq b_2$. We know this is the complementary function. Now if the absolute value of both are less than one then it is clear that as t goes to infinity, both these terms will tend towards 0. Both these terms, $A_1 b_1^t$ and $A_2 b_2^t$, both of them will converge towards 0. That means y_c will converge towards 0.

So, this is the, this is just one case. What happens if in the opposite case where the modulus of both of these roots is greater than one. In that case both b_1^t and b_2^t will be explosive. And in that case y_c will diverge. It can be like, you know, oscillatory divergence or it could be monotonic divergence, whatever it is but there will be divergence.

But what about the intermediate case where suppose $|b_1| > 1$ and $|b_2| < 1$. So, how to conclude that case?

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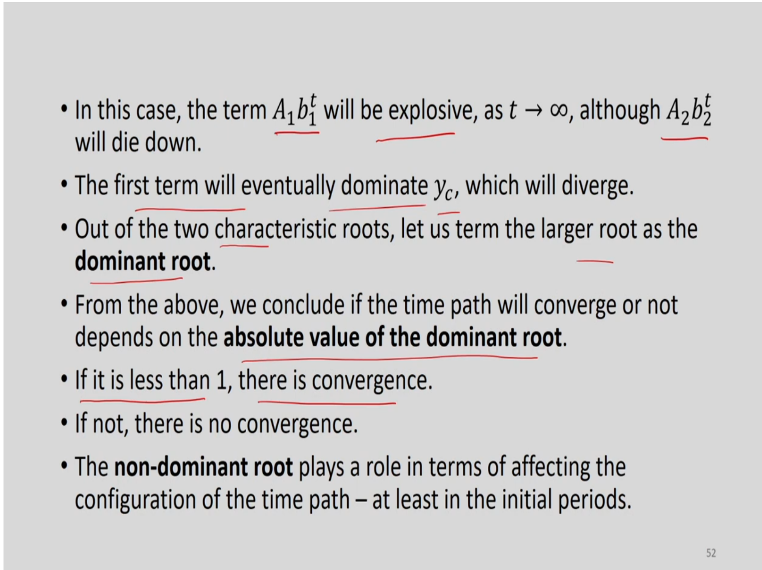

In this case obviously where you are talking about $A_1 b_1^t$, that will be explosive because the $\left|b_1\right| > 1$. But the second term that is $A_2 b_2^t$ will die down because the $\left|b_2\right| < 1$. And, so, what happens as the summation, as the net result of these two. The first term will eventually dominate y_c because you know one part is going to 0, that is fine, but the other part is explosive.

So, therefore the summation will also be explosive so y_c will diverge. Now out of these two characteristic roots let us call the larger root as the dominant root. So, from the above what we can conclude is this, if the time path will diverge or converge it depends on the absolute value of the dominant root. Dominant root means the root which is larger in absolute terms.

If the larger root is greater than one then we have a problem, there is divergence. If it is less than one there is convergence. If it is equal to one then we know there is not going to be any convergence. What about the role of the non-dominant root? The non-dominant root that is the smaller root in an absolute value plays a role in terms of affecting the configuration of the time path at least in the initial periods.

Because you know it has a kind of additive influence on y_c. Maybe it will not be able to affect the nature of y_c as t goes to infinity because that is going to be determined by the dominant root but this non-dominant root affects the configuration of the time path.

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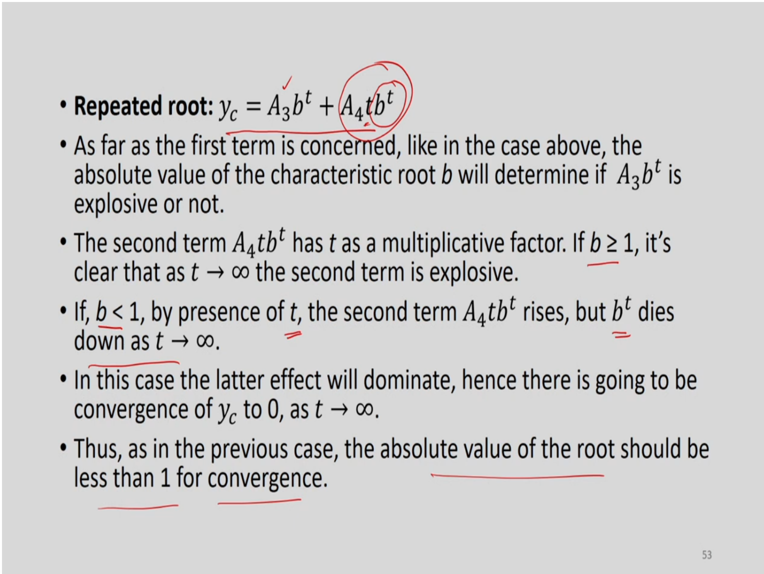

What about the stability in case of repeated roots? This was the case of repeated roots, the complementary function. Now, as far as the first term is concerned, this term. Like in the case above the absolute value of the characteristic root b will determine if $A_3 b^t$ is explosive or not. The second term that is $A_4 t b^t$ **has t as** a multiplicative factor if $b \geq 1$, it is clear that as t goes to infinity the second term is explosive.

So, if $b \geq 1$ then this term will either explode or it will remain constant. But as t is rising so the entered term is definitely going to be explosive. What happens if $b < 1$. If $b < 1$ by the **presence of t**, the second term $A_4 t b^t$ rises but b^t dies down as t goes to infinity.

It should have been written as $|b| < 1$. If the $|b| < 1$ then b^t dies down. But this is being multiplied by t and t is rising. So, in this case what will happen to the **entire term**? In this case the latter effect will dominate, that is b^t. Hence there is going to be convergence of y_c to 0, as t goes to infinity.

So, one way to understand this is that b^t, this goes t to 0, as t goes to infinity. So that pulls down the **entire** term, that is $A_4 t b^t$. Thus, as in the previous case the absolute value of the root should be less than 1 for convergence. Previous case means the case where we talked about distinct and real roots that was the previous case.

So, there we saw that it is the absolute value of the dominant root that determines the characteristic of the time path. Here also the same thing is being concluded. It matters as to what is the absolute value of b. If it is less than 1 then we have convergence.

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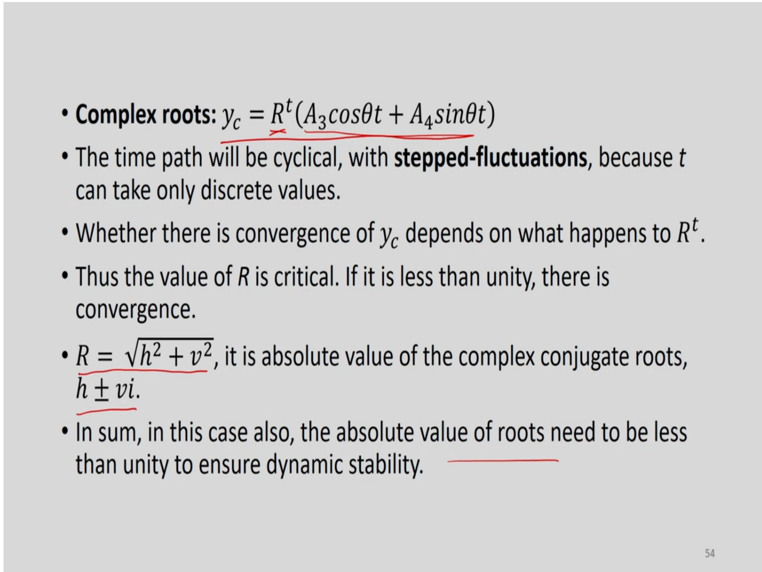

What about the complex roots? So, in the complex root we have this as the complementary function. the time path will be cyclical, we have noticed this before, with stepped-fluctuations because t can take only discrete values. **Whether** there is convergence of y_c, depends on what happens to R^t. So, this within the brackets term is just fluctuating within certain limits.

So, the convergence or divergence is purely determined by R^t. Thus, the value of R, that is R is critical if it is less than unity there is convergence. But what is R. R is given by this, $R = \sqrt{h^2 + v^2}$, it is the absolute value of complex conjugate roots, $h \pm vi$. So, it is the absolute value of the complex conjugate roots.

So, in some, in this case also the absolute value of the roots need to be less than unity to ensure dynamic stability. So, there is a common thread. Therefore, in all these three results, that the absolute value of the roots has to be less than 1.

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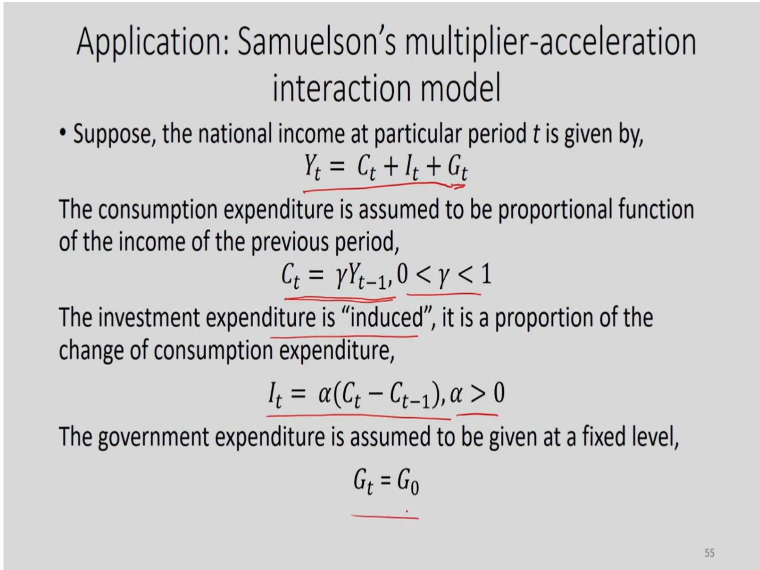

Now we talk about one important application of the second order difference equation. And this is the multiplier acceleration interaction model, proposed by the famous economist Paul Samuelson. So, some of these things that we are going to see have been introduced earlier. So, we start with the national income identity. The national income at a particular period t is given by $Y_t = C_t + I_t + G_t$.

That is the national income is the summation of consumption expenditure, investment expenditure and **Government** expenditure. So, we are assuming a closed economy, there is no export and import. Now, the consumption expenditure in this model is assumed to be proportional to the income of the previous period. That is the national income of the previous period.

So, in terms of mathematics $C_t = \gamma Y_{t-1}$, where this is important, gamma lies between 0 and 1, $0 < \gamma < 1$. So what is γ? γ is the marginal propensity to consume, but we have to be careful here because we are talking about the dependence of the consumption expenditure on the income of the previous period.

Secondly, what about the investment expenditure, what is the form of the investment expenditure? We are assuming that the investment expenditure is induced. It is a proportion of the change in the consumption expenditure. So, this is the form that it takes. It is $I_t = \alpha(C_t - C_{t-1})$, where alpha is greater than 0, $\alpha > 0$.

So, what is the justification for this? This is something new. I_t that is investment in a particular period, it depends on the change in the consumption expenditure. $C_t - C_{t-1}$, that is the change in the consumption expenditure. Now, what is the rationale for this? The rationale could be the following: the people who do the investment, that is let us suppose the producers, they look at how much consumption expenditure has been changing over the last period.

So, that tells the investors as to whether the people who are doing the consumption are feeling confident about the economy. So, if the consumption expenditure is rising then the investors feel that it is better to invest more money to build capacity so as to cater to the rising expenditure by the consumers. So, that is the logic.

Now let us talk about the last element in the aggregate expenditure which is the government expenditure. The government expenditure is assumed to be given a fixed level. Let us suppose $G_t = G_0$. So, this is just to make the model simple. We are assuming that government expenditure is fixed.

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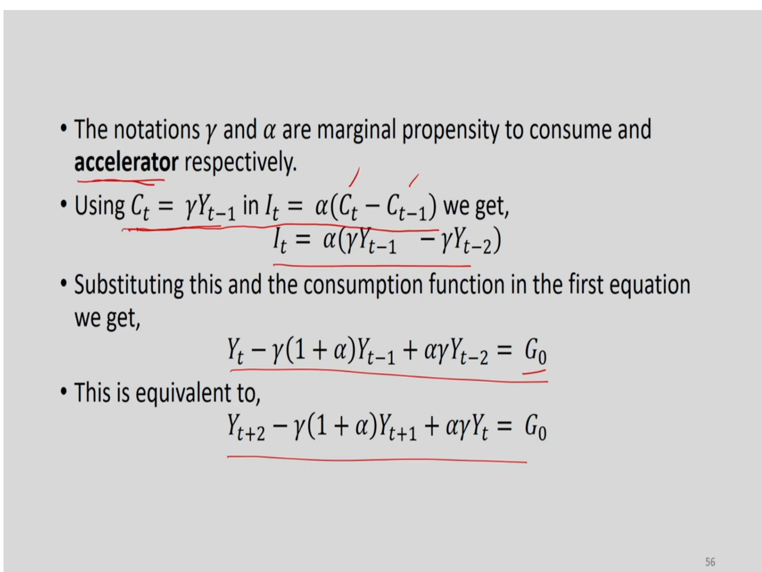

The notations γ and α, they are called marginal propensity to consume and accelerator. This is called accelerator alpha. α is positive and the logic I have just described to you that as there is more consumption expenditure, more investment will be forthcoming. So, that is the reason why $\alpha > 0$.

So, this is the reason why this model is called multiplier-acceleration interaction model. We have already introduced the consumption marginal propensity to consume, that is γ. That we shall see is an element of the multiplier. And we have introduced the accelerator. Subsequently we shall see that these two will interact to throw up an interesting configuration of the time path of income. Now, we are going to use these two relations.

$C_t = \gamma Y_{t-1}$ and $I_t = \alpha(C_t - C_{t-1})$. We can combine these two things together and we are going to get $I_t = \alpha(\gamma Y_{t-1} - \gamma Y_{t-2})$. Basically, we have just used this and that is the consumption function of the income of the previous period. We have substituted that here and here.

So, we have got it and now what we are going to do is I am going to substitute it and the consumption function in the national income identity which is this. And we are going to get this. You can see it is only a function of $Y_t s$ alone and there are three $Y_t s$ here; Y_t, Y_{t-1} and Y_{t-2}. And so, this is a difference equation of the second order. G_0 is constant.

I can change the time period a bit. I can write the same relation as in terms of Y_{t+2}, Y_{t+1} and Y_t. So, this is a second order difference equation, we know how to solve this.

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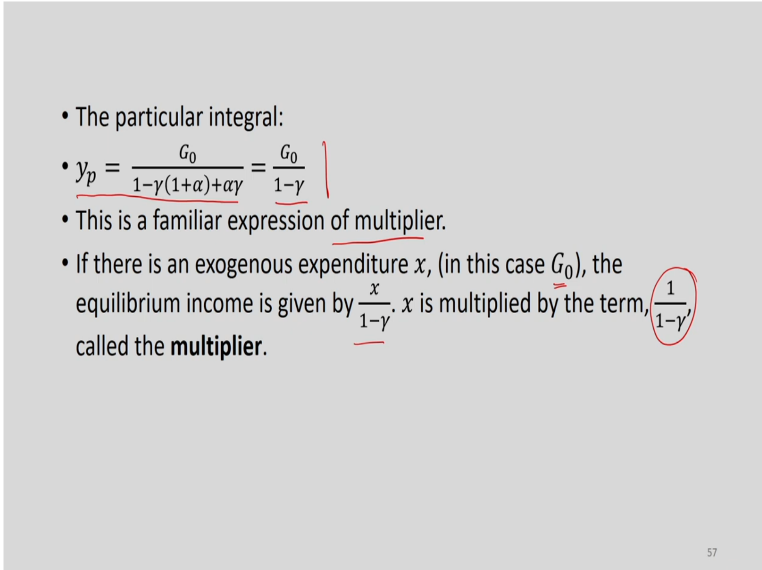

First, we look for the particular integral. So, we, I am just keeping some steps. So, Y_p the particular integral is found to be $\frac{G_0}{1-\gamma}$. This is a familiar expression of multiplier. If there is an exogenous expenditure x, in this case that x is G_0. The equilibrium income is given by $\frac{x}{1-\gamma}$.

Here the same thing is happening. $\frac{G_0}{1-\gamma}$. x is multiplied by this term and this term is called the multiplier, $\frac{1}{1-\gamma}$. And what is γ? marginal propensity to consume. So, that is what I said that MPC is an element of the multiplier.

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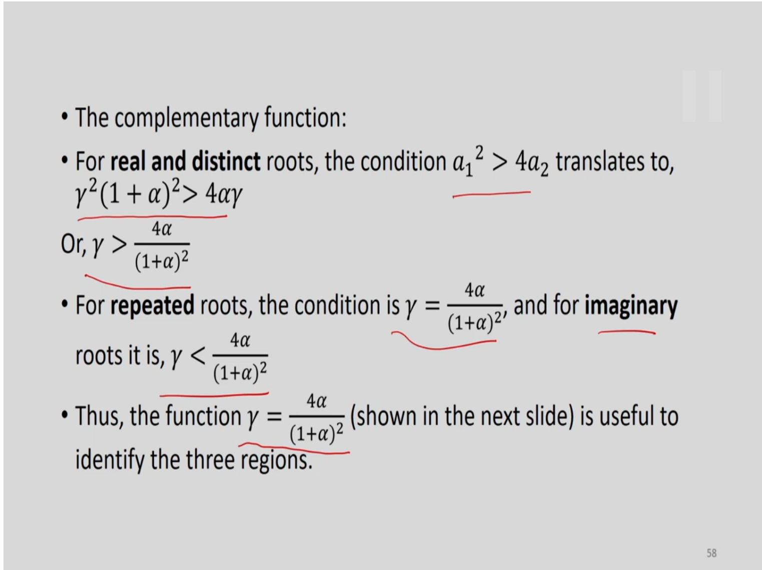

Now, let us concentrate on the complementary function. Now, as we know there could be three subcases. In the first case the roots could be distinct and real. This condition is $a_1^2 > 4a_2$. In this case it is translated into this relation, $\gamma^2(1+\alpha)^2 > 4\alpha\gamma$ and if I take the γ on the left-hand side, it becomes $\gamma > \frac{4\alpha}{(1+\alpha)^2}$.

This was the case of real and distinct roots. What about the repeated roots? That relation is this, $\gamma = \frac{4\alpha}{(1+\alpha)^2}$. And for imaginary roots the inequality sign goes the other way. $\gamma < \frac{4\alpha}{(1+\alpha)^2}$. So, actually this function, $\gamma = \frac{4\alpha}{(1+\alpha)^2}$ is useful to identify these three regions. One is the case of real distinct roots, second is the case of repeated roots and the third is the case of imaginary roots.

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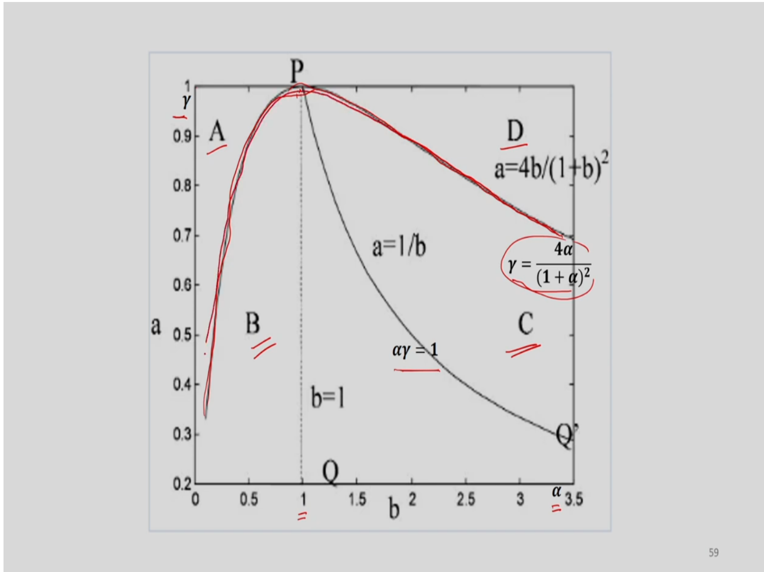

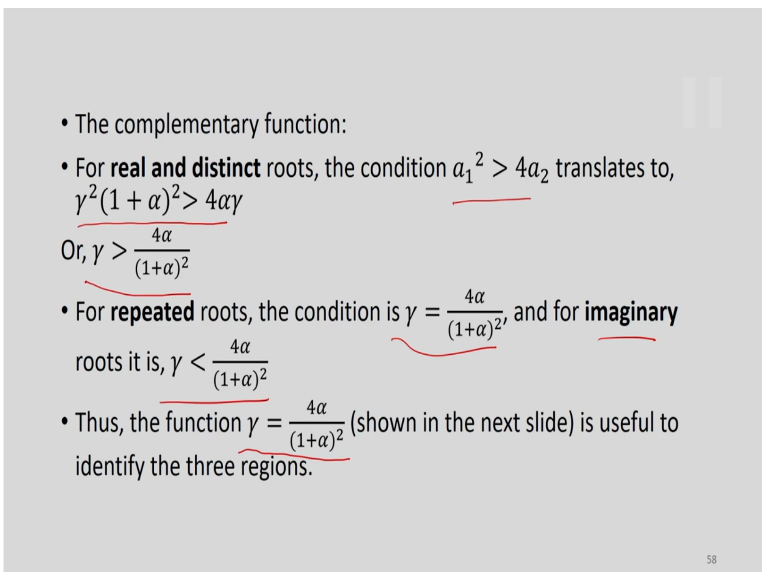

This diagram actually shows this function. So, which is this function? This line. So, it is concave to the origin line as you can see. It is going up reaching a maximum at $\alpha = 1$. Along the x axis we are representing the alpha and along the y axis I am representing the γ. γ, remember cannot **exceed** 1 so there is an upper limit here because gamma is a marginal propensity to consume.

α, however, does not have any upper bounds so it can go along the x axis. So, this line is a concave function, it is reaching a maximum at $\alpha = 1$. At that particular value, the value of

$\gamma = 1$, so this point P. And then it is coming down. As we have just seen that there are three regions here, one is above this function, so that is A and D, A and D.

So, at A and D you have $\gamma > \frac{4\alpha}{(1+\alpha)^2}$. So, this is the case of real and distinct roots, this relation. On the line, so if we are on the line then these two things are just equal. So, this is the case of repeated roots. And if we are talking about these two regions, B and C, alright, then actually $\gamma < \frac{4\alpha}{(1+\alpha)^2}$.

So, this is the case of imaginary roots. So, actually these four roots are represented in this diagram in a very neat manner. For the time being ignore this line. There is another line that I have drawn which is $\alpha\gamma = 1$. Now, the importance of that line will be made clear subsequently. Right now, we just note that this concave function which I have denoted here by the red line is the line which separates the three sub-cases of roots.

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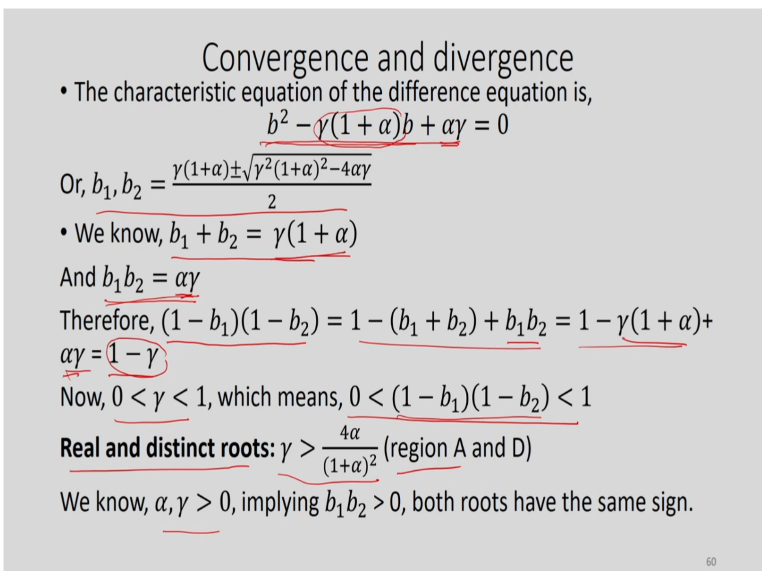

Alright, now the question is when do we have convergence, when do we have divergence and what is the nature of convergence if there is convergence. These are the important questions that we are going to deal with. First, we look at the characteristic equation. We are trying to find the complementary functions, for that we have to look at the characteristic equation. This is the characteristic equation, from this we get, these are the roots b_1 *and* b_2.

Now, what we know is that if we have a quadratic equation like this and b_1 *and* b_2 are the roots, then the summation of the roots is this term. The coefficient of the b, minus of the coefficient of the b, which is $\gamma(1 + \alpha)$. So, this is high school mathematics and the product of the roots is this term, $\alpha\gamma$. That we know. Furthermore, let us look at his expression, $(1 - b_1)(1 - b_2)$, where b_1 *and* b_2 are the roots.

Now, this simplifies into this, $(1 - (b_1 + b_2) + b_1 b_2$ and this can be written as this. Because $b_1 + b_2 = \gamma(1 + \alpha)$ that we substitute here and $b_1 b_2 = \alpha\gamma$ that we substitute here and we simplify and it turns out to be $(1 - \gamma)$. Now, $0 < \gamma < 1$, the marginal propensity to consume, so this thing also will lie between 0 and 1, $0 < 1 - \gamma < 1$.

And therefore, this will be satisfied that $0 < (1 - b_1)(1 - b_2) < 1$ because $0 < 1 - \gamma < 1$. So, these are the certain things that we will be using now. Now let us take the case of real and distinct roots. Now in this case we know this has to be satisfied. $\gamma > \frac{4\alpha}{(1+\alpha)^2}$. So, we are talking about the regions A and D, A and D.

Now we know that α and γ both are positive. Now, that basically means that this is positive; $b_1 b_2 > 0$ because $b_1 b_2 = \alpha\gamma$. Now if $b_1 b_2 > 0$ then there are only two possibilities, either both of them are positive or both of them are negative. Both the roots have the same sign.

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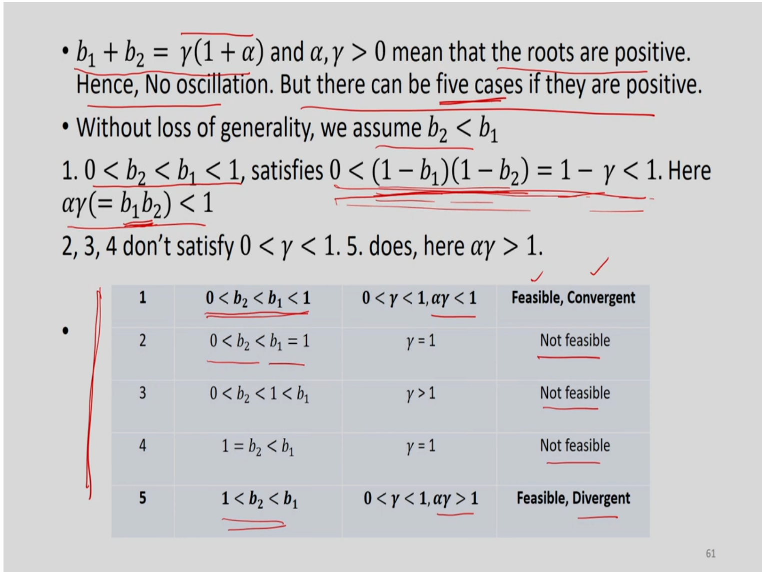

Now, can both of them be negative? We know also that the summation of the roots is given by this,$\gamma(1 + \alpha)$ and which is positive? So, the summation of the roots is positive, therefore the roots individually cannot be negative, which means there is only one possibility that the roots are positive. If the roots are positive then we know there is not going to be any oscillation.

So, roots are positive but even if the roots are positive actually there are five cases here. So, these are the cases that have been noted down. Let us take one or two to get the feel of it. So, here we are taking the roots to be, both the roots to be lying between 0 and 1. We are assuming without loss of generality that suppose b_1 is the larger root. The roots are different, so **one of the roots** is higher than the other toots. So, let us suppose b_1 is that bigger root.

So, here both of these roots are lying between 0 and 1. Now, the question is, is that feasible. That is going to be an important point. Now, if both of them lie between 0 and 1, then what is happening to this term? That is $(1 - b_1)(1 - b_2)$. That is, we know it is going to be $(1 - b_1)(1 - b_2) = 1 - \gamma$. That is known to us.

And if $0 < b_2 < b_1 1$, then $(1 - b_1)$ is going to be negative. $(1 - b_2)$ is also going to be negative. Now, the product of these two, it is possible that it is lying between 0 and 1. And which will satisfy this condition. So, this is feasible. And furthermore, we note that if this condition is satisfied then the product of these two toots is going to be less than 1 and the product of these two roots is known to be αγ.

So, here $\alpha\gamma < 1$. Here what is the time path? It is going to be convergent because both the roots are $0 < b_2 < b_1 1$. So, it is convergent and obviously it is going to be monotonic. So, this is possible. Whatever the second case, the second case is the larger root is equal to 1.

Now that is not possible the reason being that if $b_1 = 1$ then $(1 - b_1) = 0$. So, this condition is no longer satisfied. The product has to be greater than 0. But here the product is turning out to be 0. So, this is not feasible. Similarly, this is not feasible, this is the case where the larger root is greater than 1 and the smaller root is less than 1.

So that is not feasible. If the larger root is greater than 1 and the smaller root is equal to1, that is also not possible. And the last case is however feasible, where both the roots are greater than 1. Here let us suppose the roots are 1.5 and let us say 1.4. In this case 1 - 1.5 is something negative and here also there is something negative. But the product of these two might turn out to be satisfying this condition, 0 to 1, they lie between 0 to 1.

So, that is feasible. However, in this case $\alpha\gamma > 1$ because $b_1 b_2 > 1$. So, this is the case where there is going to be divergence because both the roots are greater than 1. So here from the first case of real and distinct roots we are getting two possible cases. In one case there is convergence and in the other case there is divergence.

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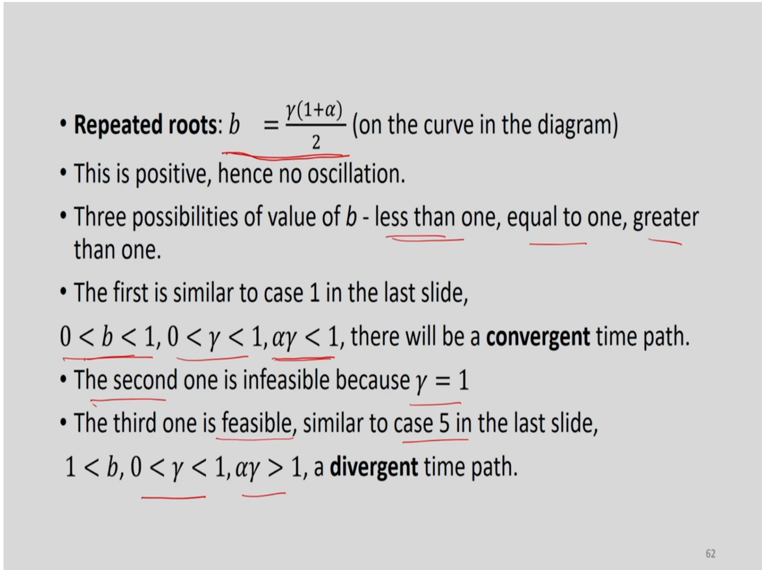

Now we come to the repeated roots case. Now, in the repeated roots, what is the root? Root is equal to $b = \frac{\gamma(1+\alpha)}{2}$. And that basically means that we are on the curve. We are neither above the curve or below the curve. And the root is positive so there is no oscillation. Now there are three possible values of the root however. Less than one, there will be convergence in this case equal to 1 and greater than 1.

The first case of less than 1 is similar to case one in the last slide, $0 < b < 1$. In this case obviously, $0 < \gamma < 1$. So, here this is the case, $(1 - b)(1 - b)$. Since $0 < b < 1$. So, $0 < (1 - b)^2 < 1$. And in this case the product of the roots will be less than 1. So, this is the case of convergence because $b < 1$.

The second case will be infeasible because that case γ will turn out to be 1, which is not possible. The third case is however feasible and this is similar to case five in the last slide, here $b > 1$. So, this is the case and we have seen that it is feasible because in this case $0 < \gamma < 1$.

In this case $\alpha\gamma > 1$ and that is a divergence time path. Now, notice one pattern that whether we are going to have convergence or divergence that is getting reflected as to what is happening to $\alpha\gamma$. If $\alpha\gamma < 1$, it is convergence and if it is greater than 1 there is divergence.

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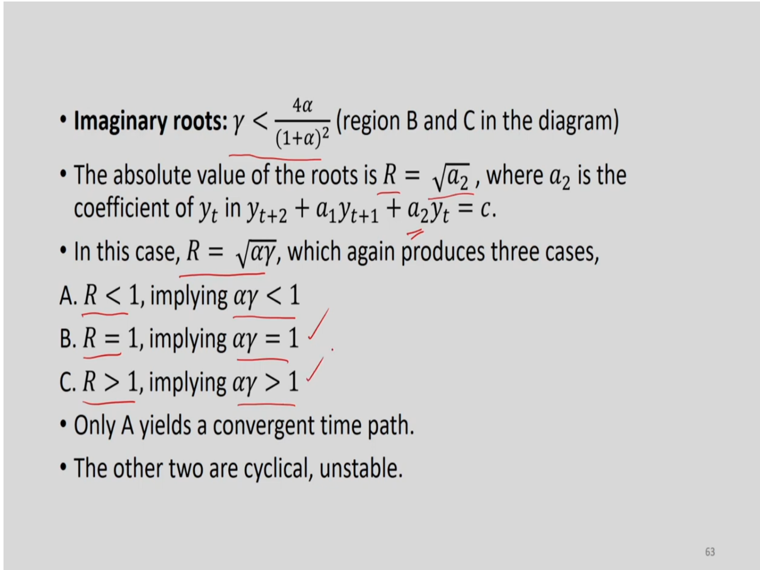

Okay, the third case of this; $\gamma < \frac{4\alpha}{(1+\alpha)^2}$. So, this is the region B and C in the diagram. So, this B and this C, this region and this region, this is the imaginary roots case. Now, as we know the absolute value of the roots, that is important. R has to be less than 1 for convergence. Here $R = \sqrt{a}$, where a is this coefficient. So, in our case $a = \alpha\gamma$.

So, $R = \sqrt{\alpha\gamma}$ and here again like before there are going to be three cases. $R < 1$ means convergence in this case. $\alpha\gamma < 1$. $R = 1$, means $\alpha\gamma = 1$ and $R > 1$ means $\alpha\gamma > 1$.

Now out of these three only the first one will be convergence, because in that case $R < 1$. The other cases will be cyclical movements obviously. But here there will be uniform oscillation and here there will be oscillation with explosive behaviour. And generally, these are unstable cases.

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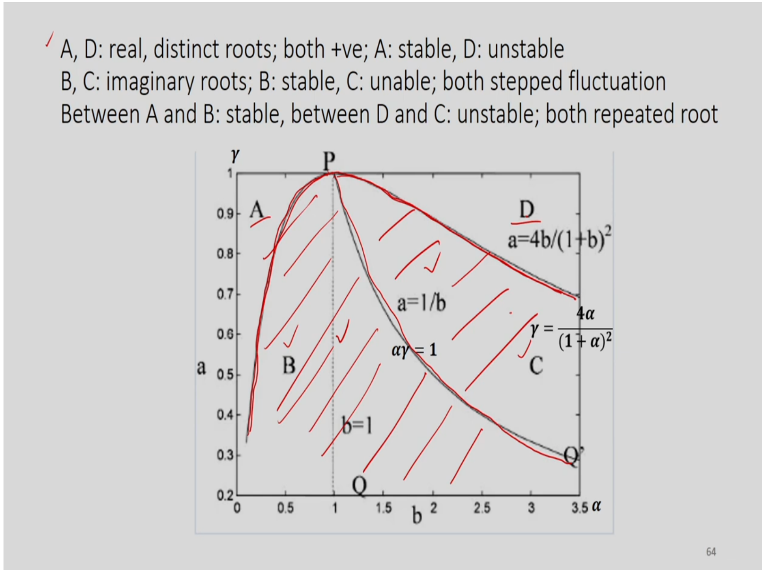

So, this diagram actually sums it up. We have talked about this figure before, this line before. What is the meaning of this line? Now we talk about his line. This line as we have noted it basically demarcates the stable cases from the unstable case. So, if we are below this line, you have all the stable cases of convergence. If we are above the line there are cases of explosive behaviour. On the line there is neither explosion nor convergence.

Basically, uniform oscillation or in the case of repeated roots it is possible at this point P there is no oscillation but it is not converging to the Y_p. So, that is what I have written here in summary, A and D are the case of real and distinct roots. Both the roots are positive. What we can say here is that A is stable and D is unstable. Second case B and C, so B and c are imaginary roots that we have seen. B is stable, C is unstable both step fluctuation, both A and B stable.

So, this and this, both of them are below the $\alpha\gamma = 1$line. Between D and C, that is between D and C you have unstable cases. So, this line, unstable but repeated roots. And here between A and B, this line you have is stable. So that basically sums it up. Now what is the economic import of all this?

The economic import of all this is that in this model what we have seen is that through the applications of second order difference equations we can actually work with a very simple Keyensian model and we can find different cases of whether there is going to be fluctuation, whether there is going to be no fluctuation and you know whether there is going to be stability or not. So, all these things can be found out in a very simple framework.

In real life the world actually sees the fluctuations or what are known as trade cycles in the national income. So, it is likely that since we are seeing actually there are fluctuations so we are basically talking about these two cases, B and C, where there is fluctuation. In C there is instability **but in B** there is no instability there is convergence.

So, it is possible that the real world is closer to B and C, where there is actually cyclical behaviour of the national income. But that is for the empirical researchers to ascertain. At this point we are through with the course of what I intended to cover in this course. Before I finish let us try to do a brief roundup of what we have covered in this course.

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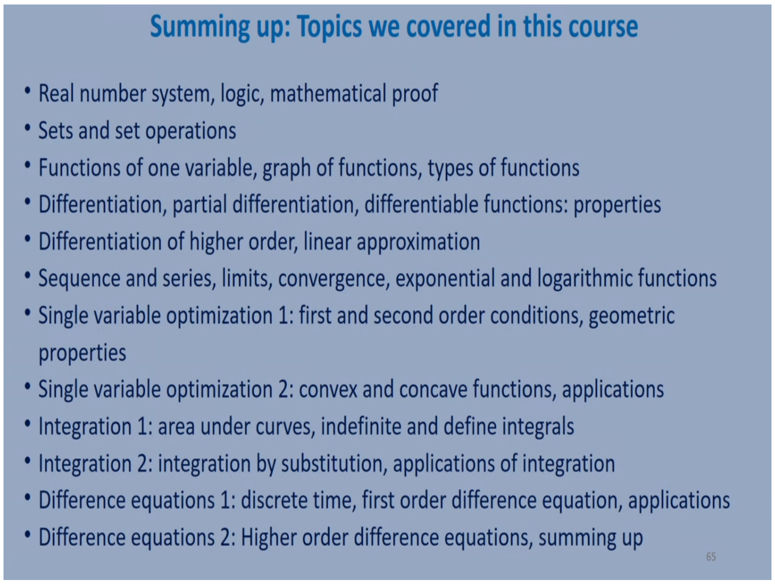

So, this is the list of topics that we have covered in this course, you can see on your screen. So, we started this course with some certain basic ideas of the **starting point of** mathematics such as real number system, logic, and mathematical proof. We talked about what is deductive logic, what is inductive logic and mathematical proof, and different ways to prove certain theorems.

Then we talked about sets, the basic ideas of sets and sets operations. Then we went into the discussion of functions of one variable and different graphs of the functions. We talked about different types of the function. You know quadratic, cubic functions, polynomials, what are polynomials. All these things were discussed in this particular module, the third module.

And then we talked about differentiation, the basic idea of differentiation. Then what are differentiable functions, properties of differentiation and importantly what is partial differentiation.

In the second part of this course if there is one mathematics for economics part two, this partial differentiation will be used a lot because we are going to be talking about functions of multiple variables where partial differentiation will be very much relevant. In the fifth module we talked about differentiation of higher order. For example, you can take the second derivative or the third derivative and what does it have to do with the nature of the graph of the function.

Those were the things that are important in this discussion and linear approximation. So, if you have a sort of complicated function which is not linear then actually you can imagine that function to be linear in the immediate neighbourhood of a point. That linearization actually makes our task quite easy at times. We also talked about sequences and series.

The idea of limits, the idea of limits was introduced in the case of differentiation itself. But here in this module, that is the sixth module, **we delve** deeper into the idea of limits and also differentiability. I talked about continuity at greater length in this particular module. And this is very useful because it gets the basic ideas clear. We talked about convergence of the series and very importantly exponential and logarithmic functions.

Exponential functions or logarithmic functions are used extensively in economics and finance because we are often talking about growth overtime. For example, you have your money in the bank and which is giving you some rate of interest over a period of time. So, on its own the money will keep on growing. If it is growing then how do you find out what is going to be the amount of money after a point of time.

Or you can talk about the present value. What is the present value of certain money which will be obtained in future. So, those things are very important when people do their calculations in finance. Then we went to a very important module, actually two modules. It is about optimization. Single variable optimization part 1 and single variable optimization part 2.

So, we talked about the first order and second order conditions. What it means in terms of geometry, these conditions. And we talked about the different kinds of functions, convex functions, concave functions and obviously a plenty of applications from the field of economics and not necessarily only economics, we talked about applications in other fields also, for example, oil extraction which has something to do with environmental economics.

Also, the population growth, you know all those things were there. Then in the ninth module the area under the curve, indefinite and definite integrals, these were discussed, this comes under the module of integration part one. So, there were two modules on integration, integration part one and integration part two. We covered these in the ninth and the tenth module.

And we talked about the economic applications of these also and integration by substitution, applications of integration in other fields, for example, when you are talking about the distribution of income in a country. So there the idea of definite integral is very important. So, those things, we talked about that. And the final topic that we covered in this course is difference equations. Again, this was covered in two modules, the eleventh and the twelfth modules.

So, here the time is discrete. And that was introduced in the eleventh module, discrete time. And first order difference equations were used and we talked about the applications of first order difference equations in real life and how to solve first order difference equations. So, we talked about Cobweb model for example. So, that is one application. And in the last module, twelfth module, we talked about higher order difference equations.

We also talked about phase diagrams. So, you have a first order difference equation but which is not linear. So, in this case you can take the help of phase diagrams to understand the stability properties of the system. And in the twelfth module we talked about the solution of second order difference equations and we talked about applications of it. For example, we talked about Samuelson's multiplier accelerator model.

This concludes our discussion and the course has been completed. I thank you for being with me in this course and all the best in your future career. Thank you.