Mathematics for Economics – I
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Module: Single Variable Optimization 2
Lecture 22: Second Derivative Test

Hello everyone, welcome to another lecture of Mathematics for Economics part one. So, the topic that we have been covering in the last few lectures is optimization with a single variable.

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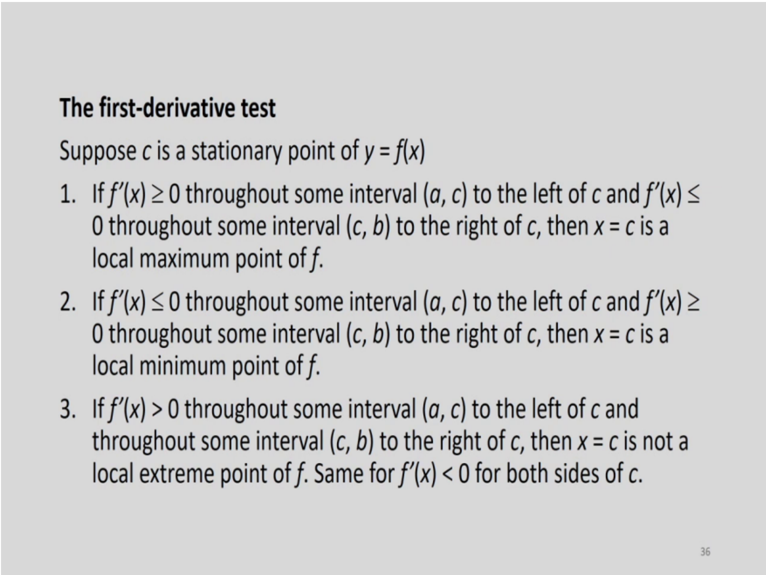

And this is the last thing that we discussed in the last lecture that you can see on your screen. We talked about the local extreme points and how to find the local extreme points, that means local maximum and local minimum and the test that we were talking about is called the first derivative test. So, here it is given on your slide and we have talked about it in the last lecture.

So, I will go quickly through it. So, what we are saying is that suppose c is a stationary point of this function $y = f(x)$ then if $f^{'}(x) \geq 0$ throughout some interval (a, c) to the left of c and $f^{'}(x) \leq 0$ throughout some interval (c, b) to the right of c then $x = c$ is a local maximum point of f.

Similarly, if $f^{'}(x) \leq 0$ throughout some interval (a, c) to the left of c and $f^{'}(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point of f.

So, these are the two conditions 1 and 2, which will help us to find out the maximum and minimum points, but these are as we have seen, these are local maximum or minimum points.

And the third condition tells us that, if the $f^{'}(x) > 0$, you know to the left or to the right of c or $f^{'}(x) < 0$ to the left and right of c, then there is no change of sign right of the first derivative and in that case, we cannot say that $x = c$ is a maximum or minimum point.

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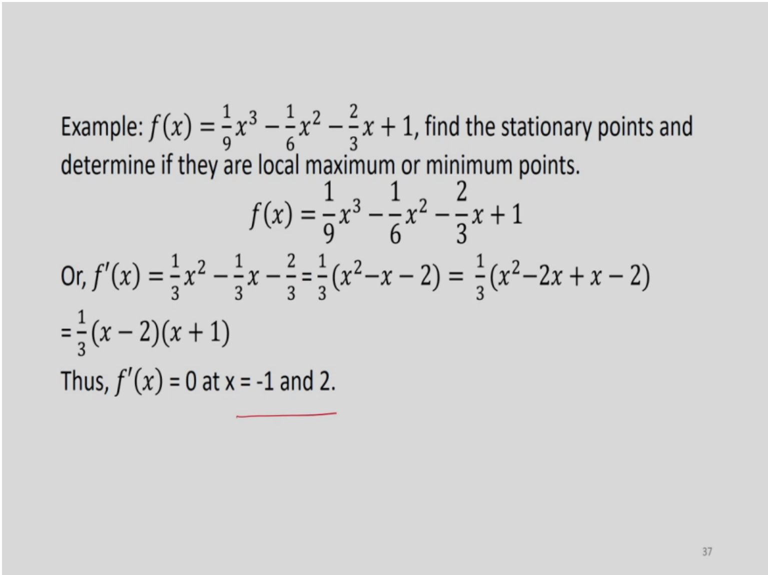

And this is the example that we discussed in the last lecture, function is given a polynomial function $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$. So, we have to find the stationary points and determine if they are maximum or minimum points. So, we have found the stationary points as -1 and +2, then we have to look at the signs of $f^{'}(x)$to the left and right of these two stationary points.

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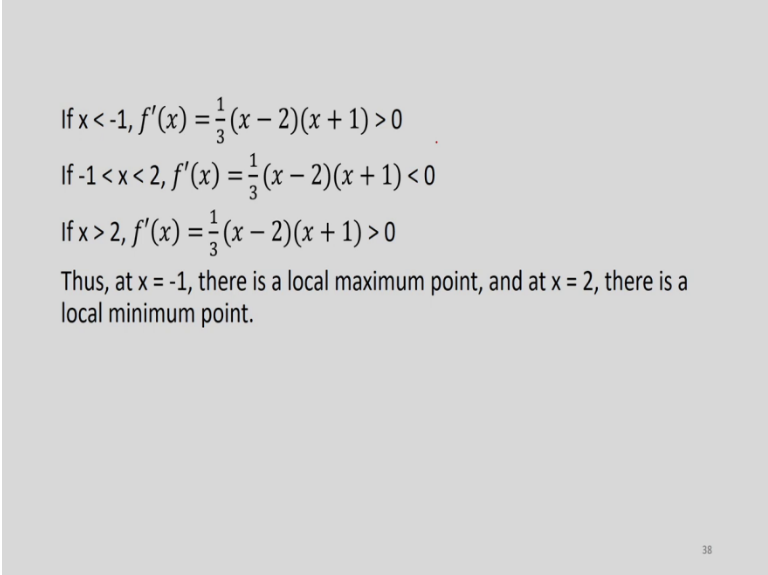

And that is what we are doing here. So, to the left of -1, the $f^{'}(x)$ is positive between -1 and +2 it is negative and to the right of 2 it is positive, which basically means first function rises and then it reaches a maximum at -1 and then it declines and reaches a minimum at $x = 2$ and then it goes on rising which basically helps us to conclude that $x = -1$ is a local maximum point and $x = 2$ is a local minimum point. So, this is about the first derivative test.

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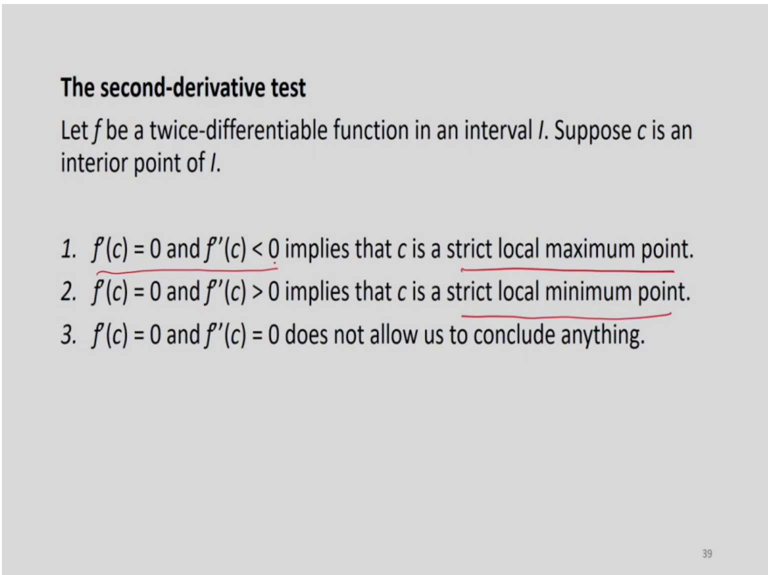

Now, what we are going to discuss today in some length is what is known as the second derivative test. So, let f be twice differentiable function in an interval capital I. Suppose, small c is an interior point of I then you have three conditions. First condition is saying that $f^{'}(c) = 0$ and $f^{''}(c) < 0$ implies that c is a strict local maximum point.

Strict local maximum point which means, if you take the value of $f(c)$, it is strictly greater than any other value $f(x)$ in the neighborhood of c and secondly, $f^{'}(c) = 0$ and $f^{''}(c) > 0$ implies c is a strict local minimum point. So, here also in the second case also you have this property of strictness, which means, in the second case that $f(c)$ is strictly less than $f(x)$ for any x in the neighborhood of c.

And thirdly, if it so happens that $f^{'}(c) = 0$ and $f^{''}(c) > 0$, then it does not allow us to conclude anything. So, in the third case, we cannot actually identify the local maximum or minimum, in the first two cases you can see that if these two conditions hold together, then you have a strict local maximum point.

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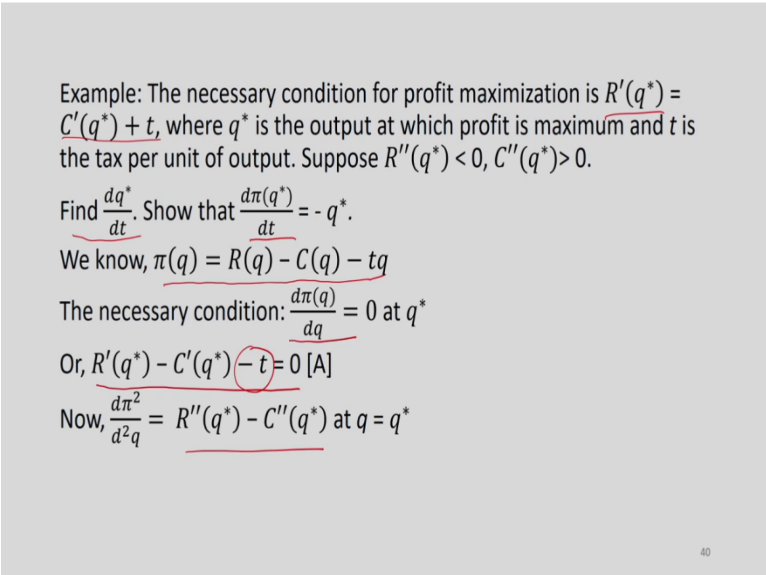

So, here is an example of how this property can be used in practical terms. So, here is the example I am reading it out, the necessary condition for profit maximization is $R^{'}(q^{*}) = C^{'}(q^{*}) + t$, where q^{*} is the output at which profit is maximum and t is the tax per unit of output. So, this is given, the first sentence is given and this is something we have seen before.

So, here the story was there is a firm which is trying to maximize the profit its revenue function is $R(q)$, its cost function is $C(q)$ plus there is a tax that the firm has to pay to the government and the tax is such that the firm has to pay tax small t per unit of output. So, the total amount of money that is paying for producing q amount of output is $C(q) + tq$ and we have seen that there is this necessary condition that if at q^* profit is indeed maximized, then the first derivative of the profit function should be equal to 0 from which this condition is obtained that $R'(q^*) = C'(q^*) + t$. So, this is what we know.

Now, suppose $R''(q^*) < 0$ and $C''(q^*) > 0$, suppose these two things are correct, then we have to find out $\frac{dq^*}{dt}$ and we have to show $\frac{d\pi(q^*)}{dt} =- q^*$. So, what does this mean $\frac{dq^*}{dt}$? What it means is as the tax changes then how much does the equilibrium output that is q^* profit maximizing output, how much does that change instantaneously and the second thing on the left hand side, it is telling us the change of the equilibrium profit as the tax rate changes by a small amount.

These two things we have to prove the first thing we have to find and the second thing we have to prove. Let us start from the beginning, which is the profit function. The profit function is $\pi(q) = R(q) - C(q) - tq$ and the necessary condition is that if you have q^* as the profit maximizing output then this condition has to be satisfied, $\frac{d\pi(q)}{dq} = 0$ at q^*.

And from here we are getting this condition, $R'(q^*) - C'(q^*) - t = 0$[A]. Now, what we do next is that we take the second derivative of the profit function at q^*. So, if we do that then actually this term will vanish because it is not a function of q and therefore, we are going to get only this, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*)$.

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Since, $R''(q^*) < 0$, $C''(q^*) > 0$, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*) < 0$ at $q = q^*$

Thus, $q = q^*$ is indeed a maximum point.

- Through implicit differentiation of [A] with respect to t we get,

$R''(q^*)\,\frac{dq^*}{dt} - C''(q^*)\frac{dq^*}{dt} - 1 = 0$

Or, $\frac{dq^*}{dt} = 1/(R''(q^*) - C''(q^*))$, which is negative since,
$R''(q^*) - C''(q^*) < 0$

The profit maximizing output falls as tax per unit rises.

The profit function at q^* is given by, $\pi(q^*) = R(q^*) - C(q^*) - tq^*$

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The second-derivative test

Let f be a twice-differentiable function in an interval I. Suppose c is an interior point of I.

1. $f'(c) = 0$ and $f''(c) < 0$ implies that c is a strict local maximum point.
2. $f'(c) = 0$ and $f''(c) > 0$ implies that c is a strict local minimum point.
3. $f'(c) = 0$ and $f''(c) = 0$ does not allow us to conclude anything.

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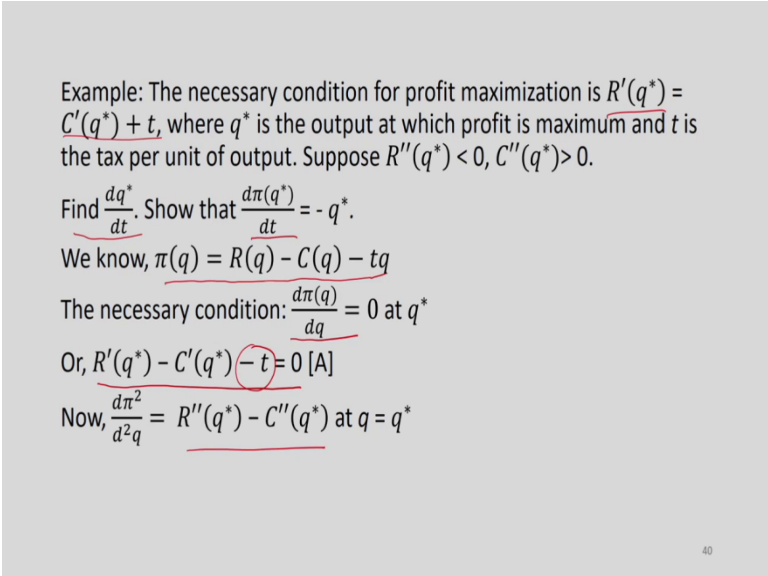

Now, we are given this information in the question itself $R''(q^*) < 0$ and further $C''(q^*) > 0$. So, if these two things are applied, then what happens to the second derivative of the profit function at q^*. So, the second derivative is this much $R''(q^*) - C''(q^*)$ at $q = q^*$.

Now, this first term,$R''(q^*)$ is negative right second term, $C''(q^*)$ is positive which means this entire thing, $R''(q^*) - C''(q^*) < 0$ and that basically tells us that $q = q^*$ is indeed a maximum point because that is we have seen in the previous slide that at c $f'(c) = 0$ that stationary point and at that very point the second derivative is negative if these two conditions are satisfied, then c is a strict local maximum point.

So, in short we started with the assumption that q^* is the maximum point and actually we have used a condition which is given and we have shown that indeed $q = q^*$is local maximum point. Now, we have to find out $\frac{dq^*}{dt}$, how do we find that out? What we do is that we take that condition A, what was this condition A is the necessary condition actually this condition this is the necessary condition, $R'(q^*) - C'(q^*) - t = 0$[A] and from this condition we do the implicit differentiation with respect to t.

Why t? The reason is that we have to find out $\frac{dq^*}{dt}$. So, it is a derivative with respect to t. So, that gives us a clue why we should use the first order condition or the necessary condition but then differentiate this necessary condition with respect to t. If we do so, then this is what we are going to get: $R''(q^*)\frac{dq^*}{dt} - C''(q^*)\frac{dq^*}{dt} - 1$ because there is small t term and that will become 1.

And let us collect these terms. So, we are jumping some steps here and if we simplify this thing then $\frac{dq^*}{dt} = 1/(R''(q^*) - C''(q^*))$ and what about the sign of this? The sign of this is negative because we have just seen that this $R''(q^*) - C''(q^*) < 0$.

So, basically what we have got is what we were required to get, we were supposed to find out what is the change in the equilibrium output with respect to change in the tax rate, which is $\frac{dq^*}{dt}$ that we have found out and it is given by $\frac{dq^*}{dt} = 1/(R''(q^*) - C''(q^*))$ and incidentally we have also been able to figure out that this is going to be negative and the implication is that as the tax rate rises, the firms will be constrained to produce less equilibrium output.

That means if they are maximizing their profits then they will be producing less output as the tax rate rises and which seems intuitive that is what I have written here profit maximizing output falls as tax per unit rises while it cuts the other way also that as the tax per unit falls the profit maximizing output will rise.

Now, what we need to find or what we need to show now is this, which is $\frac{d\pi(q^*)}{dt} =- q^*$. What it means is that the profit or the maximized profit how much does it change with respect to t and we have to show that it is equal to $- q^*$. Well, to do that, what we do is that we go back to the profit function and look at the profit function at q^*, at the maximum profit maximizing output what happens to that profit function? So, now, this is obvious so, I just have to replace q by q^* and so, $\pi(q^*) = R(q^*) - C(q^*) - tq^*$.

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By implicit differentiation with respect to t we get,

$\frac{d\pi(q^*)}{dt} = [R'(q^*) - C'(q^*) - t]\frac{dq^*}{dt} - q^*$(using the chain rule)

$= -q^*$, since $R'(q^*) - C'(q^*) - t = 0$

Hence the proof.

The maximized profit falls as tax rate rises.

Example: A tree is planted at time $t = 0$, $P(t)$ is its current market value at time t. It's a differentiable function. r is the rate of interest. $P''(t) <$ 0. When should the tree be cut down to maximize the present discounted value?

Let the present value be given by $f(t)$; we know, $f(t) = P(t)e^{-rt}$

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Since, $R''(q^*) < 0$, $C''(q^*) > 0$, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*) < 0$ at $q = q^*$

Thus, $q = q^*$ is indeed a maximum point.

- Through implicit differentiation of [A] with respect to t we get,

$R''(q^*)\frac{dq^*}{dt} - C''(q^*)\frac{dq^*}{dt} - 1 = 0$

Or, $\frac{dq^*}{dt} = 1/(R''(q^*) - C''(q^*))$, which is negative since, $R''(q^*) - C''(q^*) < 0$

The profit maximizing output falls as tax per unit rises.

The profit function at q^* is given by, $\pi(q^*) = R(q^*) - C(q^*) - tq^*$

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Now, again we are taking the help of implicit differentiation, I am differentiating with respect to t and doing so this is what I am going to get, $\frac{d\pi(q^*)}{dt} = [R'(q^*) - C'(q^*) - t]\frac{dq^*}{dt} - q^*$. So, just look at that, the expression. So, I am differentiating with respect to t. So, first I am going to take the differentiation of this with respect to q^* and $\frac{dq^*}{dt}$ and there will be $-q^*$ term here, that is basically the chain rule.

Then from here what do I get? This first term will actually drop out because this is by the necessary condition the first order condition, this is, $[R'(q^*) - C'(q^*) - t] = 0$. So, we are only left with $-q^*$ and which is the proof. This is what we are supposed to prove that $\frac{d\pi(q^*)}{dt} =- q^*$.

Now, this also has an implication it basically means that since this is a negative quantity, $-q^*$ is a negative quantity that means that as the tax rate rises the maximized profit falls here because this term is negative and we are actually able to find by how much does it fall, the decline is given by the equilibrium output that is $-q^*$.

So, here what we have done to understand why we did this exercise is that here we have actually used the condition that at the maximum point or at the point where the profit is getting maximized, the second order condition this is called the second order condition, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*) < 0$ this is getting satisfied that is this condition is getting satisfied that is why we are sure that the profit is actually getting maximized that $-q^*$.

And as side results we have also found out how much the equilibrium output and equilibrium profit change with respect to change in the tax rate. Now, let us look at another example of the same idea of the second order condition that is the second derivative. So, here is the equation a tree is planted at time $t = 0$, $P(t)$ is its current market value at time t. It is a differentiable function that is $P(t)$ is a differentiable function, r is the rate of interest and we are given this information that $P''(t) < 0$.

The question is when should the tree be cut down to maximize the present discounted value? So, we have talked about present discounted value previously. Now we are going to apply that and we shall see that we are going to be using the second order condition as well. Let us suppose that $f(t)$ denotes the present discounted value or present value of the tree and so, $f(t) = P(t)\, e^{-rt}$.

So, $P(t)$ is the current value, current market value of the tree at time t, but suppose we want to find out the present discounted value of that standing at point 0 then this will be that $P(t)\, e^{-rt}$, r is the rate of interest.

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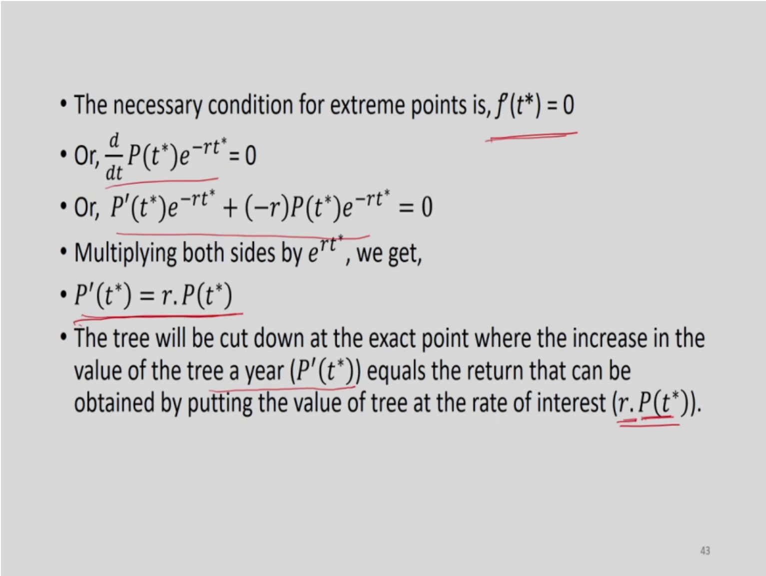

Now, the necessary condition for extreme points is that it has to be stationary point. So, $f'(t^*) = 0$ where t^* is supposed that t where the present discounted value is maximized. So, we are using this condition that the first derivative is equal to 0 and so, this is just application of derivatives, the product rule I am using here and multiplying both sides by e^{rt} we get this condition.

So, $P'(t^*) = r.P(t^*)$. Now, this condition is actually very important whenever time is involved, so some dynamics is involved and some decision maker is trying to maximize the present discounted value, then this condition makes an appearance again and again.

Let us try to understand this condition. What this condition is saying is that the tree will be cut down at the exact point where the increase in the value of the tree a year so this is the increase of the value of tree $P'(t^*)$. So, this is a change in the value of the tree as the changes by a little around that t^* point and this is equal to the return that can be obtained by getting the value of the tree at the rate of interest.

It equals the return that can be obtained by putting the value of the tree at a rate of interest $r.P(t^*)$. $r.P(t^*)$, This, $P(t^*)$ is the current value of the tree and if you are trying to get the rate of interest from that then r will be that rate of interest. So, $r.P(t^*)$ is the return from the rate of interest route whereas, $P'(t^*)$ is the change of the value of t when t changes by a little. So, these two things, the left hand side should be equal to the right hand side.

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- The sufficient condition for maximum present value is,

$\frac{d}{dt}\left[P'(t^*)e^{-rt^*} - r.P(t^*)e^{-rt^*}\right] < 0$

The LHS = $\frac{d}{dt}[P'(t^*) - r.P(t^*)]e^{-rt^*}$

= $e^{-rt^*}(P''(t^*) - r.P'(t^*))$ - $re^{-rt^*}[P'(t^*) - r.P(t^*)]$

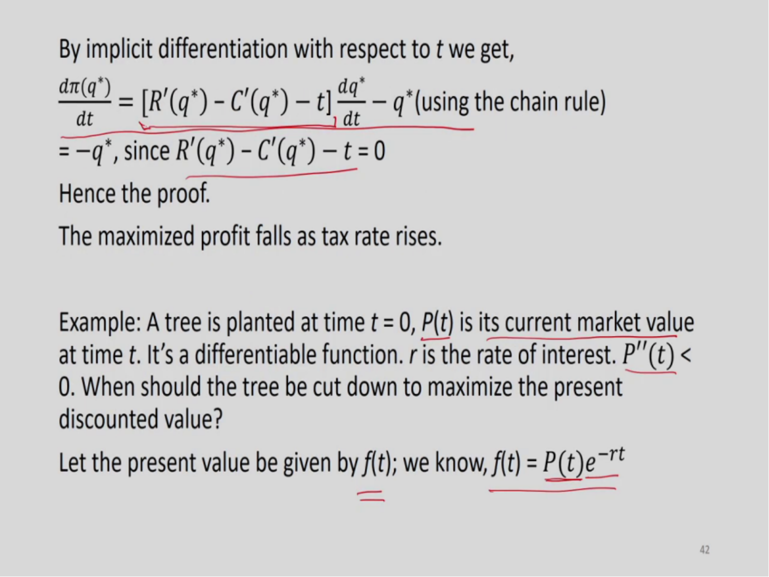
= $e^{-rt^*}(P''(t^*) - r.P'(t^*))$ (since $P'(t^*) - r.P(t^*)$ = 0)

Given that $P''(t^*) < 0$ and assuming $P(t^*) > 0$ (since $P'(t^*) = r.P(t^*)$, it guarantees $P'(t^*) > 0$),

$e^{-rt^*}(P''(t^*) - r.P'(t^*)) < 0$

Hence, t^* is indeed a local maximum point.

- The necessary condition for extreme points is, $f'(t^*) = 0$
- Or, $\frac{d}{dt}P(t^*)e^{-rt^*} = 0$
- Or, $P'(t^*)e^{-rt^*} + (-r)P(t^*)e^{-rt^*} = 0$
- Multiplying both sides by e^{rt^*}, we get,
- $P'(t^*) = r.P(t^*)$
- The tree will be cut down at the exact point where the increase in the value of the tree a year ($P'(t^*)$) equals the return that can be obtained by putting the value of tree at the rate of interest ($r.P(t^*)$).



Now this is a necessary condition. So, one is not sure whether t^* is actually maximizing or minimizing or something else. So, one has to look at the sufficient condition as well. Now the sufficient condition for maximum present value is this that I have taken the first derivative which is this one the left hand side and I have differentiated further and the second derivative should be less than 0.

Now let us look at the left hand side. I am just going to simplify this a bit. So I have taken e^{-rt^*} common which is outside the brackets and then we do the differentiation then in the process we have this term $P'(t^*) - r.P(t^*)$ that we have seen is equal to 0.

So, this part drops out. So, we are only left with this fart the first term now, is it negative that is the question. If it is negative, then we are safe; if it is not negative then we are in trouble. Now, remember in the question itself it is given that the second derivative of P function is negative, this condition is given.

So, this is negative, this term is negative, this term is positive. So, the sign actually depends on this term; this has to be positive to make the entire thing negative. So, it is given to be negative and assuming that the value of the tree always remains positive. $P(t^*)$ means the value of the tree and it can be safely assumed that it is positive.

Now, if $P(t^*)$ is positive then by this condition this is also positive. So, this term is positive and it is preceded by a negative sign so, this is negative this is negative and this we have seen is negative so, the entire term is negative. So, the sufficient condition or the second order condition is actually satisfied $t = t^*$.

Therefore, the tree should be cut down, that is our conclusion, the tree should be cut down at a point $t = t^*$ at that point of time where this particular condition is satisfied, this condition is satisfied. We have to apply this condition and try to identify the t^*. So, this is how this problem is done.

Now, as I was saying that this is a very generic sort of problem, often you have an asset here that asset is a tree which is growing in value with respect to time and at the same time you have some rate of interest which is in this case r. So, you want to find out at what point of time that asset has to be monetized, here that tree is cut down. So, that is one way of monetizing the asset.

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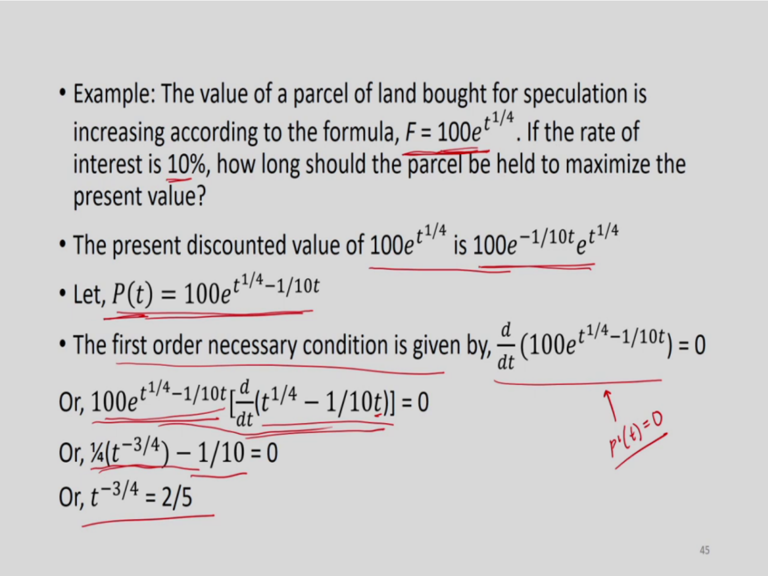

Here is another example: the value of a parcel of land bought for speculation is increasing according to this formula $F = 100e^{t^{1/4}}$. If the rate of interest is 10 percent how long should the person be held to maximize the present value?

So, this is a similar problem like we have just discussed now. So here instead of a tree you have a land, a piece of land and your purpose is not to use that land. But it means you are not going to use it for your personal use, but you are going to hold the land for speculation. That means its value might increase in future and if it increases sufficiently then you are going to sell it and get the money.

And how much the value of the land is increasing with respect to time it is given by this formula, $F = 100e^{t^{1/4}}$. Now, the first thing we need to do is to find the present discounted value of this because right now, this is not the discounted value, this is the current value of the land at that point of time, where we are looking at the value of the land.

So the first thing to do is to find out the present discounted value, let the present discounted value be given by $P(t)$. So this $P(t) = F.e^{-rt}$. Now r is given to us it is 10 %. 10 % means 0.1 and so, the present discounted value becomes this $P(t) = F.e^{-0.1t}$ and so, this is written here, $P(t) = 100e^{t^{-1/10t}}.e^{1/4}$.

So, the present discounted value of that piece of land which is given by $P(t) = 100e^{t^{1/4}-1/10t}$. Now, what we do now is we have to find out that particular point of time at which the land will be monetized, that is the land will be sold and the money will be obtained from selling the land.

Now, like before we have followed this method, we first find out that particular t which basically satisfies the first order condition that is the necessary condition and that point of t is basically the stationary point and then we have to look at the second order condition. So, that is the algorithm, first order necessary condition basically we take the first derivative of this $P(t)$.

So, basically what we have done is $P'(t) = 0$, from this we are getting this condition, $\frac{d}{dt}(100e^{t^{1/4}-1/10t}) = 0$. Now, basically we have to simplify this expression and that is how we have done it, it is basically an exponential function. So $100e^{t^{1/4}-1/10t}\frac{d}{dt}(t^{1/4} - 1/10t) = 0$.

So, this entire thing that is the first derivative should be equal to 0 and since it has to be equal to this 0 this first part is not equal to 0. So, we can only concentrate on the second part and from here we are getting this condition $\frac{1}{4}(t^{-3/4}) - 1/10 = 0$ because this t will just drop out.

So this is the condition we are getting and then we are simplifying it further and it becomes $t^{-3/4} = 2/5$. So, we are basically multiplying both sides by 4.

And from here we get at this point $t^{3/4} = 5/2$ because this was 2/5, but if I take the reciprocal it becomes $5/2$ and if I do take the reciprocal this minus sign of the power of t vanishes.

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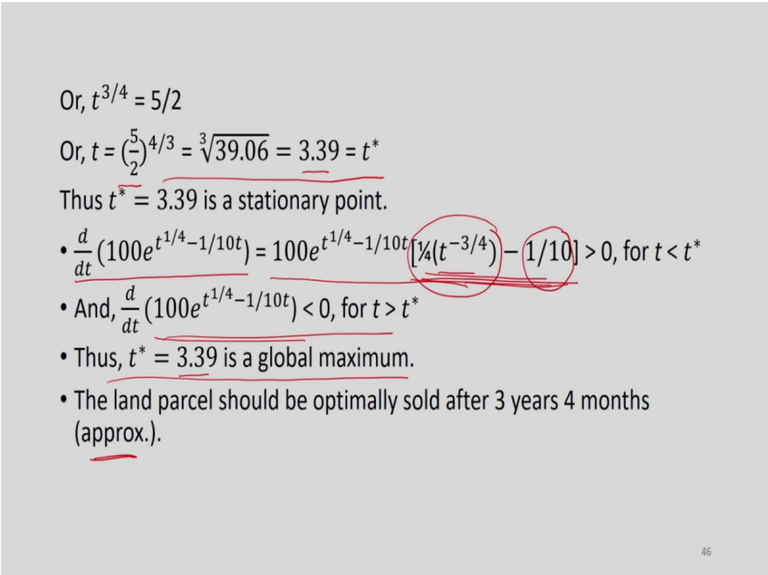

So, $t^{3/4} = 5/2$ and then I simplify it further. So, $t = (\frac{5}{2})^{4/3}$. So, it looks a little bit complicated, but you know it can be found out if you have a calculator, it can be easily found out and you get basically 3.39.

So, I have rounded off to the second decimal place. So, this is $t^* = 3.39$t star,. This is our stationary point 3.39 at this point the first order condition is satisfied, but that is not all. What we have done now is we have basically used the first derivative test of maximization. So, instead of taking the second derivative, we know that the first derivative test is that if you are to the left of t^* and the first derivative is let us suppose positive and if you are to the right of t^* and the first derivative is negative, then you are indeed getting a maximum point.

So, that is the idea we have used here. So, this is what we have seen before, this is the first derivative, $100e^{t^{1/4}-1/10t}[\frac{1}{4}(t^{-3/4}) - 1/10]$. Now, if $t < t^*$, then $100e^{t^{1/4}-1/10t}[\frac{1}{4}(t^{-3/4}) - 1/10] > 0$ and if $t > t^*$, then this first derivative $100e^{t^{1/4}-1/10t}[\frac{1}{4}(t^{-3/4}) - 1/10] < 0$. The reason why this is happening is that you can see here, t is there, but t's power is negative.

So, this will be $1/t^{3/4}$ and if you are taking a t which is less than t^*, at t^* $100e^{t^{1/4}-1/10t}[\frac{1}{4}(t^{-3/4}) - 1/10] = 0$. But if you are taking the value of t, which is slightly less than t^*, then this number, $(t^{-3/4})$ becomes higher than this number, $1/10$ and therefore, this entire term is positive.

Therefore, we are saying that if you are taking points to the left of t^*, to the left of the stationary point, the first derivative is positive and similarly for t greater than t^*, the first derivative is negative and that is what we needed to show that t^* is maximum, in fact, it is a global maximum.

We are not even talking about a local maximum here; it is a global maximum. $t^* = 3.39$, what does it actually mean? It actually means that the land parcels should be optimally sold after 3 years, 4 months, 4 months will actually take us to 3.33, 3.33 is what will give us 3 years 4 months, but it is a little bit more than slightly more than 3 years, 4 months.

So that is why I have written it approximately. So, this was a problem of selling the land at an optimal point which gives you the maximum present discounted value, where the value of the land is increasing over time.

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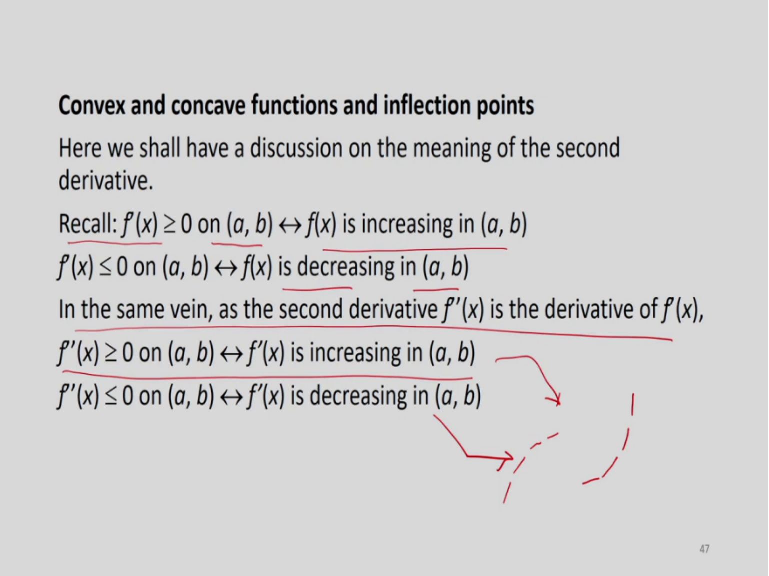

Now, we come to another important point of this optimization business. This is the case of convex and concave functions and inflection points. Here we shall have a discussion on the meaning of the second derivative. So, we have talked about the second derivative. We have used the secondary derivative to find out whether the stationary point is giving us a maximum point or a minimum point but what does it mean geometrically, second derivative positive what does it mean?

Second derivative negative, what does it mean? Or if it is 0, then what does it mean? Recall $f^{'}(x) \geq 0$ on this interval (a, b), this is ↔ $f(x)$ increasing in (a, b), this is weakly increasing in (a, b) and on the other hand $f^{'}(x) \leq 0$ in the same interval (a, b)↔ $f(x)$ decreasing in (a, b). Again this is weakly decreasing.

Now, you can take $f(x)$ to be $f^{'}(x)$and similarly, you can say the following; that $f^{''}(x) \geq 0$ on (a, b) ↔ to $f^{'}(x)$ is increasing in (a, b). So, let me show it geometrically. So, $f^{'}(x)$ is the slope and that slope is increasing like this.

So, this is like linearization of that curve or that function that we are talking about and in the second case what is happening? $f^{''}(x) \leq 0$ in the interval (a, b) ↔ $f^{'}(x)$ is decreasing in the interval (a, b). So, what does it mean when we say $f^{'}(x)$ is decreasing? So, it is something like this, the slope is going down, so this is what it means and from this we can actually get an idea of concavity and convexity.

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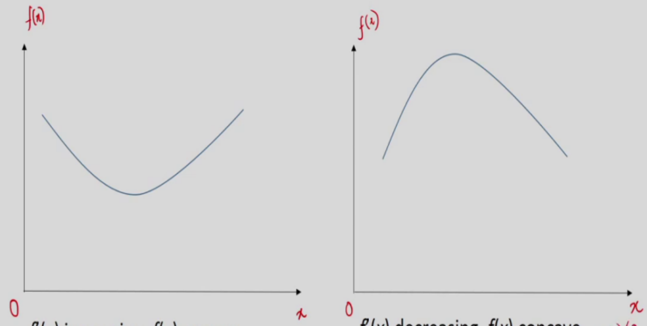

$f'(x)$ increasing, $f(x)$ convex $\qquad$ $f'(x)$ decreasing, $f(x)$ concave, $f''(x)<0$

If f is continuous in the interval I, and twice differentiable in I' (interior of I), then, we define:

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- f is convex on $I \leftrightarrow f''(x) \geq 0$, for all x in I'
- f is concave on $I \leftrightarrow f''(x) \leq 0$, for all x in I'

Example: is the function $f(x) = px^2 + qx + r$ a convex or a concave function?

From $f(x) = px^2 + qx + r$ we get,

$$f'(x) = 2px + q$$

Or, $f''(x) = 2px$

Thus, if $p > 0$, $f(x)$ is convex; if $p < 0$, $f(x)$ is concave.

If $p = 0$, it is linear function, in which case it is both convex and cancave.

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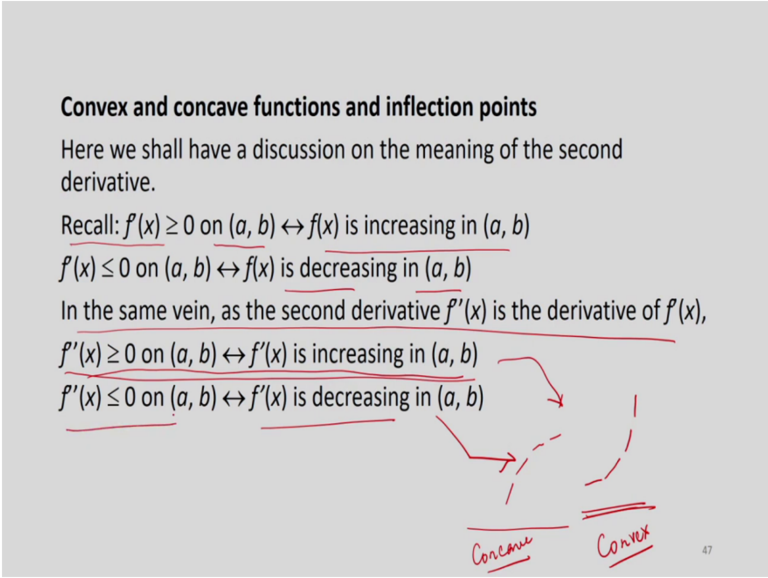

If $f(x)$ is continuous in the interval I and twice differentiable in I'. What is I'? I' is the interior of I, then we define f is convex on I $\leftrightarrow$ $f''(x) \geq 0$ for all x in I' and f is concave on I $\leftrightarrow$ $f''(x) \leq 0$ for all x in I'.

Now, remember this thing, $f''(x) \geq 0$ is something we have seen here. This is what it is and here we have seen that this is equivalent to $f'(x)$. So, $f'(x)$ increasing therefore, is equivalent to f is convex in I. Similarly, $f'(x)$ decreasing is equivalent to f is concave on I.

So, these three things are basically closely related to each other. What is happening to $f'(x)$, is it rising or declining? What is happening to the sign of $f''(x)$? Is it positive, negative? And the convexity and concavity of the function. Here is the geometric representation of the same thing.

So, let us start from the left hand panel. So, here $f(x)$ is a sort of u shaped curve and you can immediately see that this curve is something similar to what I have drawn here. So, this is convex, here $f'(x)$ is rising. It is increasing and it also means, we have just seen, it means that the second derivative is positive.

On the other hand, on your right hand side of the screen you have a function which is concave. And concave functions, we have drawn it in a rudimentary manner here, concave. So, concave functions are where $f'(x)$ is decreasing and that we have seen is equivalent to saying that the second derivative is negative.

So, here you have a second derivative negative, $f''(x) \leq 0$. Here the second derivative is positive or weakly positive, weakly negative. That is why I say that these three things are going together and these three things are going together. We shall be having a lot of occasions to talk about convexity and concavity. I think, today we will stop here itself. So I shall see you in the next lecture. Thank you.