Game Theory and Economics

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Module No. # 4

Mixed Strategy Nash Equilibrium

Lecture No. # 4

Characterisation of Mixed Strategy Equilibrium

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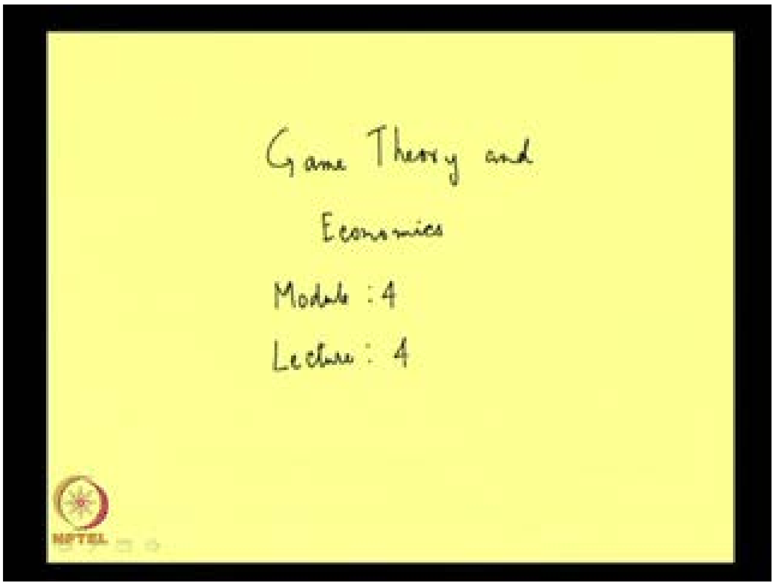

Welcome to the fourth lecture of module 4 of this course, called Game Theory and Economics. So, what we have been discussing is mixed strategy Nash equilibrium and we have done the part, where we talked about two player, two action games and how to find out the Nash equilibrium in those games; but in general, suppose, there are more than two players and more than two actions for each player, then what could be the characteristics of the mixed strategy Nash equilibrium. So, that is what we were discussing as the last topic in the last lecture.

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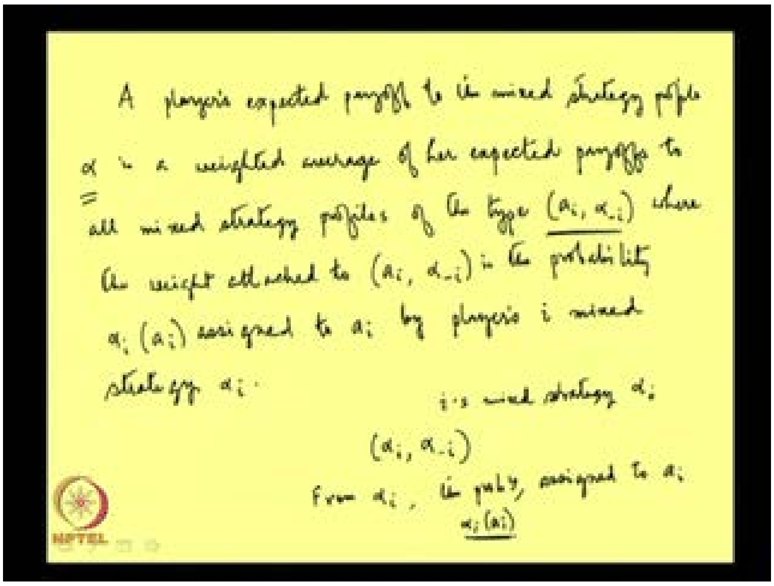

So, the main issue that we were discussing is the following: that in the mixed strategy Nash equilibrium, the payoff of each player is the weighted average of the expected payoff from his pure strategies and that is a fundamental result. Let me write it down.

So, this is the statement that I was talking about. A player's expected payoff to the mixed strategy profile alpha is a weighted average of her expected payoff to all mixed strategy profiles of the type a i, alpha not i, where the weight attached to a i, alpha not i is the probability alpha i a i is a probability assigned to a i by players i, mixed strategy alpha i.

So, to elaborate the point further, player i has the mixed strategy alpha i and other players have their mixed strategies also; the collection of such mixed strategies is alpha not i. So, the entire mixed strategies, if I club them together, I get this. This is the mixed strategy profile of all the players taken together. Now, if alpha i is the mixed strategy of player i, it means that from alpha i, I can get the probability assigned to any action a i which is given by alpha i a i. So, this is the probability with which player i plays a i.

What is being said is that the expected payoff of any player from this entire mixed strategy profile alpha is the weighted average of her expected payoff from all mixed strategy profiles of the type this - a i, alpha not i with the probability attached to a i, alpha not i given by this - alpha i a i.

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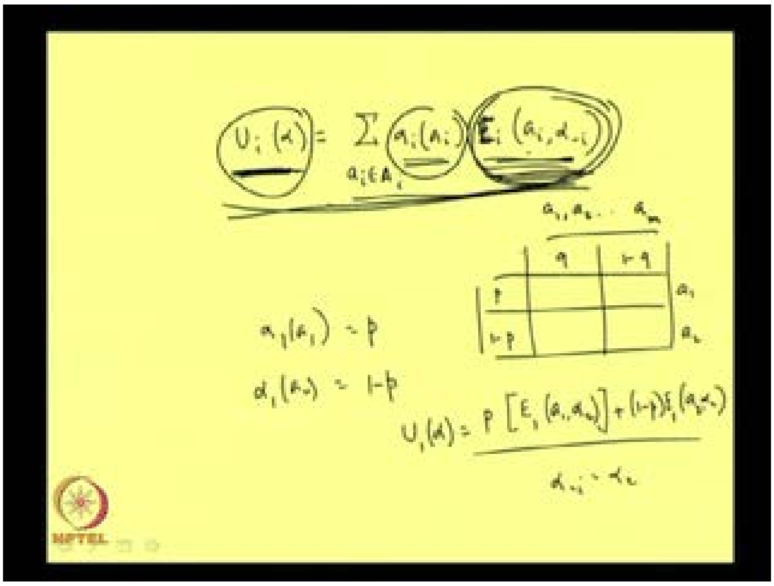

So, in short this can be written as, in mathematical terms and the rest of the formula as shown in the slide. So, this is what we are talking about. This is the expected payoff to player i, when he is playing a i and the other players are playing alpha not i. Expected payoff of this nature is multiplied with the probability of playing a i, which is given by alpha a i and then we sum over all kinds of a’s, that is small a i’s, which belong to capital A i.

Doing so, we get the expected payoff to player i from this particular mixed strategy profile alpha. So, to make the idea more clear, remember what we did in one of the previous classes. Suppose, this is p and this is q; not this, this is q. So, these are the probabilities - 1 minus p and 1 minus q.

So, this is the mixed strategy of player 1 and this is the mixed strategy of player 2. Then the total expected payoff from the mixed strategy profile to player 1 from this mixed strategy profile is given by P multiplied by suppose, this action is a 1 and this action is a 2 (and the rest of the formula as shown in the slide).

This is something we have seen before. This is the payoff to player 1, if he plays the mixed strategy alpha 1 and the other player plays a mixed strategy alpha 2. So, this is P, the probability of playing a 1 multiplied the expected payoff from playing a 1 and plus 1 minus p multiplied by the expected payoff from playing a 2. So, this is a general version of this. So, alpha a i, suppose, this is player 1, this was p; this is 1 minus p and the rest of the things are like this. alpha not i is basically alpha 2 because not i means not 1 that is 2.

So, this is something which we have seen before. This was the expected payoff to player 1 from this mixed strategy profile. Now, we are generalizing it for many players, where players have their different actions - more than two actions. Now, given this fact that given this result that we have suppose, alpha star is a Nash equilibrium.

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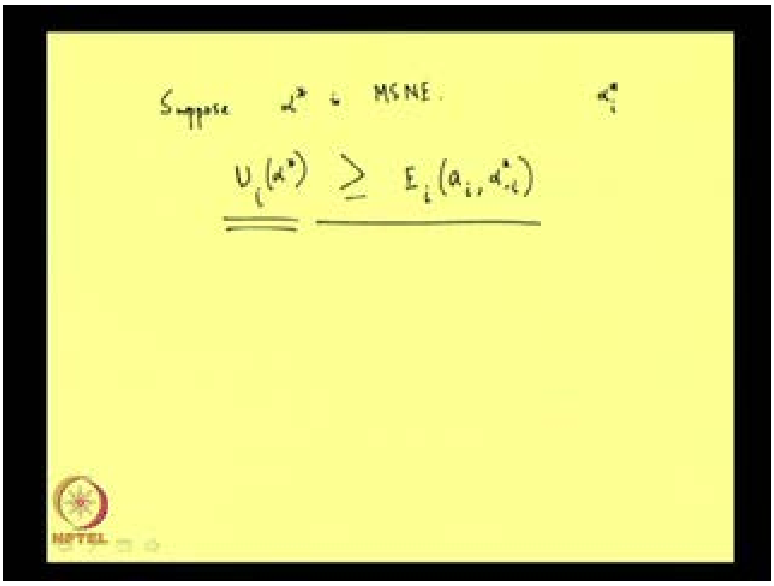

So, suppose alpha star is mixed strategy Nash equilibrium. Now, if alpha star is mixed strategy Nash equilibrium, correspondingly, there is some value of U 1 alpha star, which is the payoff to player 1 in this mixed strategy Nash equilibrium.

Now, we know that one characterizing feature of mixed strategy Nash equilibrium is that if player 1 changes her action, if he deviates from his action, which is alpha 1 here. So, let us call it i to take more general case. So, this is player i's expected payoff in the Nash equilibrium and player i's mixed strategy in the Nash equilibrium is given by alpha i star. Now, the feature of the Nash equilibrium is that if player i changes her mixed strategy then the payoff that she might get is going to be less than or equal to this payoff.

Now, which means that this is going to be greater than or equal to and the rest of the formula as shown in the slide. This I can write because a i is a pure strategy of player i which can be played by player i, but if that action is played by player I, the expected payoff to player i cannot be more than what he is getting by playing alpha i star.

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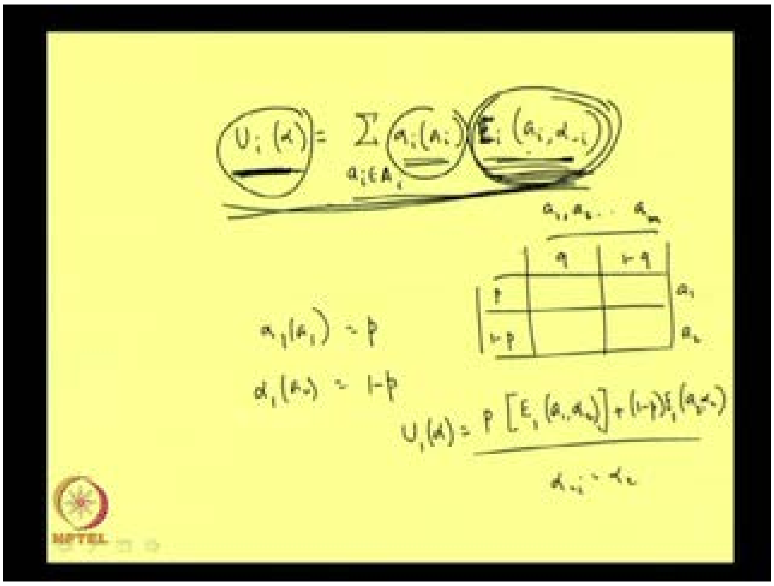

So, this is what I have written here. If he deviates and puts P 1 in action a i, then his expected payoff is going to be either less than or equal to q i's alpha star. So, this is something important. Now, in the Nash equilibrium remember, we have this thing that there are some actions which are given by a 1, a 2 etcetera. All these actions are suppose, player 1's actions; they are given by a 1, a 2,…a n and we have seen that the expected payoff from each of the action cannot be more than this value in the Nash equilibrium.

But there can be some actions for which the expected payoff - this, might be less. Now, our point is the following that if there are some actions for which the expected value, this E i, is equal to this value U i alpha and there are some values of there are some a i's, that is some actions for which this E i is less than U i alpha, then we are going to propose the following that for those actions for which these alpha's are positive, that is the probabilities attached to those actions are positive in the mixed strategy Nash equilibrium, the values of E i's must be equal and equal to this value.

So, if alpha i's are positive for some action, then the corresponding E i's should be equal for all these i actions and that equal value must be equal to the expected payoff in the Nash equilibrium to that player.

This is the first thing and the reason is the following. (Refer Slide Time: 13:58) Why they are equal and equal to the to the Nash equilibrium payoff is that we know that they cannot be more; this I know, they cannot be more.

Can they be less? Well, if one of the actions is such that the expected payoff is less, then there is no reason why the player should put positive weight or positive probability alpha on that action because if the expected payoff is less than the other expected payoff, it will be best for the player to put a weight of 0 on that action.

So, in that case, this weight cannot be positive. Yes, it will be 0. So, if it has to be positive, it must be the case that this E i's are all equal; so, that is the intuition or I should put it in the opposite way. I should say the following. (Refer Slide Time: 15:17) For the i's for which this value is 0 - alpha i is 0, for those i's, the expected payoff should be either less than this, that is either less than this or at most it can be equal to this. So, it can never be more than U i alpha star and this is clear from this relationship itself that U i alpha star is greater than equal to E i a i alpha not i star.

So, for those actions for which the corresponding probabilities are 0, obviously, this is going to be satisfied; that is what we are saying. (Refer Slide Time: 16:12)The only important thing that we are saying which is the bite of the proposition is that the actions for which the alphas are positive, it must be the case that this E i's are all equal and the equal value is equal to this.

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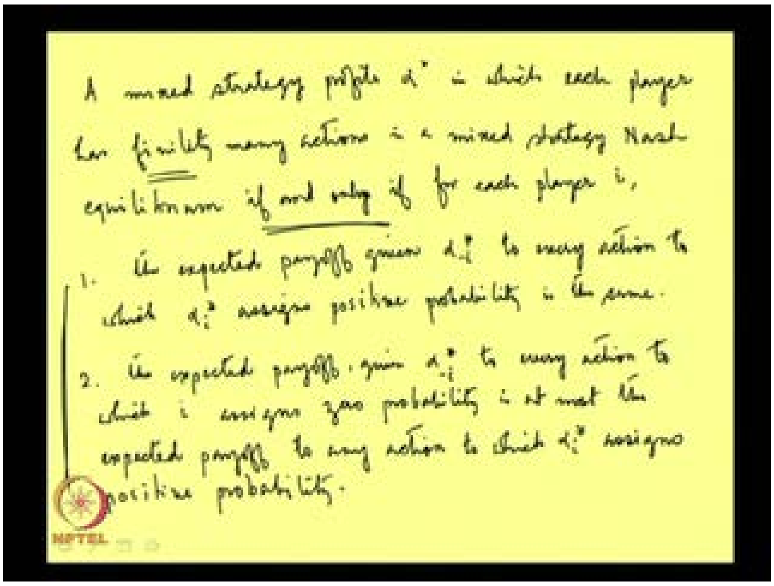

So, let me write it down in a more systematic way. It is the following. So, this is the important result that a mixed strategy profile alpha star in which each player has finitely many actions – so, this is important that the players do not have infinite number of actions, is a mixed strategy Nash equilibrium if and only if, for each player i, the expected payoff given alpha not i star to every action to which alpha i star assigns positive probability is the same.

So, in the equilibrium mixed strategy of player i, the actions to which positive probabilities are attached, their expected payoff must be same and secondly, the expected payoff given alpha not i star to every action to which i assigns 0 probability - so, it is in the equilibrium, is at most the expected payoff to any action to which alpha i star assigns positive probability.

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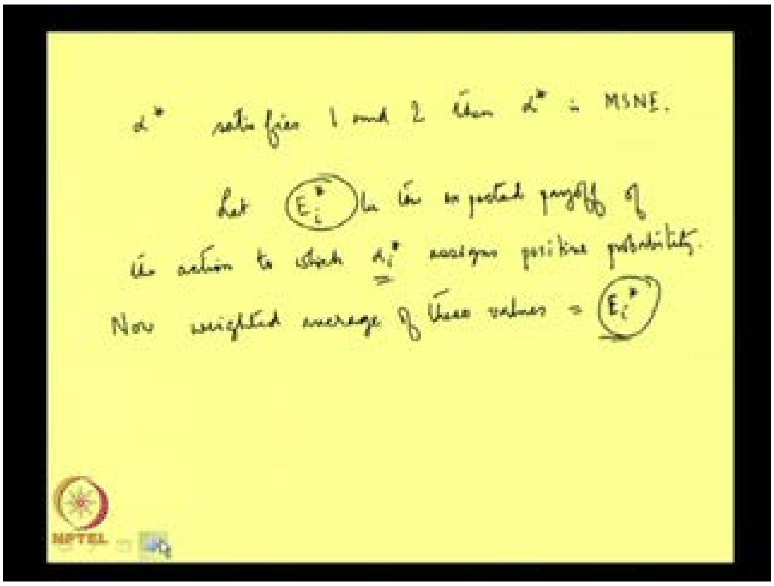

Now, here what is important is that it is an if and only if kind of statement. so if there is mixed Nash equilibrium then these two conditions must be true must be holding and that is what we have seen, but, the what is the proof of the other way. So, if these two conditions are satisfied then what is the demonstration that the corresponding alpha star is a Nash equilibrium. well this is not very difficult to show that suppose alpha star satisfies 1 and 2, these two conditions, then alpha star is mixed strategy Nash equilibrium.

The reason is the following that if alpha star satisfies 1 and 2 which basically means what? (Refer Slide Time: 21:54) Which means that for each player, let us take any player i, for him, the expected payoff to the actions to which alpha i star assign positive probability are same and there are some actions to which alpha i star does not assign positive probability; their expected payoff is less, nevermore.

Now, if that is true, then any weighted average of this expected payoffs of to the actions to which alpha i star assigns positive probability, their weighted average is going to be same as the expected payoff from each of the actions. So, it is like the following.

Let E star i be the expected payoff to the actions, this is to be, of the actions to which alpha i star assigns positive probability. Now, weighted average of these values is always going to be equal to alpha i star.

So, this is the Nash equilibrium is going to be the expected payoff in the Nash equilibrium is going to be this value and this is the payoff that player i gets in the Nash equilibrium. Now, if player i tries to deviate then obviously, he will deviate by assigning a different sets of probabilities to his actions, but whatever be the probabilities to these actions remember, what we are talking about is the weighted average of those values and in those values, the maximum value could only be alpha i star.

So, the weighted average of them can never exceed alpha i star. Hence, if player i deviates and assigns a different sets of probabilities to his actions, to his pure strategies, the expected payoff that he can get by doing so, can be at most equal to alpha i star and which means that for I, this alpha i star is a mixed strategy Nash equilibrium. So, by taking a different mixed strategy, you cannot improve his payoff. So, this is the proof that if 1 and 2 are satisfied, then the corresponding alpha star is a Nash equilibrium.

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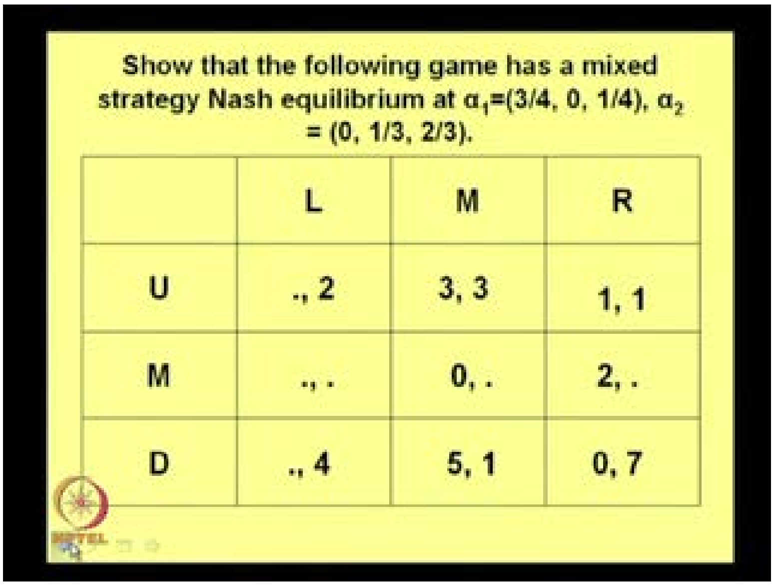

What we are going to do is that apply this result to solve some problems. So, this is the problem. Show that the following game has a mixed strategy Nash equilibrium, where alpha 1 is equal to 3 by 4, 0 and 1 by 4 and alpha 2 is equal to 0, 1 by 3 and 2 by 3.

So, what we are going to do is that we know that if alpha 1 and alpha 2 have to be the mixed strategies of player 1 and 2, where the equilibrium is attained, then it must be the case that for the actions of each player, where the positive probabilities are attached, their expected payoff to those actions must be same and for the actions, where 0 probability is attached the expected payoff must be less. So, that is what we are going to check. If it is satisfied, this condition is satisfied then we can say that it is a Nash equilibrium.

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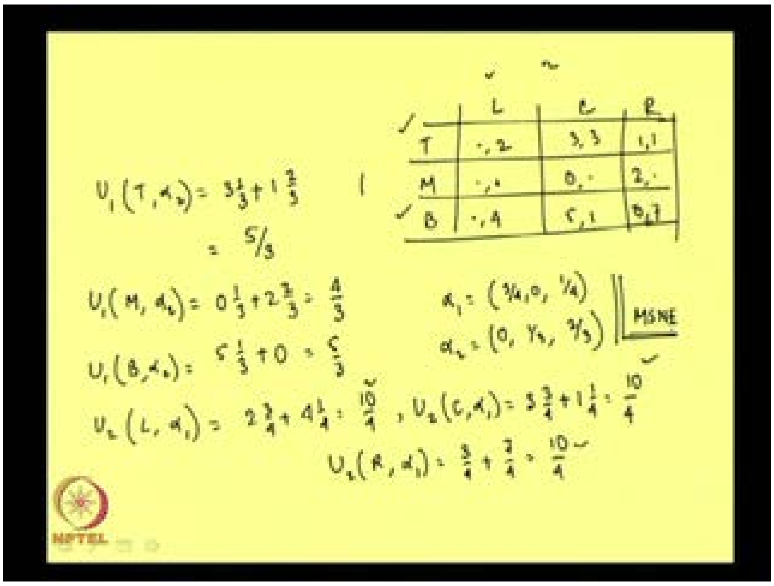

So, let us draw the diagram once again. These dots are given because this numbers at the dots are not required to answer this question. So, we have dots here and the proposition is that this and this, they constitute the mixed strategy Nash equilibrium. Now, let us look at what player 1 gets, if he plays T. This is the expected payoff that given the player 2 is playing alpha 2. So, it is not just T, but T alpha 2. Player 2 is playing this with 0 probability; this with one-third probability.

So, I have to multiply 3 by one-third and 1 with two-third. So, this is 5 divided by 3. The expected payoff to player 1, if he plays M and the other player is playing alpha 2 is equal to again 0 multiplied by one-third plus 2 multiplied by two-third which is 4 divided by 3. Again, it is going to be 5 divided by 3 plus 0; so, this is 5 divided by 3.

We see that for player 1 at least that condition is satisfied. On T and on B, he is assigning positive probability and on M he is assigning 0 probability. The expected payoff from T and B are equal and from M it is less. Let us now look at player 2's expected payoff. If he plays L and the other player is following alpha 1 then he is getting 2 with probability 3 by 4, then it is 0, then 4 with probability 1 by 4; so, it is 10 by 4. If he plays C, then he is getting 3 with probability 3 by 4 and 1 with probability 1 by 4 which is again 10 by 4 and finally, if he plays R, then his expected payoff is 3 by 4 plus 7 by 4, which is again 10 by 4.

So, the expected payoffs are all equal, but it is true that in the first action on L, he is attaching 0 probability, but that does not matter. We have seen that even if the probability attached to a particular action is 0, the expected payoff of that action can be equal or less. So, here it is equal and it is satisfying the conditions 1 and 2 and hence, these are indeed mixed strategy Nash equilibrium.

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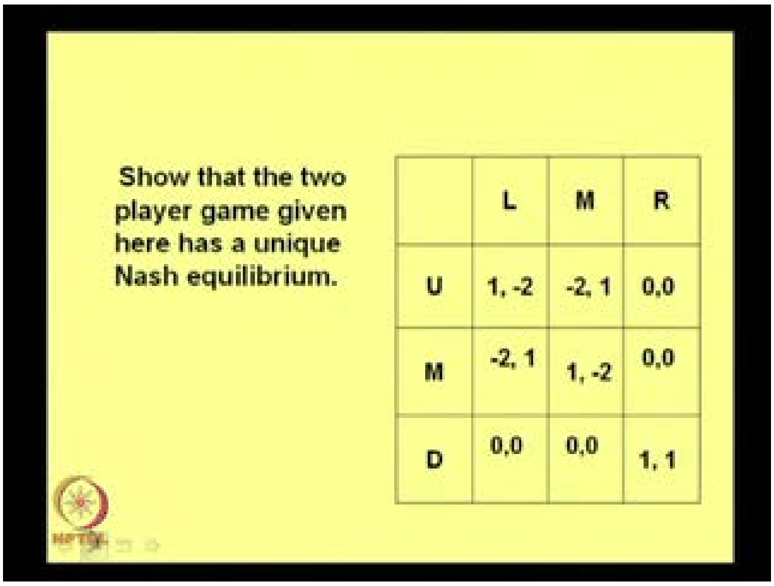

So, this is how we can check whether a particular set of mixed strategies of different players constitute a mixed strategy Nash equilibrium. So, another application that we can talk about is this game. In this game, we have to show that there is a unique Nash equilibrium. So, there is only one Nash equilibrium and there is no other Nash equilibrium in this game. That is what we have to show.

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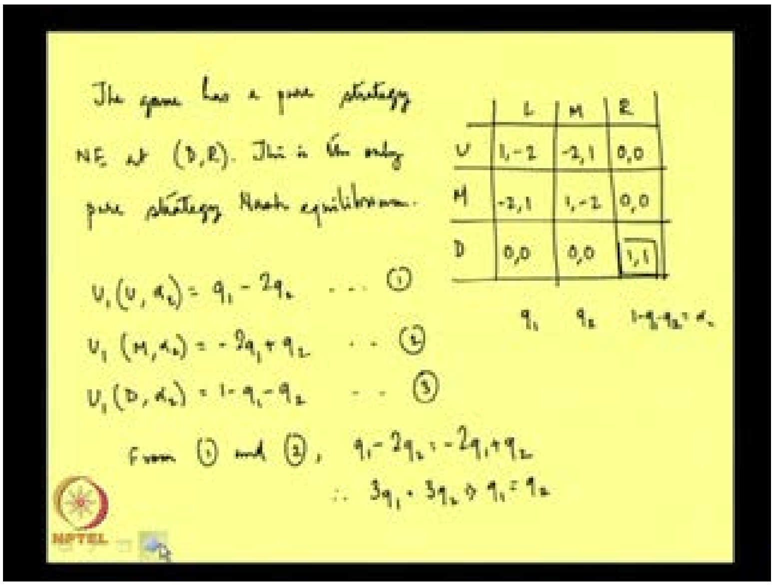

We are going to use this property that we have just discovered. So, let me draw the diagram first, the game first. Now, before looking for proper mixed strategy Nash equilibrium, it is obvious that the game has a pure strategy Nash equilibrium at D, R and this is the only pure strategy Nash equilibrium. So, this is the pure strategy Nash equilibrium; there is no other. This is the only pure strategy Nash equilibrium.

If there is no other pure strategy Nash equilibrium, there can be mixed strategy Nash equilibrium. Let us suppose that player 2 plays N with probability q, M with probability q 1, q 2 and R with probability 1 minus q1 minus q 2, then it must be the case that for player 1, the expected payoff from U and M and D must be equal.

So, what is the expected payoff of player 1 from U, given that let us suppose, this is alpha 2, player 2 is playing alpha 2. So, it is given by q 1 minus q 2; that is all. If he plays M, then what is his expected payoff? Minus 2 q 1 plus q 2 and if he plays D, then his expected payoff is just 1 minus q 1 minus q 2. From 1 and 2 Let us call them 1, 2, 3. From 1 and 2, what do I get? From 1 and 2, what I get is q 1 minus 2 q 2 is equal to minus 2 q 1 plus q 2 or 3 q 1 is equal to 3 q 2 implies q 1 is equal to q 2. This is something which we have got here.

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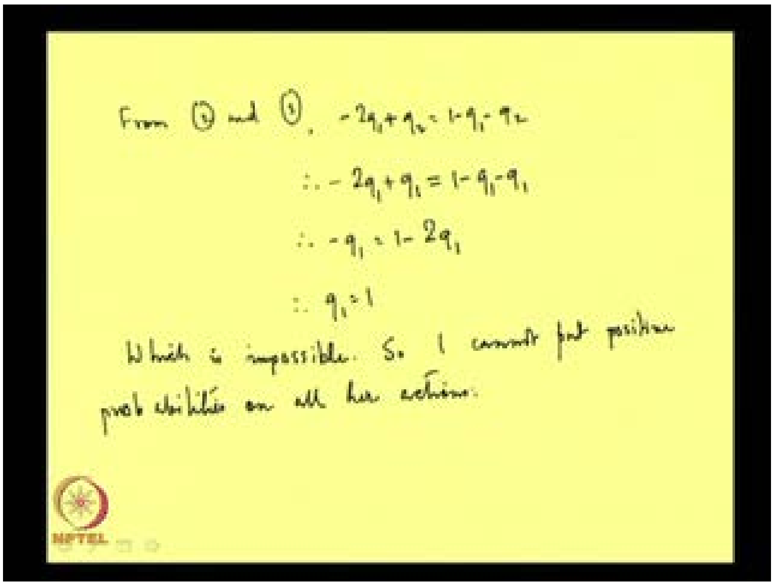

Now, from 2 and 3, what do I get? (Refer Slide Time: 36:15) Mind you, here, we are assuming that player 2 is playing each of this L and M what we are assuming is that player 1 is playing U, M and D with positive probabilities. If she is playing U and M and D with positive probabilities, then their expected payoffs must be equal.

So 1 must be equal to 2 and 2 must be equal to 3. From equality of 1 and 2, we have got q 1 is equal to q 2; from equality of 2 and 3 what do I have? I have minus 2 q 1 plus q 2 is equal to 1 minus q 1 minus q 2 or if I take q 1 is equal to q 2 and so, in place of q 2, I am writing q 1.

Here, I have minus q 1. Here, I have 1 minus 2 q 1, which means q 1 is equal to 1. So, in this case, if q 1 is equal to 1, then q 2 is also equal to 1 and which means that q 1 plus q 2 is going to be greater than 1, which is impossible. So, it can never be the case. So, one cannot put positive probabilities on all her actions. So, this is what we are ruling out that it is not possible that all the actions are having positive probability; there cannot be any such mixed strategy equilibrium.

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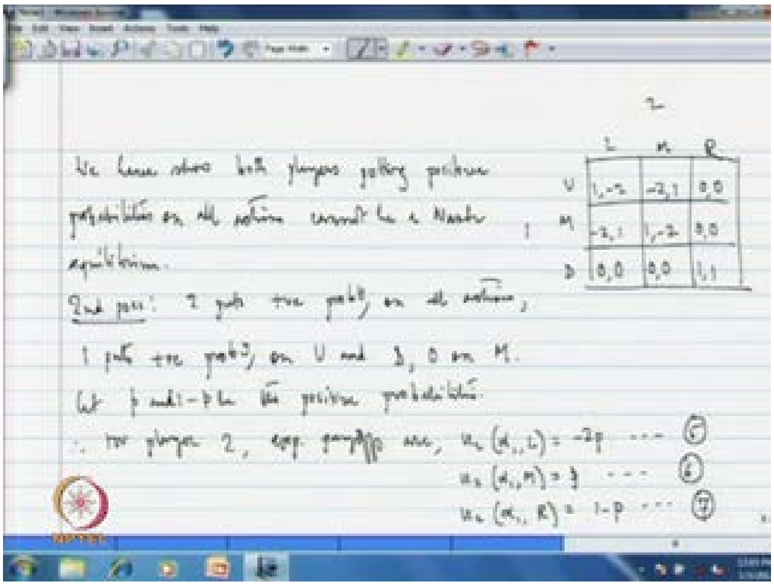

So, this is the game and we have shown both players putting positive probabilities cannot be a Nash equilibrium. The second possibility is that 2 puts positive probability on all actions, 1 puts positive probability on U and D; so, 0 on M. Can this be a Nash equilibrium? Now, if this is the case, then what we are going to have is the following. Remember, in this case, U and D are having positive probability. Let p and 1 minus p be the positive probabilities. So, what we are going to have for player 2. The expected payoffs are and the rest of the formula as shown in the slide. Let us call this 5, 6, 7. It is minus 2p and then you have p and then you have 1 minus p.

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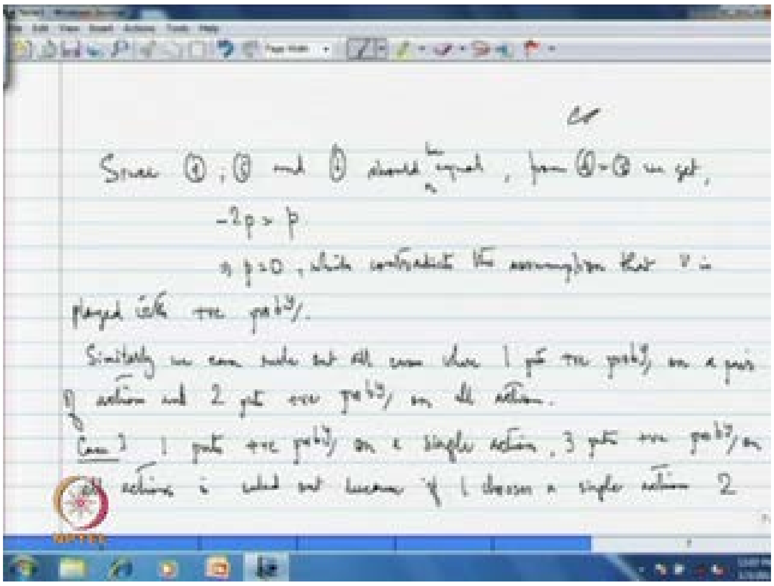

Now, since 4, 5 and 6 should be equal, from 4 equal to 5, we get They should be equal because player 2 is putting positive probability on all these three actions. We shall get minus 2p is equal to p, which means p is equal to 0 and which contradicts the assumption that U is played with positive probability.

So this case is also getting ruled out. Similarly, we can show we can rule out all cases where 1 puts positive probability on a pair of actions and 2 puts positive probability on all actions. Can there be another case? Case 3, where 1 puts positive probability on a single action and 3 puts positive probability on all actions, but this is also ruled out because if 1 chooses a single action, 2 has an unique best response.

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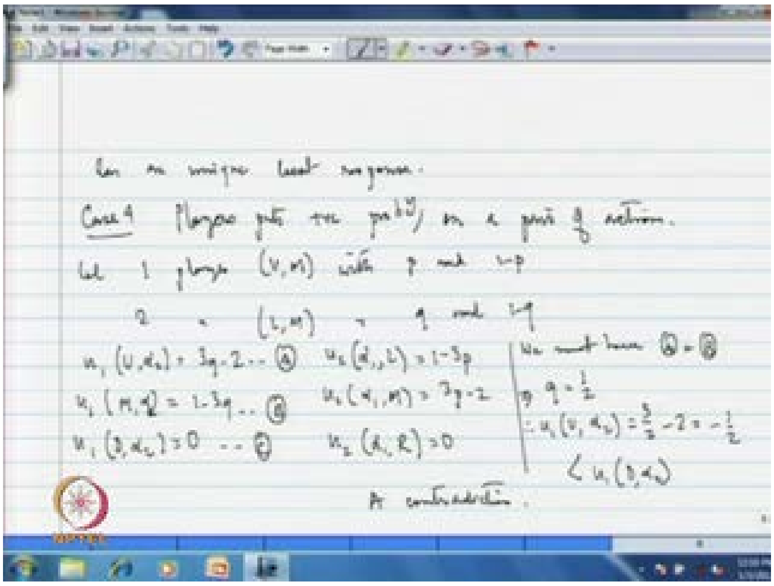

So, all these cases are ruled out. Case 4: players put positive probability on a pair of actions. Can this be a case? Now, let 1 play U, M with p and 1 minus p and 2 plays let us say, L and M with q and 1 minus q. Now, if that is the case then what are the payoffs? In this case, u 1 U, we shall see that it turns out to be 3 q minus 2; this will turn out to be 1 minus 3 q, 0. Let us call it A; let us call it B; let us call it C.

Now, we must have A is equal to B because 1 is putting positive probability on both U and M, but if we equate A and B what we shall get is q is equal to half. This implies q is equal to half, which means that u 1 U alpha 2 is equal to 3 by 2 minus 2, which is minus half and which is less than u 1 D, alpha 2, which is a contradiction.

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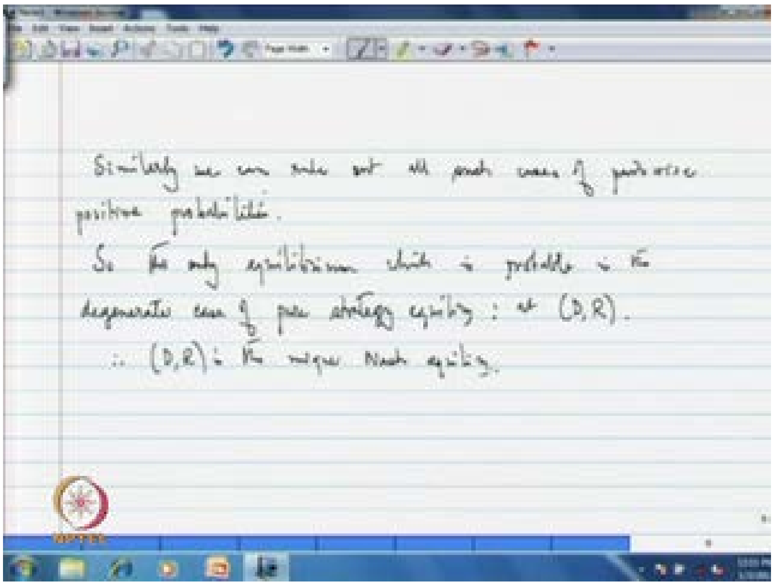

So, this case is being ruled out. Similarly, we can rule out all such cases. These are pair wise positive probabilities. So, the only equilibrium which seems to be probable which is probable is the degenerate case of pure strategy, which we have already seen is at D, R, which is basically meaning that this is the unique Nash equilibrium. That is it.

So, before we end this lecture, let me take you through what we have done in this lecture. What we have essentially done in this lecture is that we have looked into one important aspect of mixed strategy Nash equilibrium that in the mixed strategy Nash equilibrium, the actions for which positive probabilities are put, those actions must be giving the player same expected payoff and for those actions, where the probabilities are 0, the expected payoff can be same as the expected payoff of the other actions or can be less; it can never be more. This characteristic of mixed strategy Nash equilibrium we have seen helps us to check whether a particular mixed strategic profile is Nash equilibrium or not. Thank you.

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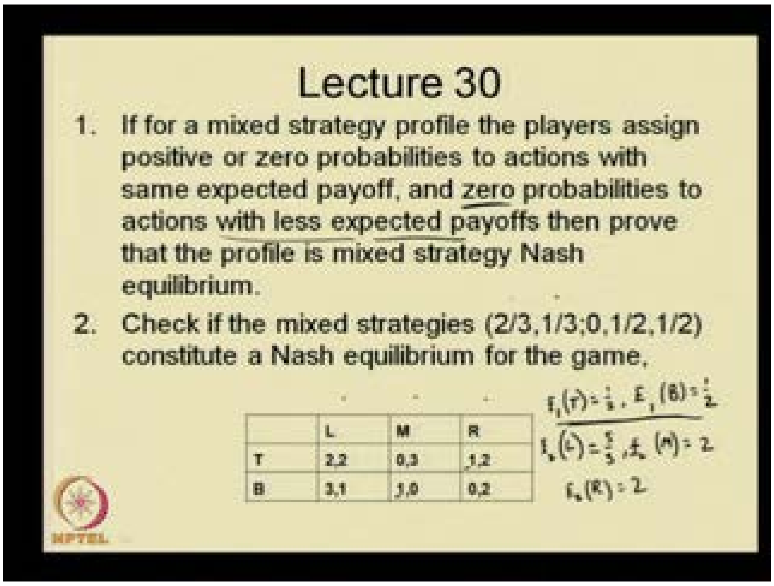

If for a mixed strategy profile, the players assign positive or 0 probabilities to action with same expected payoff and 0 probabilities to actions with less expected payoffs, then prove that the profile is a mixed strategy Nash equilibrium.

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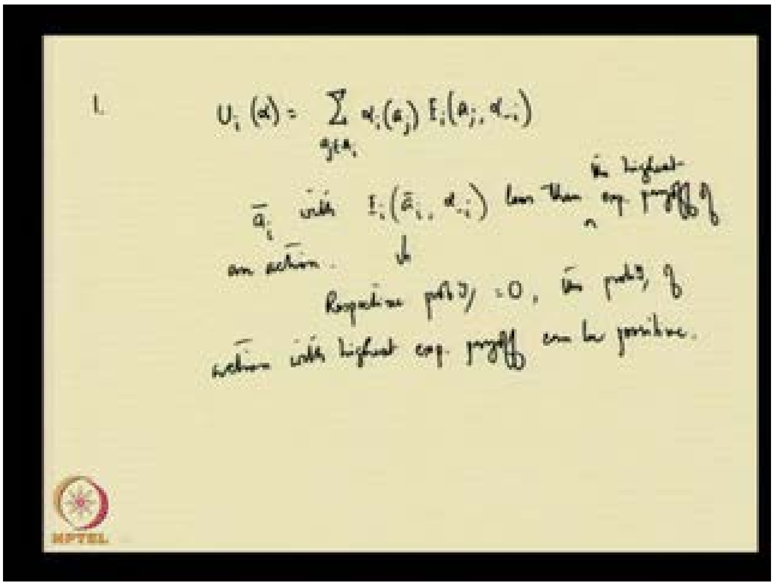

This is what we have done. We have seen before also that we know the following. That the expected payoff of a player, if the mixed strategy profile is alpha is given by the following. It is given by this.

Now, the thing that we have to prove is the following. Players are assigning positive or 0 probabilities to actions with the same expected payoff and 0 probabilities to the actions with less expected payoff. Then prove that the profile is mixed strategy Nash equilibrium.

Suppose, the players are assigning positive probabilities, instead of 0 to actions which have less expected payoff. Now, suppose, there is an action a bar with expected payoff less than expected the highest of an action. Then obviously, if the player is playing optimally then the respective probability has to be less than or it has to be equal to 0, in fact. compare and The probability of actions with highest expected payoff can be positive or it can be 0 also because even if you assign 0 probability to an action, all the other the probabilities which was attached to it, the positive probability goes to other actions, which have the same payoff. So, the total expected payoff remains the same. So, that is optimal.

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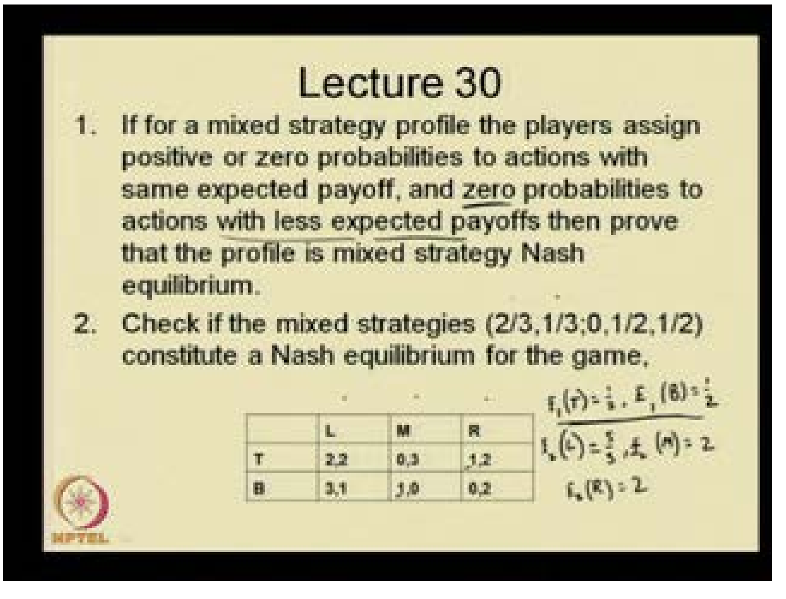

Check if the mixed strategies two-third, one-third; 0, half, half constitute the Nash equilibrium for the following game. Here, we are going to use the result that if you attach positive probability to some action, then the excepted payoff of those actions must be same and if you attach 0 probability to some action, then the expected payoff of that action can be either less or equal to the expected payoff of the other actions.

So, here, let us check if this is true or not. Expected payoff to player 1 that is the row player from T given the other player is playing with this probability. So, what is the expected payoff that this player is getting from T? It is half and from B, again, it is half. This and this we are talking about.

What about player 2? From L, it is getting 5 by 3 - 4 by 3 plus 1 by 3 which is 5 by 3. Yes and this is 2 and this is again 2. Now, obviously, we are satisfying that property that the expected payoffs are equal here and the corresponding probabilities are positive. For player 2, the expected payoff from this and this are equal and here, it is a little less, which is alright. So, that is it. Thank you.