Real – Time Digital Signal Processing
Prof. Rathna G N
Department of Electrical Engineering
Indian Institute of Science - Bengaluru

Lecture – 21
DFT, DTFT, Twiddle Factor, Properties, Circular Convolution and Examples

(Refer Slide Time: 00:34)

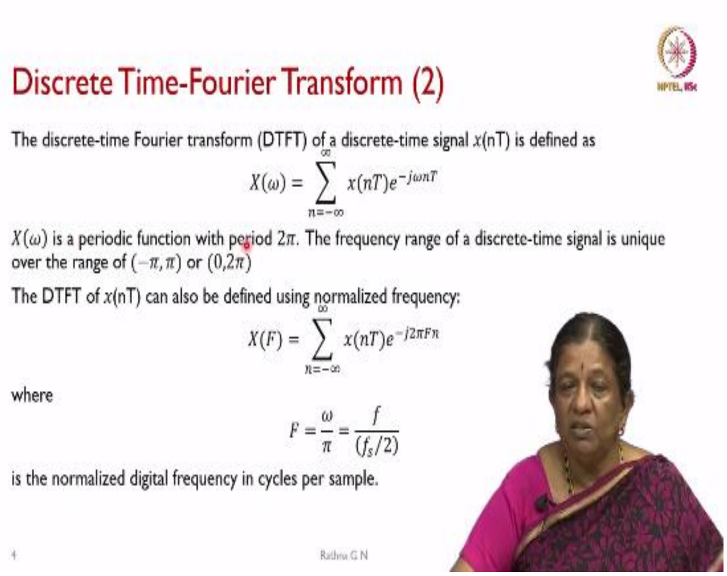
Welcome back to real time digital signal processing course. So, today we will discuss discrete Fourier transform in detail. So, coming to the previous class, so, we covered FIR and IIR filters in the previous module. So, we saw how quantization affects the frequency response for the IIR filter, and then how we have to do the scaling and other parameters what we have to take it into consideration to design our IIR filter.

(Refer Slide Time: 01:01)

So, coming to now, we will go to the frequency domain, why we have to go to the frequency domain? So, first of all what we have to say from analog time domain we have to move to frequency. So, the time domain the components which we are unable to see basically, so, whether the other domain that is in the transform domain, which is going to give us the parameters what we are looking for so, for that we go to the Fourier domain basically that is frequency domain.

So, in this case today we will be discussing about discrete time Fourier Transform first and then we will see that how Discrete Fourier Transform thing is developed. So, that is what, what we are going to tell, that is we introduce the discrete time Fourier transform for the theoretical analysis of our discrete time signals and systems and the Discrete Fourier transform which can be computed by our digital hardware for practical applications.

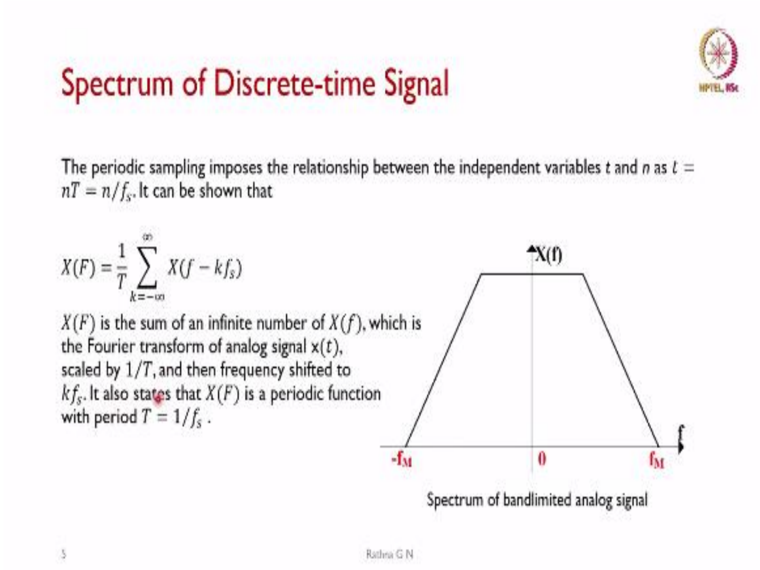
(Refer Slide Time: 02:13)



So, coming to the definition, so, we know that discrete time Fourier transform DTFT is for the discrete time signal x(nT) is defined with this equation that is $X(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$ and we say $X(\omega)$ is periodic function with period π. So, the frequency range of discrete time signal is unique over the range that is -π to π or we can consider 0 to 2π. So, DTFT of x(nT) can also be defined using the normalized frequency.

So, if it take that then it is going to be $X(F) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j2\pi Fn}$ where our F is a frequency is given by $\frac{\omega}{\pi}$, which is nothing but $\frac{f}{(f_s/2)}$, F is the signal what we are interested and f_s is the sampling frequency. So, we say that normalized digital frequency in cycles per sample.

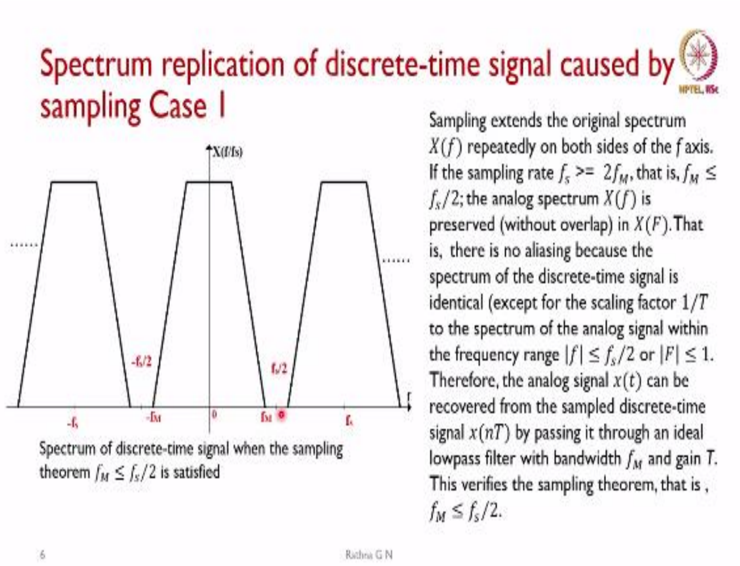
(Refer Slide Time: 03:41)



So, how we are going to represent the spectrum of discrete time signal? So, we said that it is periodic sampling imposes relationship between independent variables t and n as $t = nT = n/f_s$, so, that we can show that $X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - kf_s)$. So, we will be seeing that $X(F)$ is the sum of the infinite number of $X(f)$ what we are going to say, which is the Fourier transform of our analogue signal x(t), it is scaled by $1/T$, and then frequency shifted to kf_s.

So, it also states that $X(F)$ is a periodic function with period $T = 1/f_s$. So, we will be seeing the spectrum here. This is our $X(f)$ what we are considering it and this is the maximum frequency in our input signal varies between $-f_M$ to f_M, and then f is the x axis.

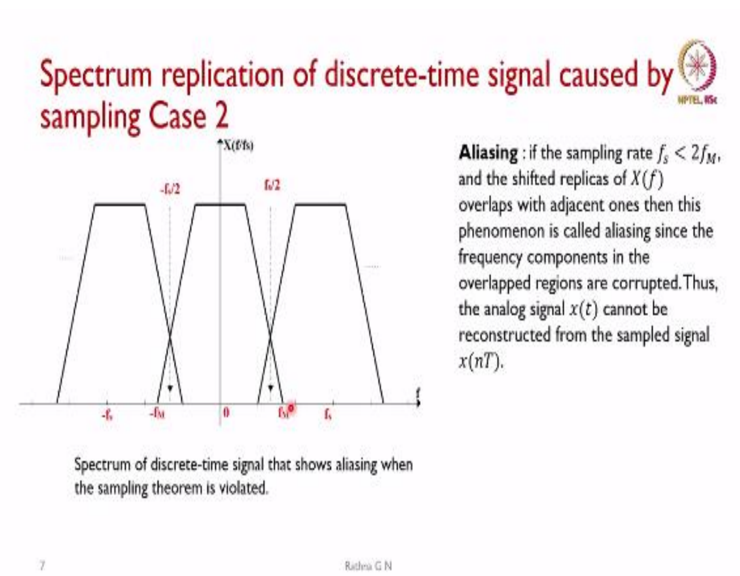
(Refer Slide Time: 05:08)



So, coming to the next one, if why we are going to have the replication? That is discrete time signal caused by sampling, basically. So, this sampling extends original spectrum, whatever we have a consider $X(f)$, repeatedly on both sides. So, we will be seeing this is extended on both sides. So, we call this as $-f_s/2$ and then $f_s/2$. So, we will be getting the images of the spectrum basically. So, what happens if our maximum frequency is less than or equal to our half the sampling frequency according to Shannon sampling theorem.

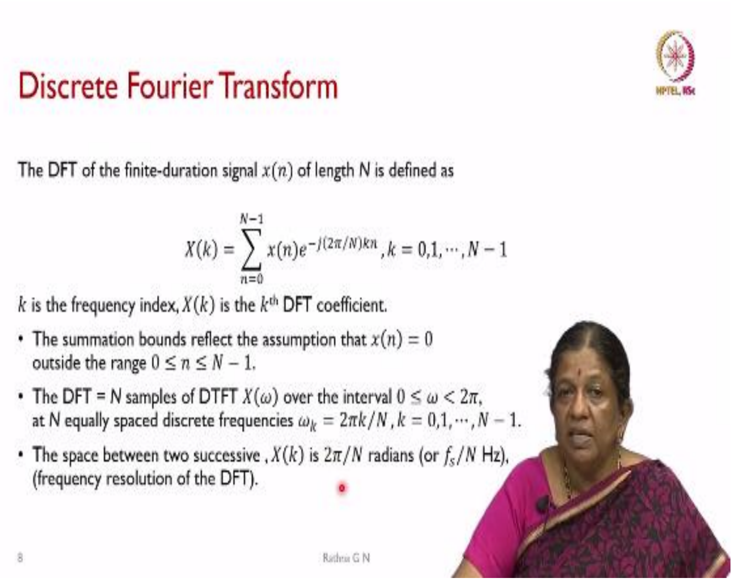
So, then what happens so we will not be seeing any overlap in the thing. So that is what the theory says that our f_M is going to be much away from our $f_s/2$.

(Refer Slide Time: 06:14)



So in the next case, if f_M is what we say is more than our $f_s/2$, or f s sampling frequency is less than twice the maximum frequency component present in our signal, then what we are going to have is we will be seeing that overlaps of it. So, $f_s/2$ is here, $-f_s/2$ is here, our f_M has more than $f_s/2$. So, you will be seeing that we will be getting the aliased signal. So, this we may not be able to reconstruct to the original signal, it may map it to some other signal as we have seen aliasing in our sine wave generation also.

(Refer Slide Time: 07:03)

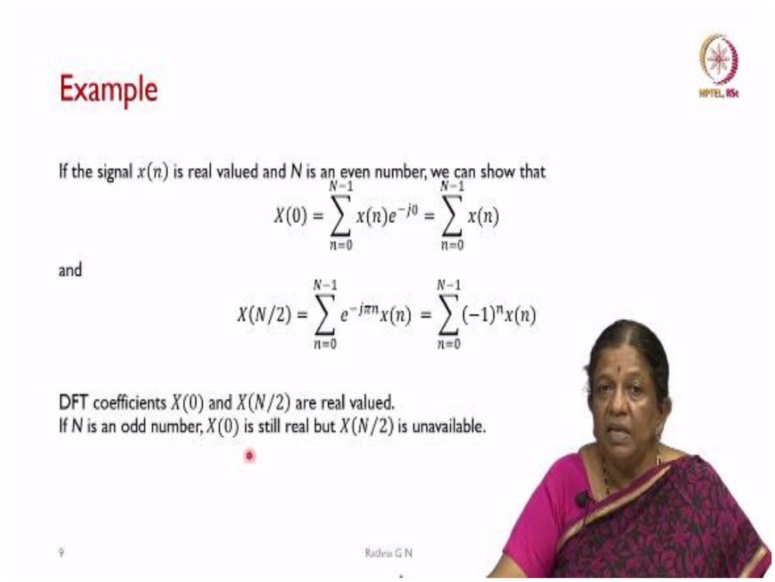


Coming to discrete Fourier transform. So, what we have is it this is we call it as a finite duration. Whereas DTFT was discrete in time, but frequency was continuous in omega, here both frequency and then time have been discretized. So, $x(n)$ of length N is defined as given by $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}, k = 0,1,\cdots,N-1$. And then we will be seeing that n is varying 0 to n - *1*. So, we call k is the frequency index.

So, we will be saying $X(k)$ is the kth DFT coefficient, the summation bonds reflect the assumption that $x(n) = 0$ outside the range 0 to 0 less than or n which is less than or equal to N - *1*. So, we call DFT that is N samples of DTFT of $X(\omega)$ over the interval is ω in between 0 and then 2π at N equally spaced discrete frequencies that is ω_k what we call it $2\pi k/N$ here k is going to be 0 to n - *1* and the space between 2 successive $X(k)$ is nothing but $2\pi/N$, we call it as a resolution of the DFT.

So, we say the unit for this is radians, and then if we represent in terms of sampling frequency it is going to be f_s/N hertz.

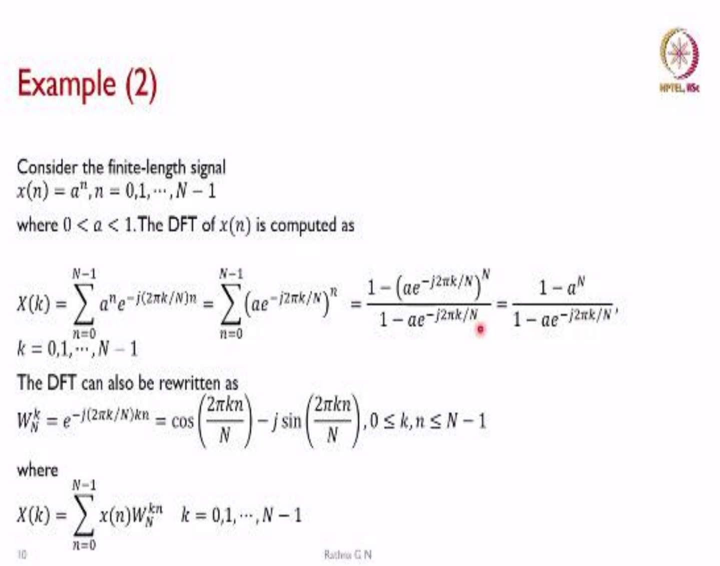
(Refer Slide Time: 09:01)



So, then what we can see that, as an example, if the signal n is real valued and N is even number, so, we can show that $X(0)$ if we substitute that is $n = 0$ to $n - 1$ which is nothing but $x(n)e^{-j0}$, which is nothing but $x(n)$ and then $X(N/2)$ any other N divided by 2 is represented as this way $\sum_{n=0}^{N-1} e^{-j\pi n}x(n)$. So, this is we know that $e^{-j\pi n}$ is nothing but $(-1)^n x(n)$. So, what we observe from this is the DFT coefficients $X(0)$ and $X(N/2)$ are real valued.

That means, if $x(n)$ is real valued then output even the DFT coefficients are real valued. So, if N is an odd number, $X(0)$is still real, but $X(N/2)$ is not available.

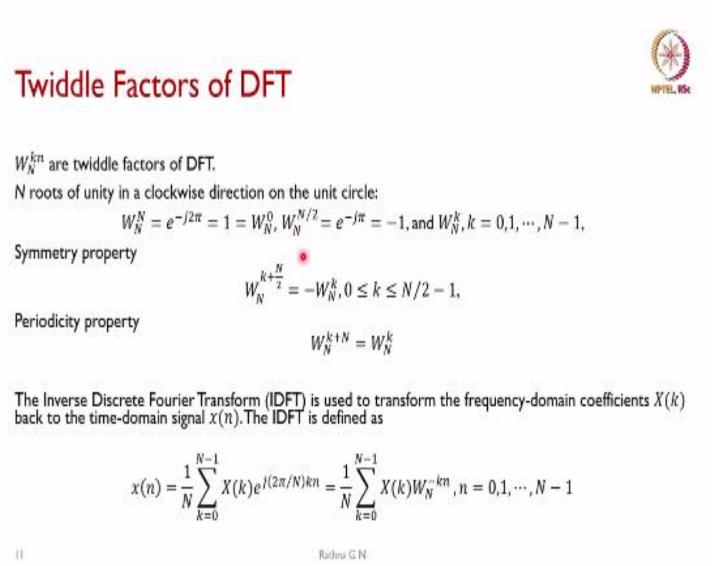
(Refer Slide Time: 10:14)



Now we will consider the sequence or signal of finite length, which is given by $x(n) = a^n, n = 0,1,\cdots,N-1$ and then a is inbetween 0 and 1, then DFT of $x(n)$ is computed as that is $X(k)$, substitute $x(n)$ with $a^n e^{-j(2\pi k/N)n}$. So by simplifying, so what we will be getting is $1-\left(ae^{-j2\pi k/N}\right)$, and k is going to vary between 0 to *n - 1*. So, we can represent this DFT as W_N^k, we call this as a twiddle factor, which is given as $e^{-j(2\pi k/N)kn}$.

So, we know that we can split our exponential into cos and sine function. So, $\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$, $0 \le k,n \le N-1$. So then, if we substitute W_N^k in our equation, so $X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}$ k is varying between 0 to *N - 1*.

(Refer Slide Time: 11:46)

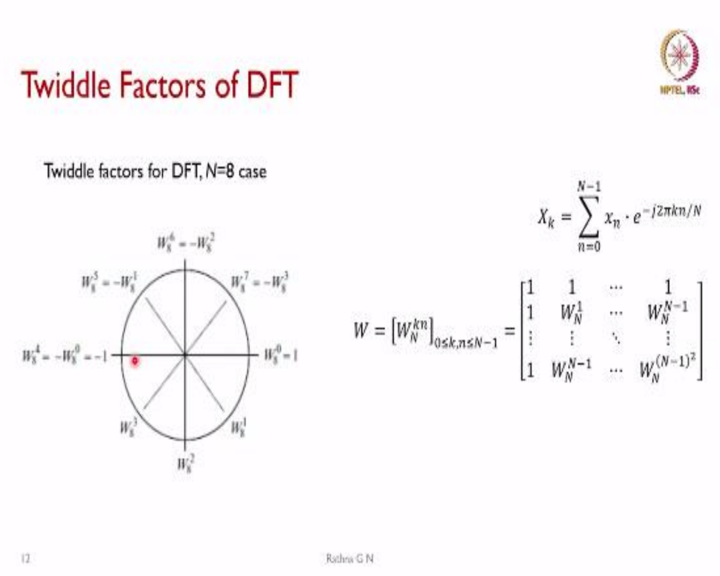


So, that is what, what I said twiddle factors of DFT W_N^{kn} are called, and then we know that *N* roots of unity is a clockwise direction on the unit circle, we will see it in a while hold on. So, then what we call W_N^N is nothing but $e^{-j2\pi}$, which is equal to 1 which is nothing but equal to W_N^0. So, we can see that the DFT is periodic in nature. So, that is the reason why we take 0 to *N - 1* when we substitute $x(n)$is equal to *N* basically, that is *N* is *N*, then what happens.

kn becomes N becomes $e^{-j2\pi}$, which is 1 which is nothing but W_N^0. So, and then the central point what we will check it up $W_N^{N/2}$, which is equal to $e^{-j\pi}$. So, which is nothing but -1 and W_N^k, what we call it between the other values as $k = 0,1,\cdots,N-1$. So, if we extract the symmetry property, then what happens we can split this into 2 parts, that is $W_N^{k+\frac{N}{2}}$ which is equal to $-W_N^k$. So, k is in between 0 to $N/2 - 1$ and if we consider the periodicity property, then what happens $W_N^{k+N} = W_N^k$.

So, we know that the inverse discrete Fourier transform we call it as IDFT. So, is used to transform the frequency domain coefficients $X(k)$back to time domain signal $x(n)$, the IDFT is defined with this equation $x(n)$ is equal to $\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)kn}$. So, if we represent with twiddle factor, it will be $\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}$ in this case, *n* will be varying between 0 to *N - 1*.

(Refer Slide Time: 14:26)

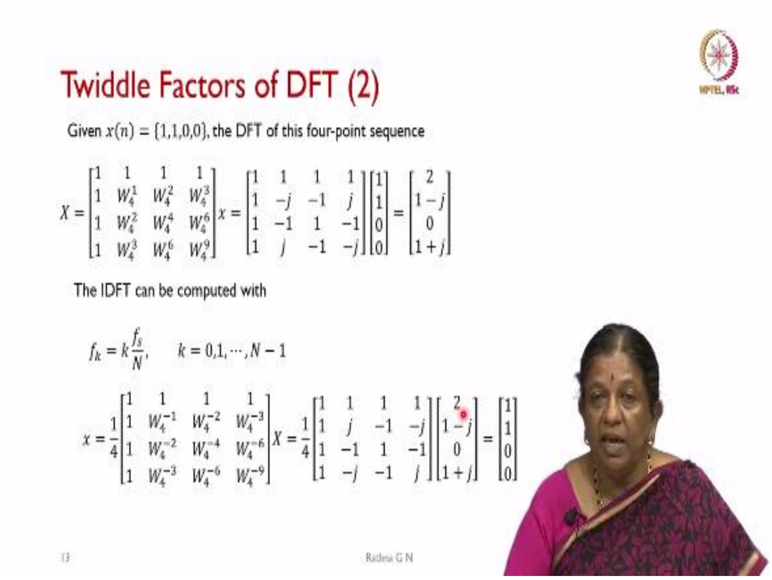


So, we can see that how the twiddle factors of DFT in case of $N = 8$ is assumed, so on a unit circle, so, we will be seeing that this is W_8^0 which is equal to 1 and then what we have is W_8^1 and then W_8^2, W_8^3 and then $\frac{N}{2}$ what we just now saw that W_8^4 which is nothing but minus W_8^0 which is equal to -1. And then we know W_8^5 is equal to $-W_8^1$, and then W_8^6 will be equal to $-W_8^2$. And we know that when we come to W_8^7, it comes out to be $-W_8^3$, you will be seeing that this is what it is $-W_8^3$.

And when it becomes W_8^8, just now we said any W_N^n = W_N^0, here it is W_8^8 which is equal to W_8^0 = 1. So, these are the twiddle factors, depending on power of 2, if you are considering it, it will be on the unit circle this way, and then it is going to be repeated after a period. In this case, we have assumed $N = 8$. So, the equation again repeated for DFT, what it is shown, and then in terms of twiddle factors, W what we call it W_N^{kn} where 0 is less than or equal to k, and then n is less than or equal to $N - 1$, both k and then n are in 0 to $N - 1$.

How we can write the in a matrix form? That is, all these coefficients are first row is 1, and first column is 1. After that, we will have W_N^1. And then the last one in this row is W_N^{N-1}. So that way, what you are represent it. So last row, what you can see, W_N^{N-1} $W_N^{(N-1)^2}$.

(Refer Slide Time: 16:52)



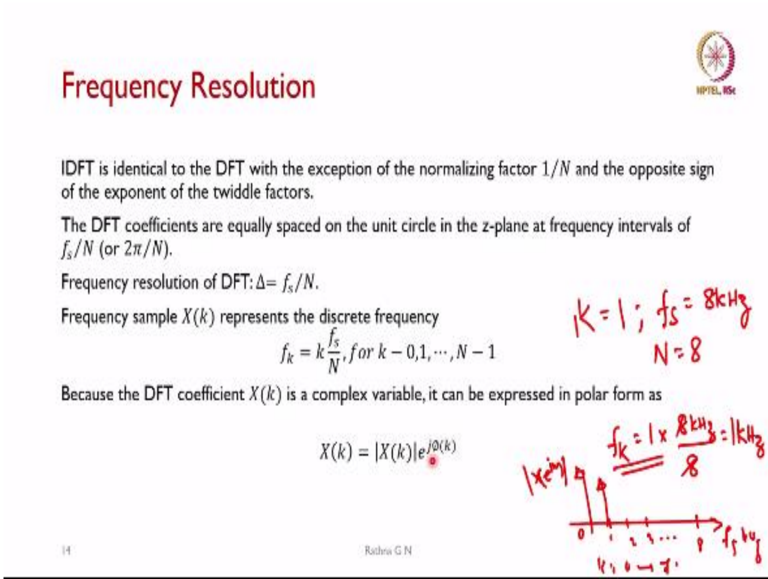
So, just to show that how twiddle factors are represented with W_n weight, we will assume given $x(n) = \{1,1,0,0\}$. So, I think it should be striking to your mind that here we have assumed $n = 4$. So, we will see the DFT 4 point sequence what it is going to be. So, we represent x is equal to all 4 1s here, and then 4 1s in the column, then this is W_4^1, W_4^2, W_4^3, and W_4^2, W_4^4 and W_4^6, so on in the last line all you will be seeing it when you expand them.

So, in terms of e power, then you will be seeing that W_4^1 is nothing but $-j$ and W_4^2 is -1, and W_4^3 is $-W_4^1$, which is $+j$. So, here also you will be seeing the thing, this is how we will enter the thing. And we have been given $x(n)$ is this value. So, this is our coefficient. And then these are our $x(n)$ when we do the matrix multiplication. So, the DFT of the sequence is nothing but 2, $1 - j$, 0 and $1 + j$, to see that whether our DFT is correct or not, we will take the IDFT. So, we assume that $f_k = k\frac{f_s}{N}$, k will be varying between 0 to N - 1.

So, then what happens to our X, which is nothing but $\frac{1}{4}$, that is $\frac{1}{N}$ in this case, so we will substitute our matrix is going to be just our DFT matrix, but you will be seeing that the twiddle factors are negative in this case. So, we will be substituting all this values when we do that. So, $\frac{1}{4}$ this is our DFT coefficients in the IDFT basically. So, you will be seeing that it is a complex conjugate of this DFT coefficients which you will be resulting in. what is it? After multiplication with our $X(k)$ that is 2, 1 – j, 0 and $1 + j$.

So, we will be getting back our $x(n)$ that is 1 1 0 0 so you will be seeing that this is the DFT and IDFT with an example how they are what I will call it as analysis and then synthesis equations what we call them.

(Refer Slide Time: 19:50)



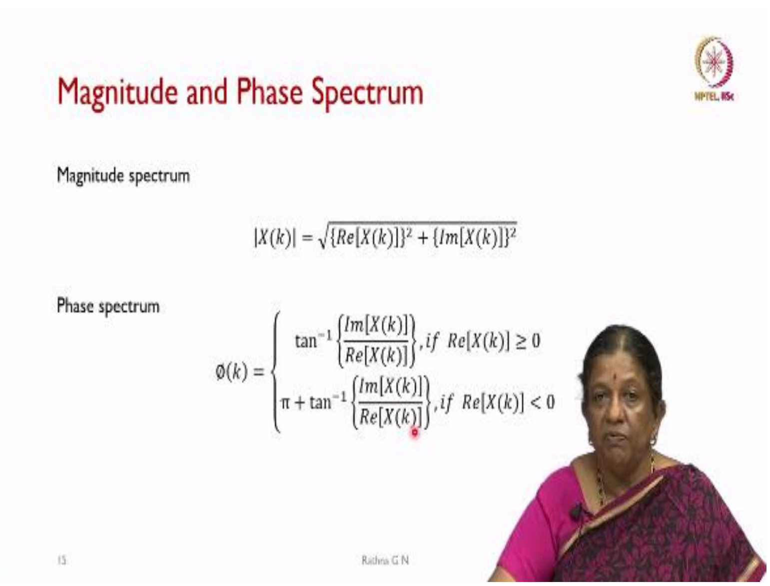
So, what is the frequency resolution in this case? So, we said identical to DFT, with exception of the normalizing factor $\frac{1}{N}$, and the opposite sign of the exponent of the twiddle factors, that is what, what we said. So the DFT coefficients are equally spaced on the unit circle, we saw that usually we call it as the z plane at frequency intervals of $\frac{f_s}{N}$, or it can be $2\pi/N$, because that is what we map in the ω, that is in the frequency domain, 0 to 2π. And then here sampling frequencies f_s up to 2π.

And we say that frequency resolution of our DFT delta is nothing but $\frac{f_s}{N}$ this is the samples, or the period between 2 samples. Frequency sample $X(k)$ represented discrete frequency. So, where f_k is given by $k\frac{f_s}{N}$ for k = 0 to $N - 1$. So, we assume that if I do not know what is the frequency I am getting at the output, if I know that k is equal to I will put it as 1. And we have assumed f_s sampling frequency is 8 kilohertz.

And then I am going to pass it through $N = 8$. To be simpler to calculate this, then what is the frequency f k component in this case is going to be k is 1. And I have chosen 8 kilohertz as my f_s divided by number of samples is 8. So, then you will be seeing that f_k is what I am representing at $k = 1$ is 1 kilohertz. So, if I draw the thing, so this will be my magnitude, x of $x(e^{j\omega})$, what I can put the thing magnitude of it, and this is the f_s in kilohertz what I will put it, and this is my 0, this is 1, 2, 3, I will put it as 8.

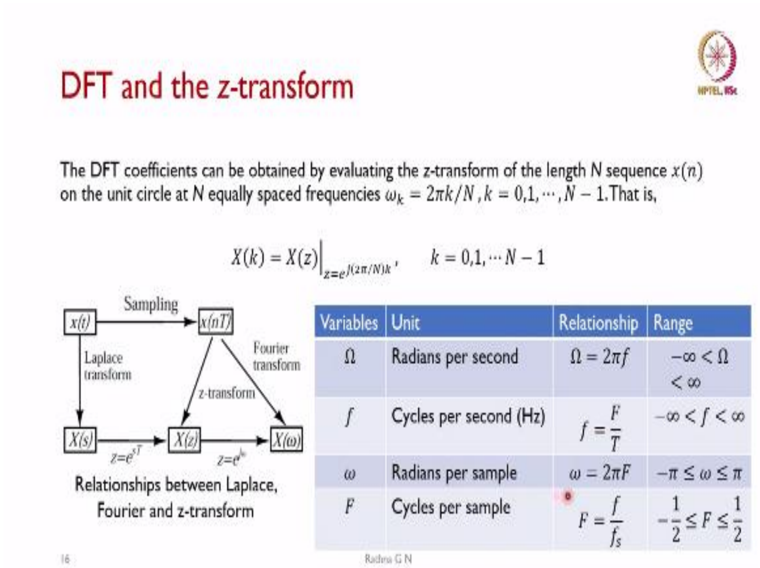
So, I will be getting the peak here, which is my f_k in this case, what I am representing. So, this is how you will be calculating depending on k value is going from 0 to here 7, in this case, 8 will be f_s by 2 point, so it will be repeated. So, this is how you calculate. And then, we know that DFT coefficients X, is complex variable. So, it can be expressed in polar form as $X(k) = |X(k)|e^{j\emptyset(k)}$. So, this represents my phase and this represents my magnitude.

(Refer Slide Time: 23:00)



So that is how we will be representing it here, how we can compute our magnitude and phase spectrum for a given signal. So, the first is the magnitude spectrum, we will be calculating $|X(k)| = \sqrt{\{Re[X(k)]\}^2 + \{Im[X(k)]\}^2}$, so will give me the $|X(k)|$. And then the phase spectrum always be represent in terms of a tan inverse. So, $\emptyset(k) = \begin{cases} \tan^{-1}\left\{\frac{Im[X(k)]}{Re[X(k)]}\right\}, if\ Re[X(k)] \geq 0 \\ \pi + \tan^{-1}\left\{\frac{Im[X(k)]}{Re[X(k)]}\right\}, if\ Re[X(k)] < 0 \end{cases}$.

(Refer Slide Time: 23:58)



So now, how we are going to represent our DFT and then z transform, what is the relationship we will see in this slide. So that DFT coefficients can be obtained by evaluating the N sequence $x(n)$ on the unit circle, we have seen it already at N equally spaced frequencies w_k, which is given by $2\pi k/N$, $k = 0,1,\cdots,N-1$. So, what is it? $X(k) = X(z)|_{z=e^{j(2\pi/N)k}}$, $k = 0,1,\cdots N-1$.

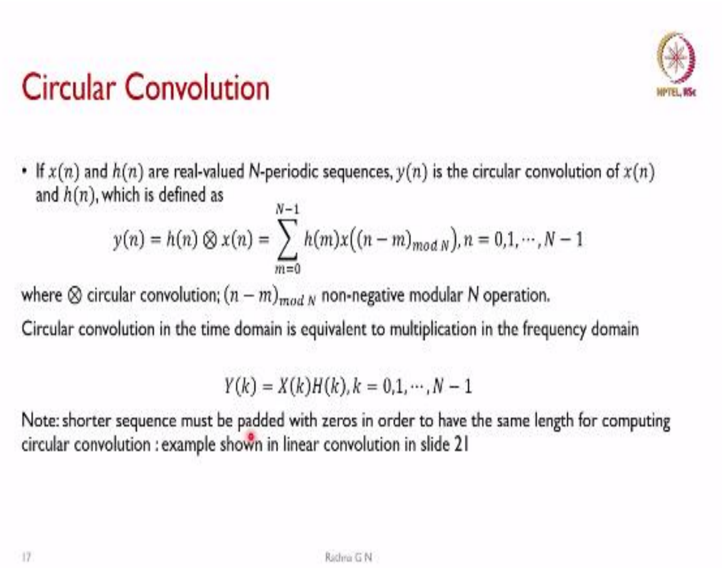
So, some of the terms have will be going from one domain to the other domain, what is shown in this figure, what is it *x* of t is my time domain by doing the sampling of the signal, I will be entering into digital domain that is $x(nT)$. And then if I take the Laplace transform for in the time domain, then I will be going into Laplace domain which is represented as $X(s)$, and then how I can traverse to my digital zee domain by substituting a simplest impulse invariant method that is that is equal to e^{sT}.

I can enter into zee transform basically in the digital domain, or from the digital signal, I can use the z transform to calculate my $X(z)$. And if I calculate the Fourier transform, I will be entering into the Fourier domain that is $X(\omega)$. And then how these 2 are related as you can see, by substituting $z = e^{j\omega}$, I can get the frequency component from zee domain, so the sum of the units and then variables and relationship and ranges shown in this table.

So, if we represent it as Ω, we call it as radians per second. So, we say that $\Omega = 2\pi f$, and this is the range for our Ω that is $-\infty$ to ∞ and f we call it as cycles per second in hertz. So, which is $f = \frac{F}{T}$ which is f is assumed in this case as sampling frequency. As you can see cycles per sample that is $\frac{f}{f_s}$ and then this also varies between $-\infty$ to ∞ and ω, usually we represent it in radians per sample.

And $\omega = -2\pi F$ which varies between $-\pi$ to π ω, or it can vary between 0 to 2π. And this is between $-\frac{1}{2} \leq F \leq \frac{1}{2}$. So, this is what the relationship with respect to z transform and then DFT.

(Refer Slide Time: 27:07)



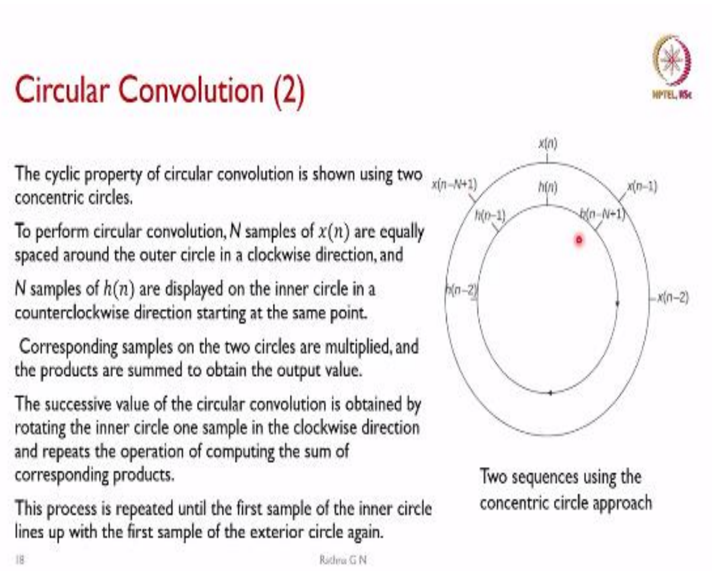
So, now we will assume that how to go with the; we said that DFT is a periodic function, we will see how to calculate the circular convolution. So, all of you must be conversant with your linear convolution and circular convolution, usually DFT will be represented as circular convolution. So, we have $x(n)$ and $h(n)$ are real valued N periodic sequences, y of n is a circular convolution of $x(n)$ and $h(n)$. So, which is represented $y(n) = h(n) \otimes x(n)$.

Which is given as $h(m)x((n-m)_{mod\,N})$ N what we will be taking it that means to say $(n-m)_{mod\,N}$ non negative modular N operation will be considering it n is varying between 0 to $N - 1$ in this case. So, how we represent in the, this is what we have in the time domain and in the frequency domain. So, it results in the multiplication of 2 Fourier transform of that is discrete

Fourier transform of x and then h. So, $Y(k) = X(k)H(k)$, where k will be varying between 0 to $N - 1$.

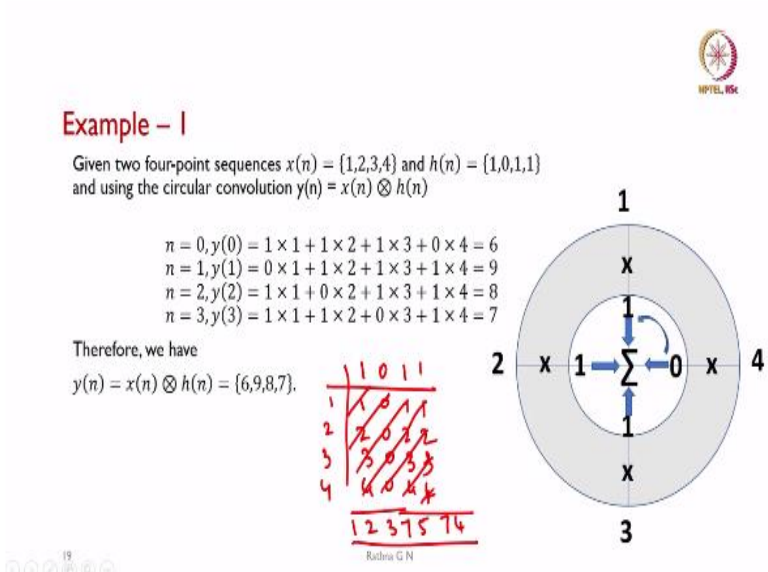
So, that is what, what it says if the shorter sequence must be padded with 0s in order to have the same length for computing circular convolution. So, what do we mean by that, so, $X(k)$ and $H(k)$ are of different length. So, to make them equal length, one of them had to be padded with 0s. So, we will see that linear convolution in slide 21 so, how we have padded with 0s and then made power of 2 and then used in our circular convolution.

(Refer Slide Time: 29:09)



So, how we are going to compute our circular convolution, so, usually it is represented with 2 concentric circles. And we will be seeing that this goes in the clockwise direction $x(n)$ and $h(n)$ are aligned actually, then we will be moving $h(0)$ in this direction $n - N + 1$ and then so on. And then next last one will be $h(n-2)$ $h(n-1)$, whereas you will be seeing that $x(n)$ goes in this direction anticlockwise, $x(n-N+1)$ and then we will be coming to $x(n-2)$ and $x(n-1)$. So, this is how we will represent. We will see with an example how we are going to compute this.

(Refer Slide Time: 30:01)



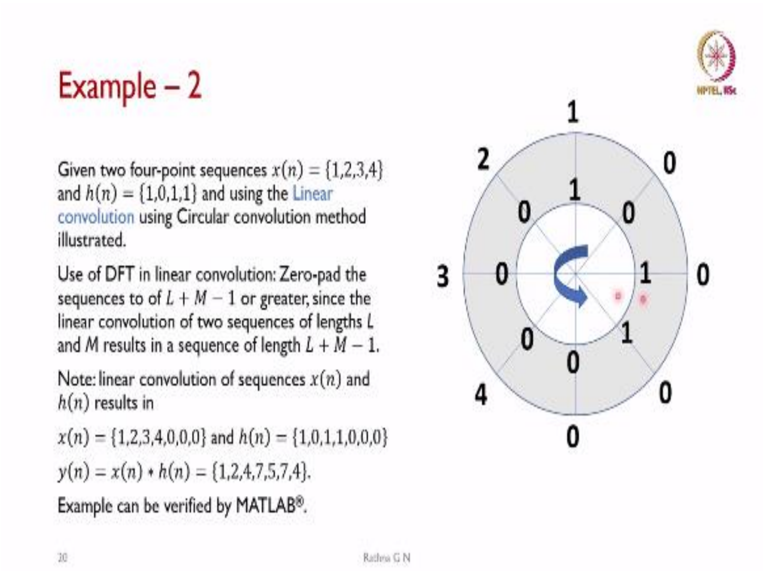
So, as an example, as I have been mentioning $x(n)$ is given as 1 2 3 4 and our h of n is 1 0 1 1. In this case, as you can see, both are of the same length we will do the circular convolution. So, the steps have been written here. So, how to arrive at these steps using our concentric circles we will see the thing. So, my $x(n)$ is 1 2 3 4 it is written in this way anti clockwise and then $h(n)$ is 1 0 1 1 is in the anti clockwise sorry, this is in the clockwise the $x(n)$ is in the anti clockwise.

o, when I do the first time, we have aligned the $x(n)$ and then $h(n)$ together and we multiply these 2 numbers and then all of them we multiply you will be seeing 1×1 is 1, 2×1 is 2, 0×4 is 0, and then here 1×3 is 3. So, when you add it up, this is what the step what it is given at $n = 0, y(0)$ is given by $1 \times 1 + 1 \times 2 + 1 \times 3 + 0 \times 4 = 6$. So, then what we are going to do is, I can move this as the arrow shows 1 step each time my $h(n)$ in this thing, and I keep this one as it is.

So, when I do that next value is going to be $0 \times 1 + 1 \times 2 + 1 \times 3 + 1 \times 4 = 9$. So, this is 6 9 8 7 when I do the circular convolution. So, you will be seeing that how we can implement linear convolution just if you have done the thing. So, just I will write it here most of you would have used this method to compute your linear convolution. So, when I put the thing, all of us know that we put $x(n)$, I can put it here 1 2 3 4.

This is 1 0 1 1 what I can represent, then I will be multiplying with the 1 into this number, this is 2 0 2 2 and then 3 0 3 3, 4 0 4 4. So, how we are going to get the thing, so, you will be knowing that this is the way we will be adding it up and then result in the linear convolution. So, you will be seeing that this is 1 2 3 7 and then 5 7 and then 4, this is the equivalent of linear convolution. So, whereas circular convolution, you can see that 6 9 8 7, can I use this method to implement linear convolution that is what we will check in the next slide.

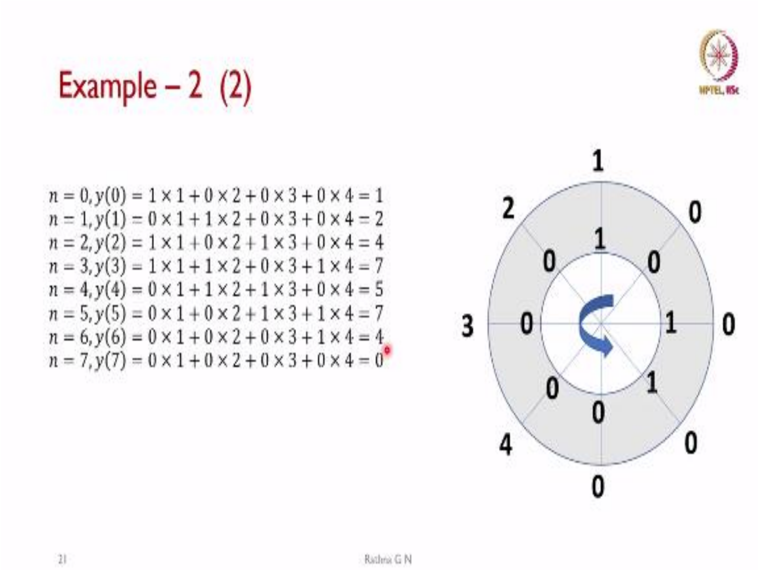
(Refer Slide Time: 33:09)



So, you will be seeing that, what we have done is we have these are the 2 values. So, that is for the linear convolution using circular convolution, we have to do 0 pad of $L + M - 1$ is the length of the sequence what the result is going to be. So, we have to pad both of them with these 0s $L + M - 1$, L is the length of $x(n)$, M is the length of $h(n)$. In this case, we have both of them are equal 4 + 4, 8 – 1, 7 will be the length, what we have is 4 lengths, so we will be padding with 3 zeros. And then the other one also will be padding with 3 zeros.

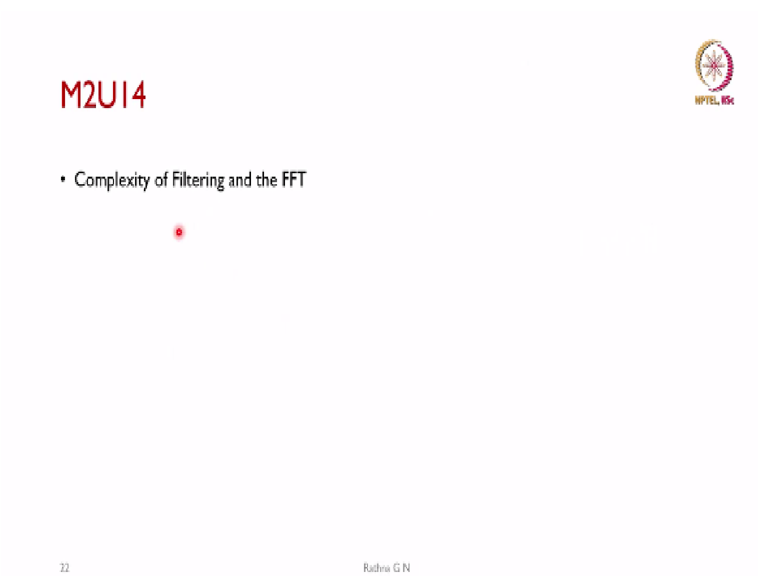
So, now we have although we have the equal length, to simplify, if I want to compute power of 2, then what I have to do is I had to make them 8, as you will be seeing that 2 concentric circles I have divided into 8 parts. So, we will be putting 1 2 3 4 and then 4 zeros, what will be padding instead of see 3 zeros. Here also we will be doing 1 0 1 1 and 4 0s same way as the previous one, you can now shift the thing the resulting value with circular convolution, you will be seeing that 1 2 4 7 5 7 4 is achieved, which can be evaluated verified using MATLAB.

(Refer Slide Time: 34:42)



Or the next slide shows that how it has been implemented. So, I showed you how it has been done in the previous example, same way, if you do the thing, you will be resulting with this value the last one will be 0 which you can discard and then keep these $L + M - 1$ values for your result.

(Refer Slide Time: 35:06)



So, you can see that we have taken little bit of DFT in to show that circular convolution property, other properties, you can look into the book and then come out with it. In the next class, we will be seeing the complexity of filtering and how we will be deriving from DFT, FFT equation. Thank you.