Mathematical Aspects of Biomedical Electronic System Design
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Lecture No. 09
Discrete Time Fourier Transform and Sampling

Hello, everyone, welcome to the fourth lecture of the course mathematical aspects of biomedical electronic system design.

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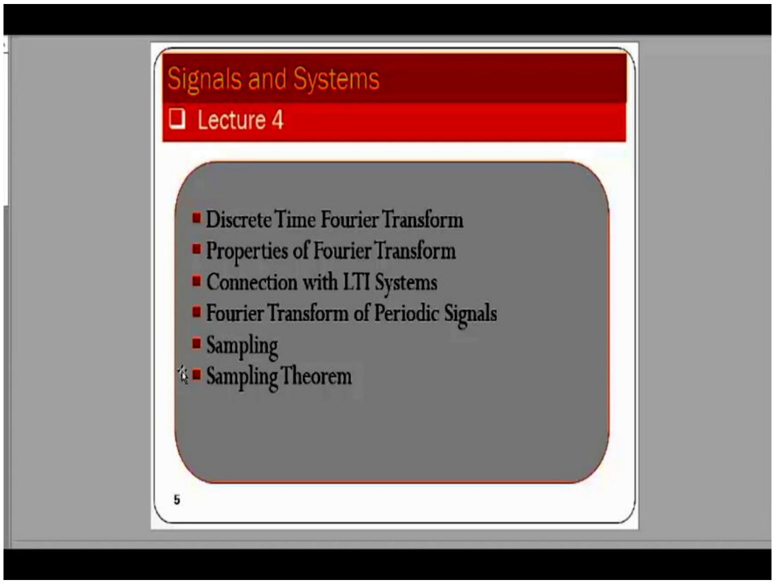

In this lecture, we will cover discrete time Fourier transform, properties of Fourier transform, connection with LTI systems, Fourier transform of periodic signals. Moreover, we will also introduce sampling and we will see sampling theorem. So, let us begin with the lecture.

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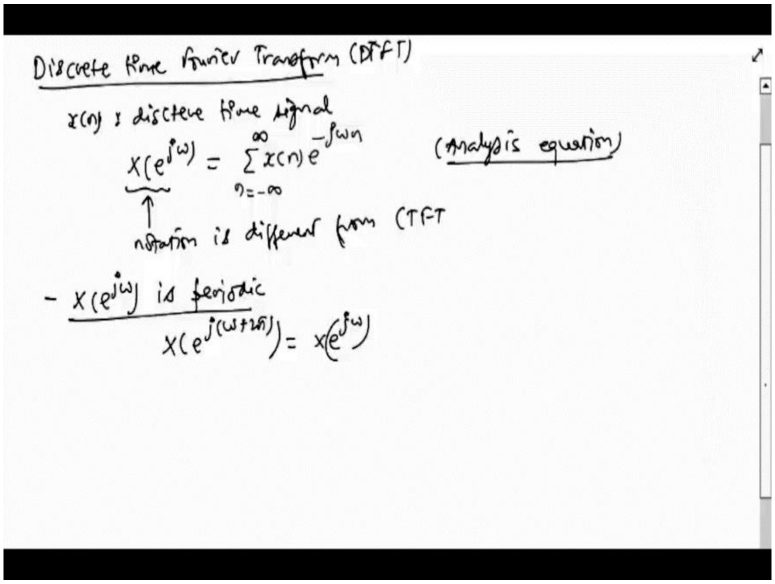

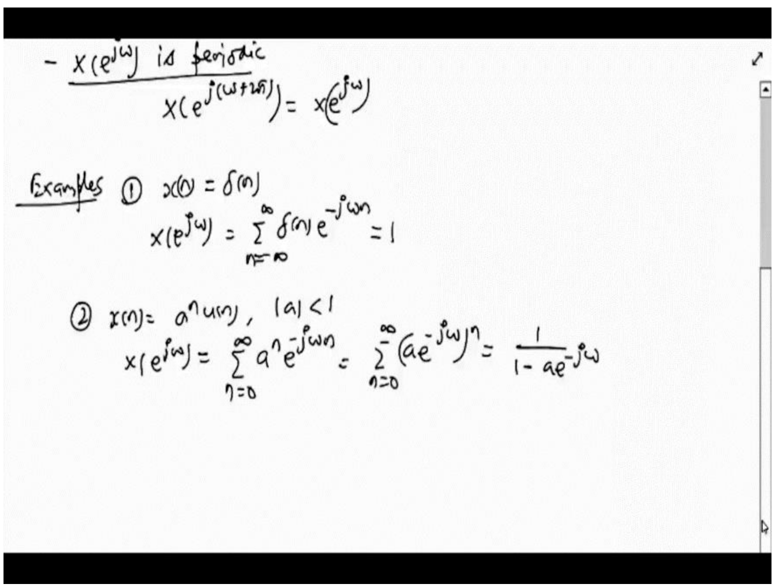

We will start with defining discrete time Fourier transforms. Recall that, discrete time Fourier series are defined only for periodic signals. However, like continuous time Fourier transform, discrete time Fourier transform in short DTFT are also defined for a periodic signals. Let us see what these are. DTFT is for a discrete time signal, x(n) is defined as x in an discrete time signal.

Its DTFT is defined as

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Notice that, unlike continuous time fourier transform where we used x(ω) to denote fourier transform, here we are using a slightly different notation. I am calling the DTFT $x(e^{j\omega})$, so this notation over here, notation is different from continuous time Fourier transform.

There is a feature that distinguishes this discrete time Fourier transform from continuous time, and that is this Fourier transforms are periodic. So, what I mean that, $x(e^{j\omega})$ is periodic, with period 2π. So,

$$x(e^{j(\omega+2\pi)}) = x(e^{j\omega})$$

This is something that is easily, can be easily verified by using the definition of Fourier transform. As in continuous time case, this equation is called analysis equation.

Let us see a few examples. Example 1 is, the x(n) is dirac delta function, that is it is δ(n). Then what about $x(e^{j\omega})$? If you just apply the definition, it becomes

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n} = 1$$

Let us see another example. Now, I take x(n) = a^nu(n), where $|a| < 1$.

Now let us see what DTFT is, $x(e^{j\omega})$ is equal to, since u(n) = 0 for all negative n, I can start the summation from 0,

$$x(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

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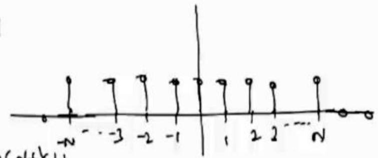
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

③ $x(n) = \begin{cases} 1 & \text{if } |n| \leq N \\ 0 & \text{o/w} \end{cases}$



$$X(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n} = \begin{cases} \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\omega/2)} & \text{if } \omega \neq 0 \\ 2N+1 & \text{if } \omega = 0 \end{cases}$$

Inverse Fourier Transform

$$x(n) = \frac{1}{2\pi}\int_{2\pi}$$

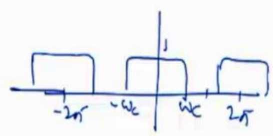
$n=-N$ ... $2N+1$ if $\omega = 0$

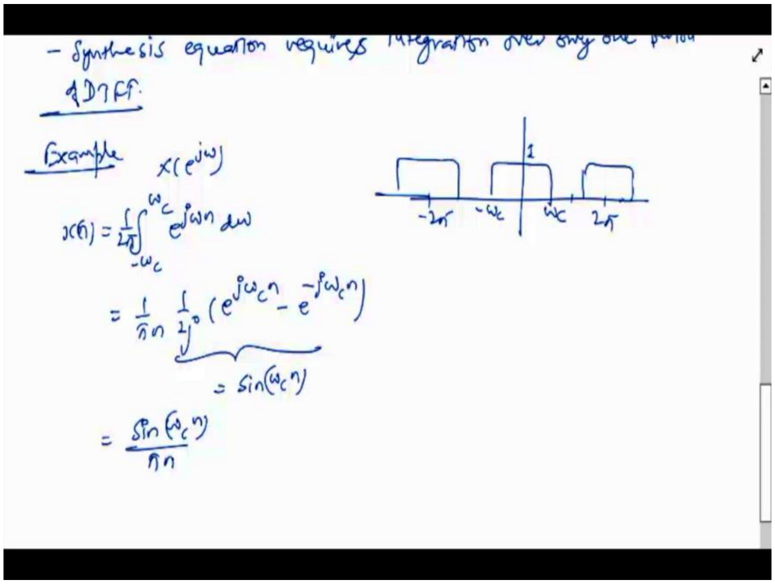
Inverse Fourier Transform

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{(synthesis equation)}$$

- synthesis equation requires integration over only one period of DTFT.

Example $X(e^{j\omega})$





Let us see one more example. Now my $x(n) = \begin{cases} 1, & |n| \leq 0 \\ 0, & otherwise \end{cases}$. So, if I plot this signal, it would look as follows, look like follows 1 2 3 dot, dot, dot up to N. Here, -1 -2 -3 dot, dot, dot, up to -N. Signal is 1 at all these points. And it is 0 outside this. So, what about the Fourier transform of this signal? Let us see, $x(e^{j\omega})$.

Now, I only need to $\sum_{n=-N}^{N} e^{-j\omega n}$. Because the value is 1 in this range. And if we solve, it is easy to do, it turns out to be $\frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$, if $\omega \neq 0$, and it is simply $2N + 1$, if $\omega = 0$. So, having seen these examples, the next question is, if we are given Fourier transform of a signal, how can we retrieve the signal from this Fourier transform? So, that is how do we obtain so called inverse Fourier transform of a signal.

So, let us see, just like continuous time case, we use synthesis equation for this purpose. So, next we are looking at inverse Fourier transform. So, for a signal, for a, for a fourier transform $x(e^{j\omega})$ its inverse is obtain using the following equations,

$$x(n) = \frac{1}{2\pi} \int_{2\pi} x(e^{j\omega}) e^{j\omega n} d\omega$$

So, as in continuous time case, this equation is called synthesis equation.

Again, notice one difference visa-vise continuous time case, that is, the synthesis equation requires integration work only over one period. Notice that, 2π was the period of the Fourier transform. So, synthesis equation requires integration over only one period of DTFT. Let us see an example. Suppose, $x(e^{j\omega})$ is a periodic pulse function. So, that is $x(e^{j\omega})$, the following function.

It is a function, that takes value 1 between $-\omega_c$ and ω_c. And where $\omega_c < \pi$, so π will be somewhere here and then I have, then this pulse repeats. So, 2π and so on. So, what about the inverse Fourier transform of this? How do we obtain x(n)? As I said above, we only need to integrate the range in the, in over one period. So, here we can do it by integrating from $-\omega_c$ and ω_c.

So, let us see $-\omega_c$ and ω_c and in this range the value is 1, so I have $e^{j\omega}$. And at the front, we have $\frac{1}{2\pi}$, if we solve it, it turns out to be

$$= \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

which we can readily this thing we can readily recognize to be sin $(\omega_c n)$. So, the signal is

$$= \frac{\sin\,(\omega_c n)}{\pi n}$$

So, now, having seen the inverse Fourier transform as well, we will now look at a few properties of discrete time Fourier transform.

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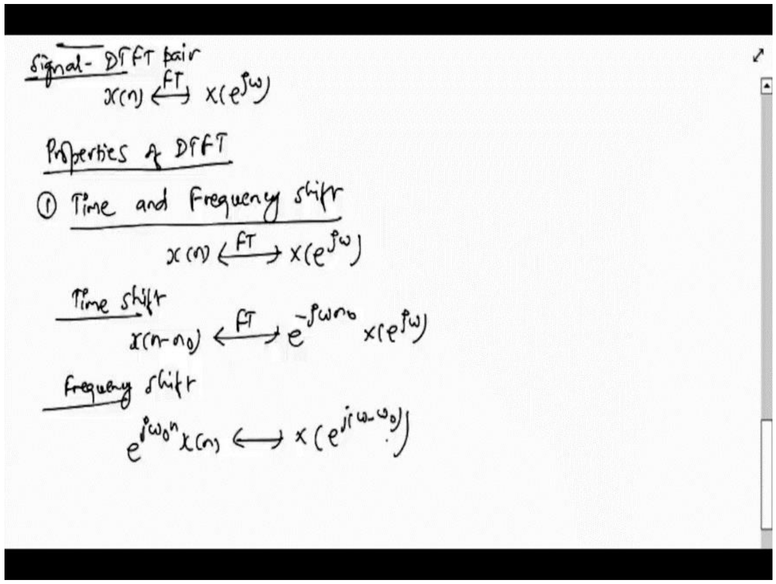

While stating these properties, I will use the same convention that I used for continuous time case, that is for any function and its fourier transform pair I will use the following notation x(n) is the signal, and its fourier transforms $x(e^{j\omega})$. So, this is signal the DTFT pair. So, with this convention, let us look at properties of the DTFT. The first property that we will see is time and frequency shift.

We will see that, all these properties are counterparts of similar properties in continuous time case. So, similar duality observation holds here as well, time and the frequency shift. So, if x(n) had fourier transform $x(e^{j\omega})$, then the time shift property says that, fourier transform of

$$x(n-n_0) \overset{FT}{\leftrightarrow} e^{-j\omega n_0} x\left(e^{j\omega}\right)$$

Similarly, the frequency shift property says that Fourier transform of a

$$e^{j\omega_0 n} x(n) \leftrightarrow x\left(e^{j(\omega-\omega_0)}\right)$$

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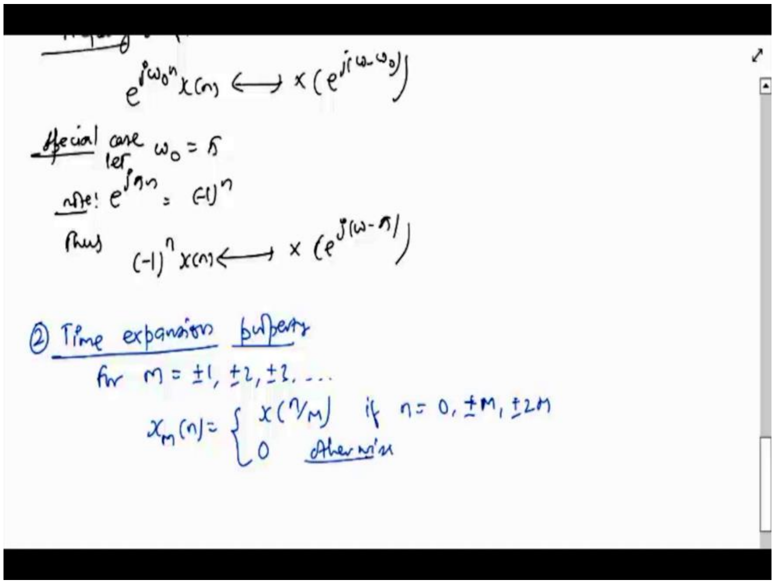

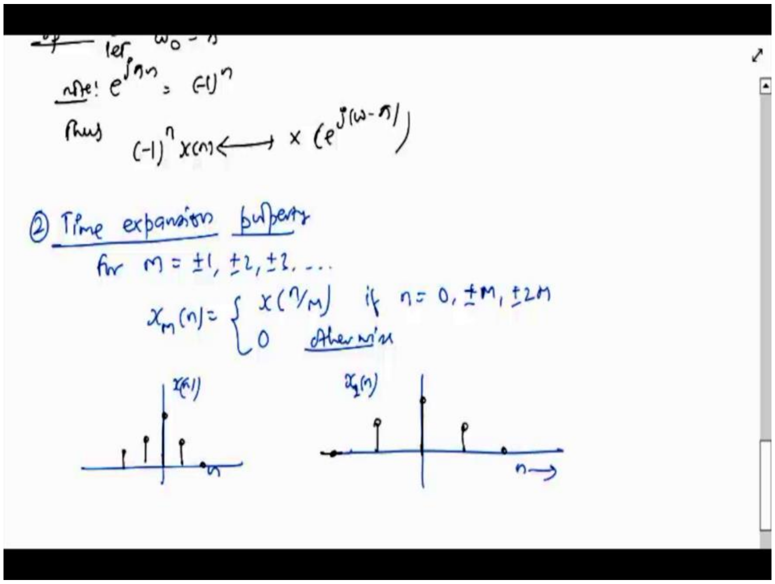

Let us see a special case of the frequency shift property, which is often of interest special case. Here I set $\omega = \pi$, notice that, note that $e^{j\pi n} = (-1)^n$. Thus, from the above frequency shift property, we get that Fourier transform of

$$(-1)^n x(n) \leftrightarrow x\left(e^{j(\omega-\pi)}\right)$$

So, these were time and frequency shift properties. The next property, that we will see is time expansion property.

Expansion property, before I state the time expansion property, let me define what it means by an expanded signal. So, for any nonzero integer m, say m =±1, ±2, ±3 etcetera, so the signal

$$x_m(\text{n}) = \begin{cases} x(\frac{n}{m}), & if\ n = 0, \pm m, \pm 2m, etc \\ 0, & otherwise \end{cases}$$

Well, so x_m(n) is non zero only if $\frac{n}{m}$ is an integer. Let us see an illustration of this. So, let us say my x(n) is as follows.

Then, the expanded signal, say twice expanded signal x to n will look like as follows. So, it will have non zero values only at even values of n. So, I will rub this, this, this, this and all other positions it is value will be 0.

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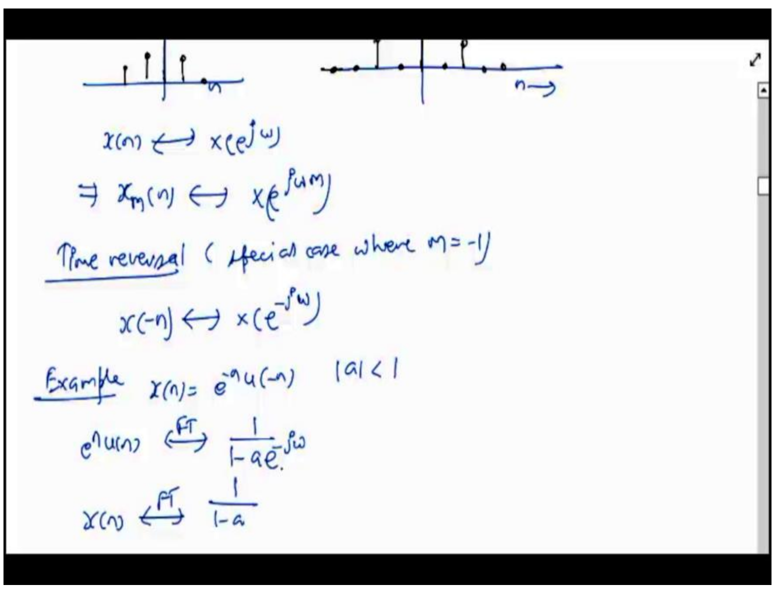

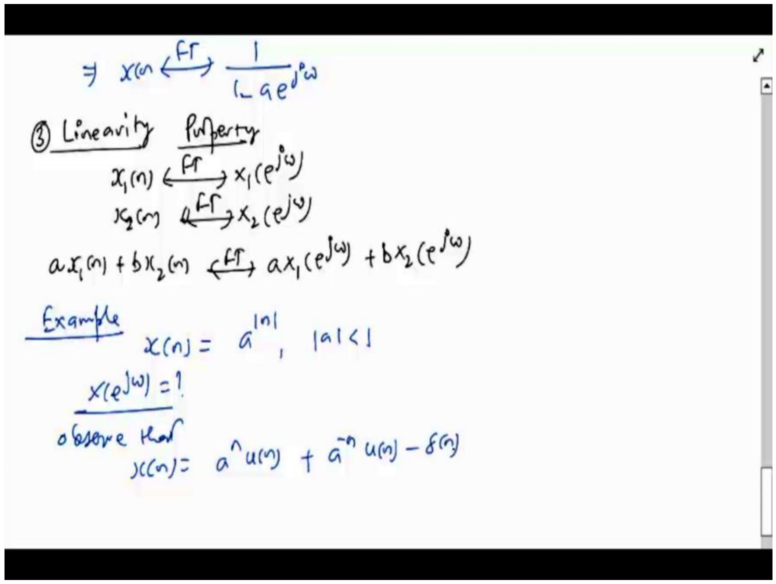

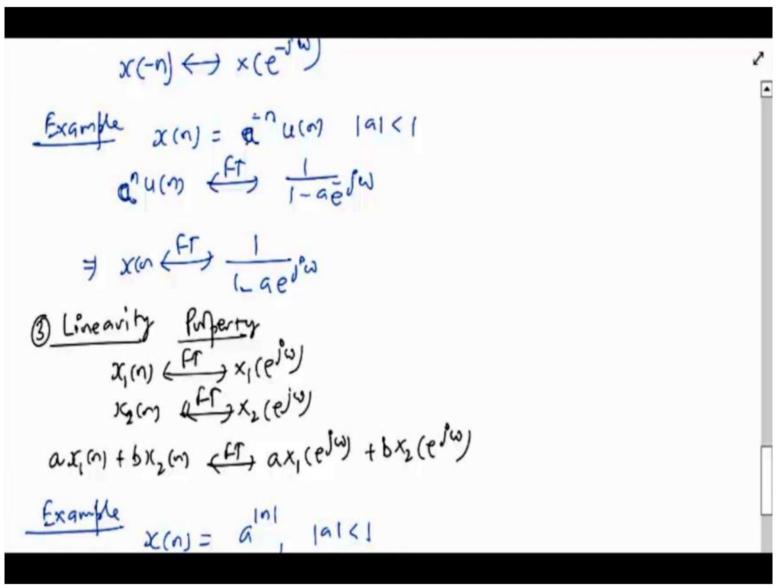

Now, time expansion property says that if x(n) has Fourier transform $x\left(e^{j\omega}\right)$, then $x_m(n)$ will have Fourier transform $x\left(e^{j\omega m}\right)$. Again, an special case of this property that is quite useful is what we call time reversal property, time reversal. This is a special case, when, where M, M=-1. Now, time reversal property says that Fourier transform of x(-n) will be $x\left(e^{-j\omega}\right)$.

So, let us see an example illustrating Fourier, illustrating time reversal. Let us consider

$$x(n) = e^{-n}u(-n)$$

where as before $|a| < 1$. We have readily seen that, the Fourier transform of

$$e^n u(n) \leftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

So, using time reversal property, Fourier transform of $x(n) \leftrightarrow \frac{1}{1-a}$. Now, $-j\omega$ will be replaced with $+j\omega$, $-\omega$ will be replaced with $+\omega$.

So, this is time reversal property. After time reversal property, time action, expansion property, the next one is linearity property. So, number 3, is linearity property. So, this property is similar to its counterpart, in continuous time case. For instance, let us assume that $x_1(n)$ has discrete time Fourier transform $x_1(e^{j\omega})$. And $x_2(n)$ discrete time Fourier transform $x_2(e^{j\omega})$.

Then linearity property says that, for some numbers a and b, $ax_1(n) + bx_2(n)$ will have fourier transform $ax_1(e^{j\omega}) + bx_2(e^{j\omega})$. Let us see an illustration of this property with an example. So, let us say that x, let us say that we want to evaluate Fourier transform of $x(n) = a^{|n|}$, where again mod, $|a| < 1$. Observe that, so the, what we wanted is $x(e^{j\omega})$.

Observe that,

$$x(n) = a^n u(n) + a^{-n} u(n) - \partial(n)$$

Now, we know the Fourier transforms of each of these. Notice that, we just computed the Fourier transform of. So, this is a, this is a we just computed Fourier transform of $a^{-n}u$. We had earlier computed Fourier transforms of $a^n u(n)$ and $\partial(n)$ respectively. So, $x(e^{j\omega})$ can be obtained by combining appropriately, combining the Fourier transform of the individual functions.

So, it becomes

$$x(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{j\omega}} - 1$$

And this can be simplified, to be

$$= \frac{1 - a^2}{1 + a^2 - 2acos\omega}$$

I will skip the details, but it is something that is can be easily seen. So, this was an illustration of the linearity property.

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$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}} - 1$$

$$= \frac{1-a^2}{1+a^2-2a\cos\omega}$$

④ Conjugation and conjugate symmetry properties

$$x(n) \xleftrightarrow{FT} X(e^{j\omega})$$

then $x^*(n) \xleftrightarrow{FT} X^*(e^{-j\omega})$

* If $x(n)$ is real valued then

$$x(n) = x^*(n)$$

$$X^*(e^{-j\omega}) = X(e^{j\omega})$$

$$x(n) = x^*(n)$$

$$X^*(e^{-j\omega}) = X(e^{j\omega})$$

$$X(e^{j\omega}) = X^*(e^{j(2\pi-\omega)})$$

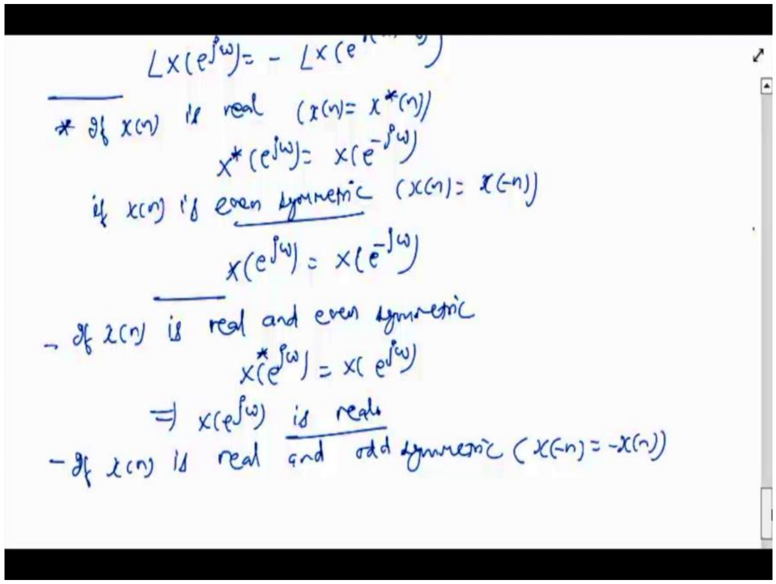
↑ periodicity of $X(e^{j\omega})$

$$|X(e^{j\omega})| = |X(e^{j(2\pi-\omega)})|$$

$$\angle X(e^{j\omega}) = -\angle X(e^{j(2\pi-\omega)})$$

* If $x(n)$ is real $(x(n) = x^*(n))$

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$



Next, we look at conjugation, and conjugate symmetric properties, conjugation and conjugate symmetry properties. It says that, if x(n) has fourier transform, $x\left(e^{j\omega}\right)$, then the complex conjugate of x(n), that is x*(n) will have fourier transform given by $x^{*}e^{-j\omega}$. So, this has several implications for instance, if x(n) is real valued, then we know that x(n)=x*n.

And so, the above relation implies that $x^{*}e^{-j\omega} = x\left(e^{j\omega}\right)$. We can combine this property along with the time reversal property to get many of the, many interesting results. In fact, if we use periodicity of $x\left(e^{j\omega}\right)$, then we observe that this can further be written as xe raise to see, you see that, that you can write is the following

$$x\left(e^{j\omega}\right) = x^{*}(e^{j(2\pi-\omega)})$$

And this is using periodicity of $x\left(e^{j\omega}\right)$.

And the, this equality has a couple of implications. Namely, the absolute value

$$\left|x\left(e^{j\omega}\right)\right| = \left|x^{*}(e^{j(2\pi-\omega)})\right|$$

which in turn will be same as

$$= \left|x(e^{j(2\pi-\omega)})\right|$$

Moreover,

$$\angle x\left(e^{j\omega}\right) = -\angle x(e^{j(2\pi-\omega)})$$

So, both these properties now follow from periodicity of $x(e^{j\omega})$.

We can combine the conjugation, conjugate symmetry property with time reversal property to get interesting inferences. For instance, if x(n) is again, if x(n) is real, then x, that is x(n)=x*(n), we have readily seen that $x^*(e^{j\omega}) = x(e^{-j\omega})$. Moreover, if x(n) is even symmetric, even symmetric. That is $x(n) = x(-n)$ then $x(e^{j\omega}) = x(e^{-j\omega})$.

Now, if we combine these two properties, we see that if x(n) is real and even symmetric and $x^*(e^{j\omega}) = x(e^{j\omega})$. That is to $x(e^{j\omega})$. We can follow similar arguments, to claim that if x(n) is real and odd symmetric, what do I mean by odd symmetric x(-n)=-x(n) $\forall n$.

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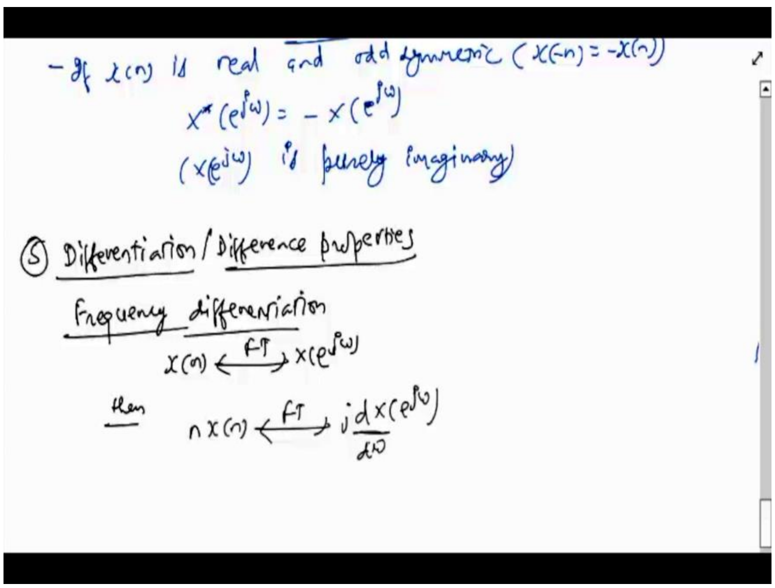

So, if x(n) is real and odd symmetric, then $x^*(e^{j\omega}) = -x(e^{j\omega})$. In other words, in this case $x(e^{j\omega})$ is pure imagined, which is purely imaginary. So, these were conjugation and conjugate symmetry properties. The next property that we will see are, differentiation or difference properties, differentiation or difference properties slash difference property.

So, there are two properties here, one is frequency differentiation property, the first property is frequency differentiation which says that if x(n) has discrete times $x(e^{j\omega})$, then nx(n) will have discrete time fourier transform, which is $j\frac{dx(e^{j\omega})}{d\omega}$. Notice that, in time domain we are talking of discrete time so, there cannot be an analogue of differentiation in time property.

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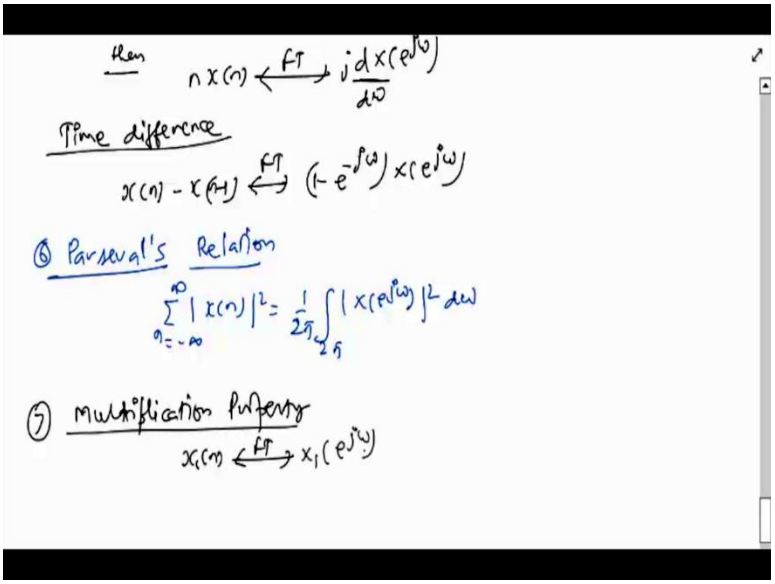

But we can talk of a time difference property, namely we can talk of Fourier transform of x(n)-x(n-1). So, from the time shift property, we see that Fourier transform of

$$x(n) - x(n-1) \overset{FT}{\leftrightarrow} \left(1 - e^{-j\omega}\right) x\left(e^{j\omega}\right)$$

This property is called time difference property, time difference property. We will see uses of these properties in a while, but, for now, let us look at the next property, which is Parseval's relation.

The is similar to the relation in case of continuous time signals. In particular, we have that

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} \left|x\left(e^{j\omega}\right)\right|^2 d\omega$$

Again, notice that on the right hand side, the integral, integration is limited to one period only.

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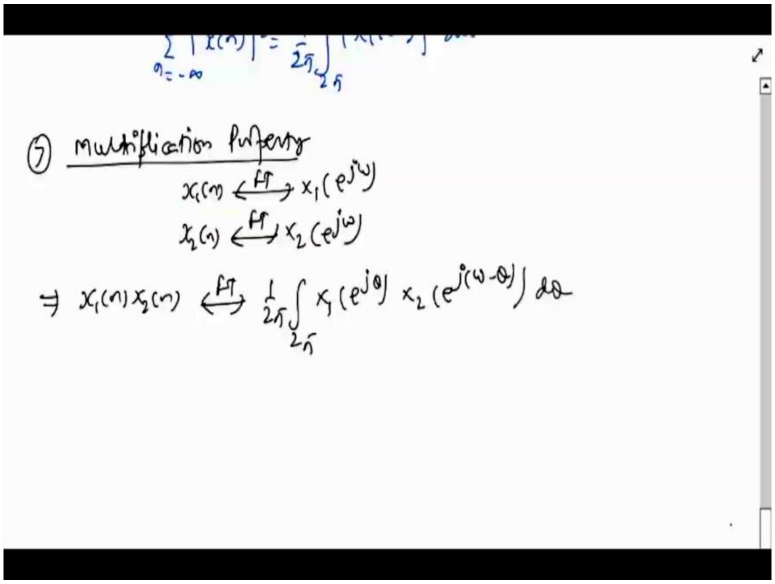

So, next property is, multiplication property. This says that if $x_1(n) \overset{FT}{\leftrightarrow} x_1(e^{j\omega})$. And $x_2(n) \overset{FT}{\leftrightarrow} x_2(e^{j\omega})$, then the product of these two signals, that is

$$x_1(n)x_2(n) \overset{FT}{\leftrightarrow} \frac{1}{2\pi}\int_{2\pi} x_1(e^{j\theta})x_2(e^{j(\omega-\theta)})d\theta$$

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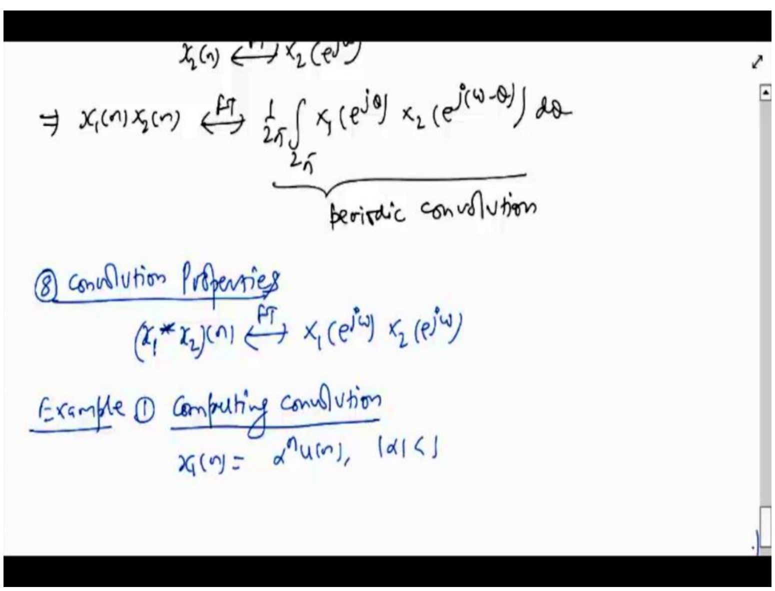

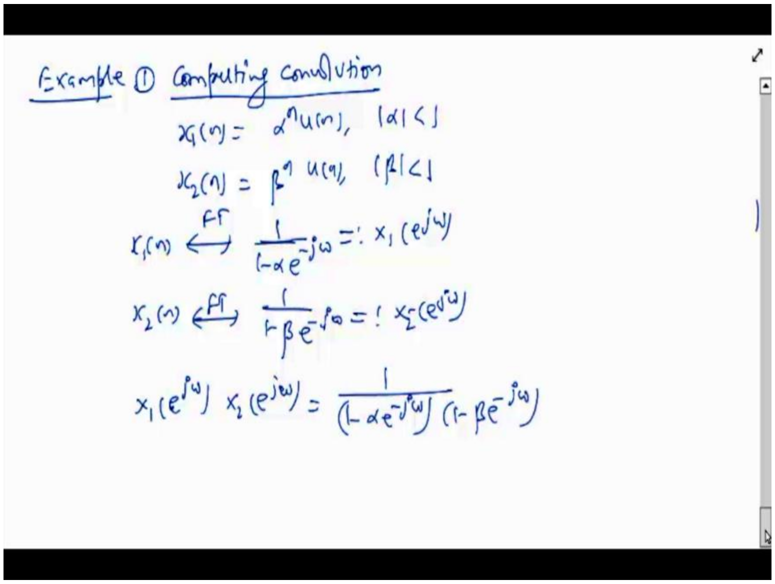

Notice that, the integral on the right hand side reminds of convolution. However, this time it is called, this operation is called periodic convolution, because the integrand is periodic and integration is over one period. So, this is called periodic convolution. Let us now go to the next property, which is very important property and we will see several uses of it, and this is convolution property, convolution properties.

This property says that, the Fourier transform of convolution of x_1 and x_2, that is $(x_1*x_2)(n)$, its Fourier transform is simply a product of individual Fourier transforms. That is $x_1(e^{j\omega})x_2(e^{j\omega})$. We will now see several examples or applications of this property. So, we will show in this, in the first example, we will show how we can use this convolution property to calculate convolution of two signals, so computing convolution.

So, let us assume that $x_1(n)=\alpha^n u(n)$, where $|\alpha| < 1$. And $x_2(n)=\beta^n u(n)$, where again $|\beta| < 1$. We have already seen that,

$$x_1(n) \overset{FT}{\leftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

And,

$$x_2(n) \overset{FT}{\leftrightarrow} \frac{1}{1-\beta e^{-j\omega}}$$

Now, from convolution property, the convolution of x_1 and x_2 is nothing but inverse Fourier transform of multiplication of the terms on the right hand side.

That is, we would be interested in computing the inverse Fourier transform of let us say, this is $x_1(e^{j\omega})$ and $x_2(e^{j\omega})$, then our interest is in computing the inverse Fourier transform of this product. That is

$$x_1(e^{j\omega})x_2(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

Like in the similar example, for continuous time case, we can use partial fraction expansion to simplify the right hand side, to get it in a convenient form.

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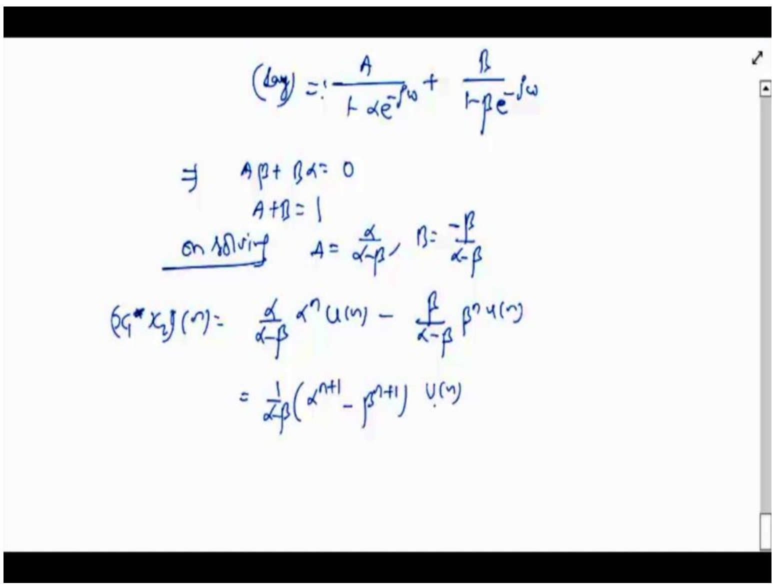

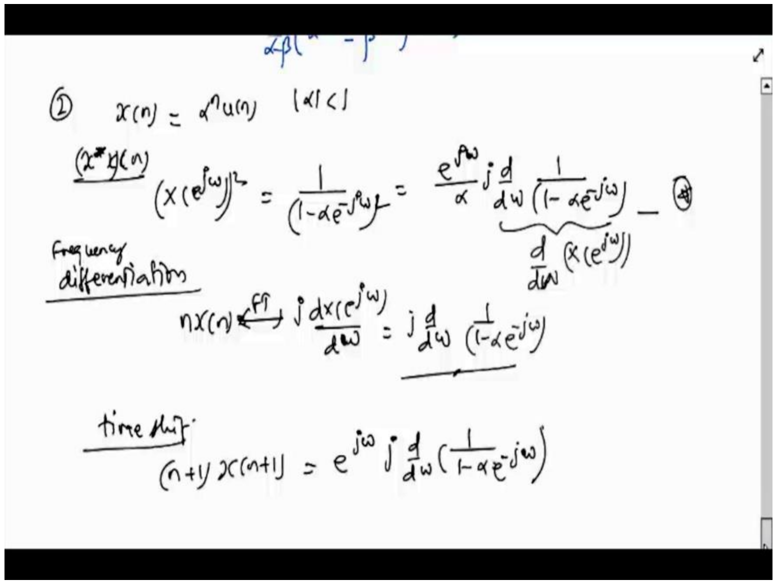

So, let us say that this

$$= \frac{A}{1 - \alpha e^{-j\omega}} + \frac{1}{1 - \beta e^{-j\omega}}$$

We can solve this partial fraction expansion to get $A\alpha + B\beta$ rather $A\beta + B\alpha = 0$ to be equal to 0. And $A + B = 1$. And on solving, we get

$$A = \frac{\alpha}{\alpha - \beta}$$

and

$$B = \frac{-\beta}{\alpha - \beta}$$

This says that convolution of x_1 and x_2 is

$$(x_1{}^* x_2)(n) = \frac{\alpha}{\alpha - \beta} \alpha^n u(n) - \frac{\beta}{\alpha - \beta} \beta^n u(n)$$

Which in turn can be written as,

$$= \frac{1}{\alpha - \beta} (\alpha^{n+1} - \beta^{n+1}) u(n)$$

Let us see another example, where we cannot use simple partial fraction expansion, but a huge convolution property in conjugation with differentiation in frequency property to compute convolution.

So, again I consider $x(n) = \alpha^n u(n)$, where $|\alpha| < 1$. And now my interest is in computing convolution of x with itself. So, this is what I am interested in. Towards that, I will start with recognizing that discrete time fourier transform of this convolved signal is $\left(x\left(e^{j\omega}\right)\right)^2$, which is

$$\left(x\left(e^{j\omega}\right)\right)^2 = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

Which can be seen to be

$$= \frac{e^{j\omega}}{\alpha} j \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

Notice that, this is $\frac{d}{d\omega} x\left(e^{j\omega}\right)$. This will help us in proceeding further. Now, before we take the inverse fourier transform, let us notice that, fourier transform of nx(n) will be

$$nx(n) = j\frac{dx\left(e^{j\omega}\right)}{d\omega}$$

Which is

$$= j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

Now, using, but notice that the terms here on the right hand side in this equation are quite different from this, that is I see an additional $e^{j\omega}$. This is where I will invoke time shift property. Now, this was differentiation property, differentiation, to be precise frequency differentiation. Now, I will invoke time shift property.

Notice that, Fourier transform of

$$(n+1)x(n+1) = e^{j\omega} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

So, this is star.

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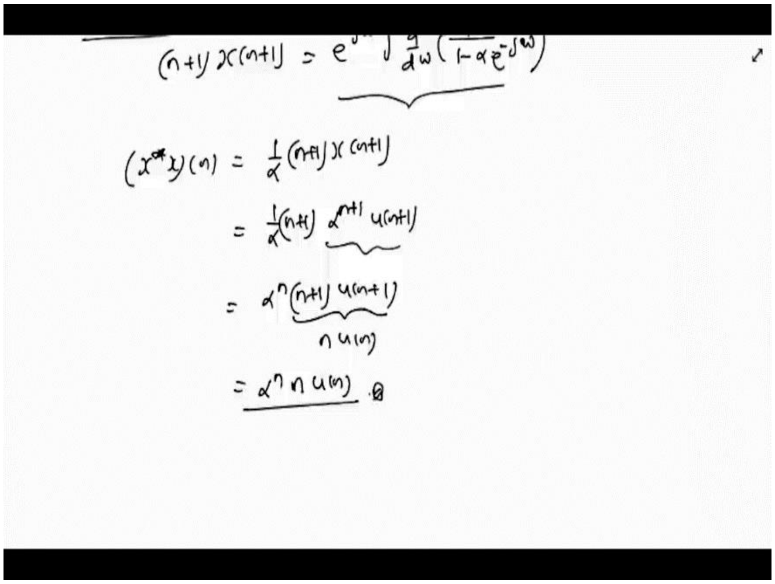

Now, I can compare this right hand side with star to recognize that the inverse fourier transform of $\left(x_1\left(e^{j\omega}\right)\right)^2$ that is

$$(x^*x)(n) = \frac{1}{\alpha}(n+1)x(n+1)$$

By substituting the value of x(n), which each $\alpha^n u(n)$, I see that the right hand side becomes

$$= \frac{1}{\alpha}(n+1)\alpha^{n+1}u(n+1)$$

So, this becomes

$$= \alpha^n (n+1)u(n+1)$$

Notice that, this is same as nu(n), because only point where the two could differ was n=-1, but at n=-1 both of these functions are achieved. So, this can be written as $= \alpha^n nu(n)$. So, we see how we use several properties to compute the convolution of x with itself. So, this was all about the properties of discrete time Fourier transform. Now, we will see how to, we can use this convolution property to compute the frequency response of a discrete time LTI system.

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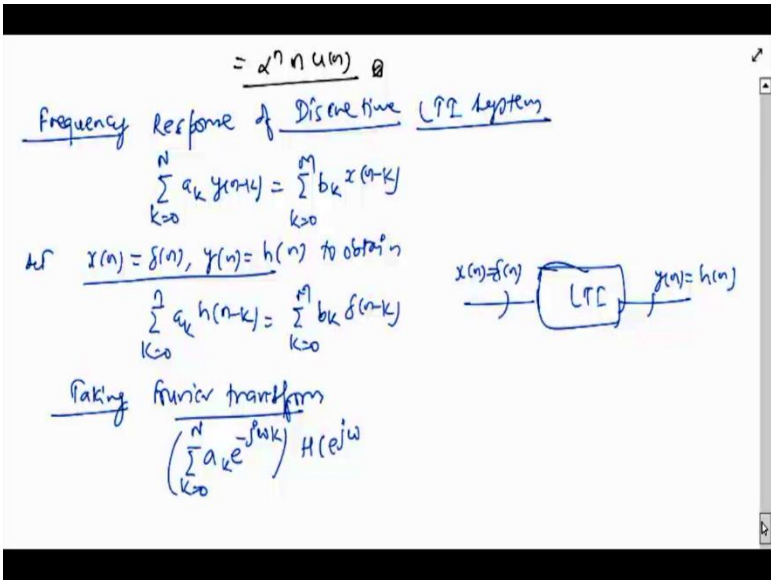

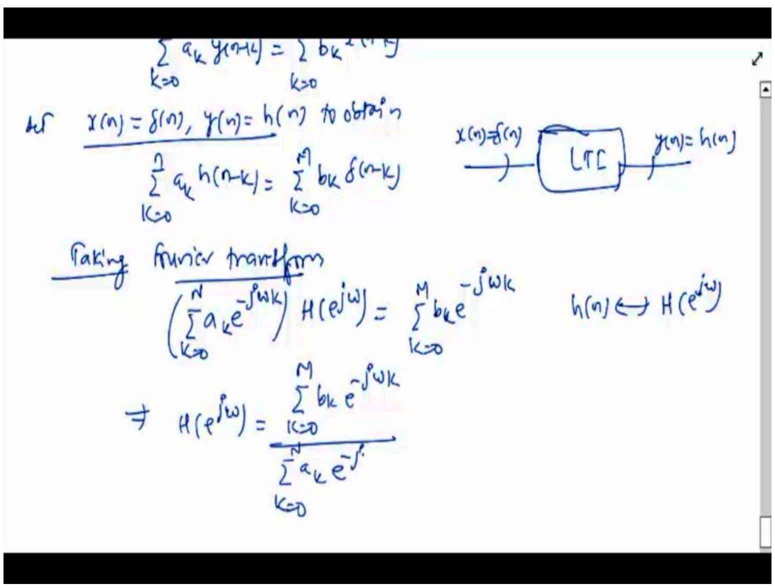

So, this is yet another application of discrete time Fourier transform. And this is in computing frequency response, response of discrete time LTI system. Notice that, a discrete time LTI system is characterized by the equation,

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Now, we will substitute $x(n) = \delta(n)$ and y, $y(n) = h(n)$, that is impulse response, to obtain

$$\sum_{k=0}^{N} a_k h(n-k) = \sum_{k=0}^{M} b_k \delta(n-k)$$

Notice that, we did so because by definition, when we input $\delta(n)$ to this LTI system, the output $y(n) = h(n)$. So, it was perfectly valid to substitute $x(n) = \delta(n)$ and $y(n) = h(n)$. Now taking Fourier transform of both the sides, we find that

$$\left(\sum_{k=0}^{N} a_k \, e^{-j\omega k}\right) H(e^{j\omega})$$

Where $H(e^{j\omega})$ is Fourier transform of h(n). And we have used time shift property here.

This becomes

$$= \sum_{k=0}^{M} b_k e^{-j\omega k}$$

This says that,

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k \, e^{-j\omega k}}$$

We will now give a couple of examples, to illustrate how we can use this relationship compute impulse response of a discrete time LTI system. So, let us see examples.

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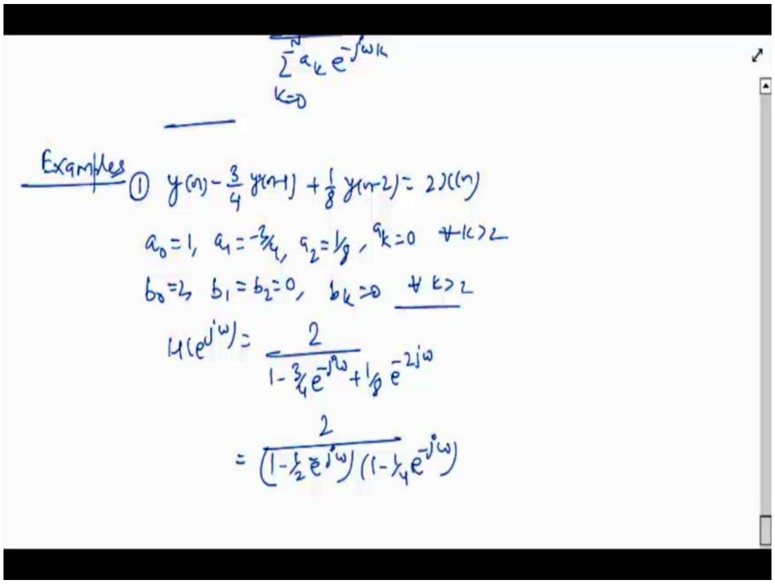

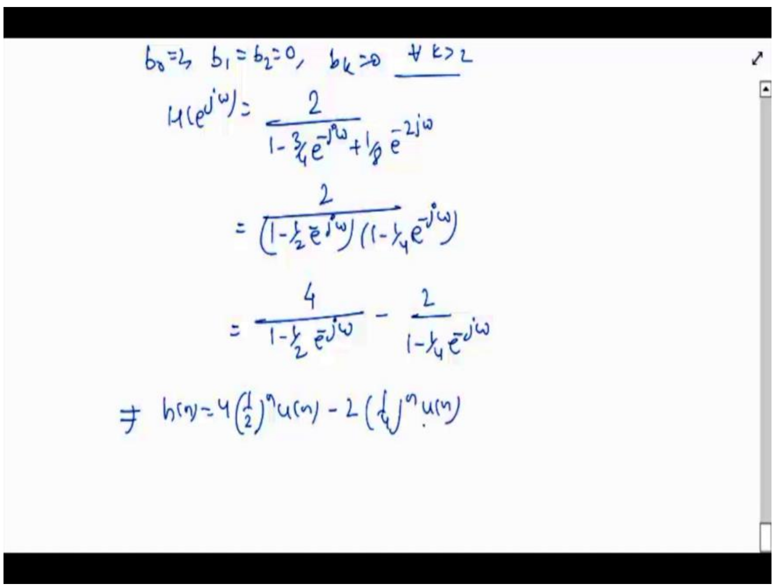

So, first example is say y(n) is input and output of the discrete time system as related as follows,

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 2x(n)$$

Question is, what is the impulse response of this system? Let us observe that in this system $a_0 = 1, a_1 = -\frac{3}{4}, a_2 = \frac{1}{8}, a_k = 0 \;\; \forall k > 2$. Similarly, $b_0 = 2, b_1 = b_2 = 0, b_k = 0 \;\; \forall k > 2$ as well.

Hence, from the above relation we can see that

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

Which can further be written as

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

We can use partial fraction expansion, to see that the right hand side is

$$= \frac{4}{(1-\frac{1}{2}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})}$$

Now, we can compute, we can easily compute the inverse Fourier transform of the two terms on the right hand side to obtain

$$h(n) = 4\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{4}\right)^n u(n)$$

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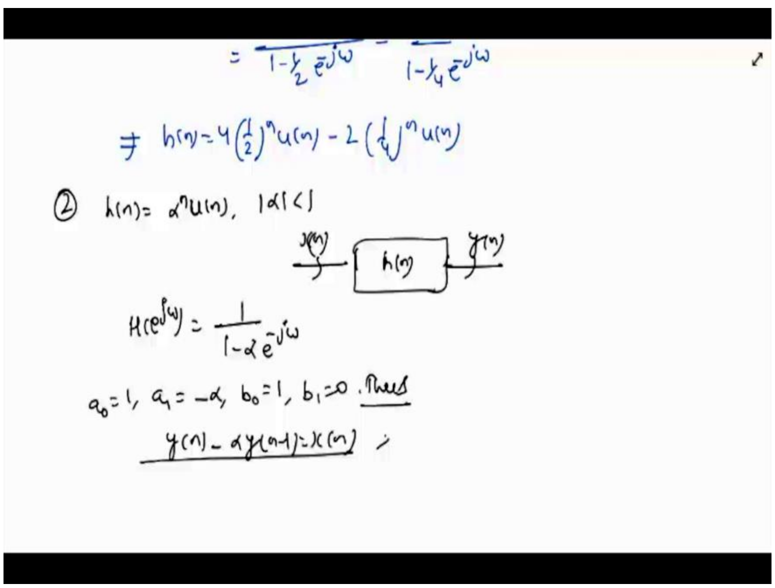

Now, we are given an LTI system, which impulse response is, $h(n) = \alpha^n u(n)$, where $|\alpha| < 1$. So, we have an LTI system, which impulse response h(n) is as given here, and we need to find a relation between input and output of this system. Notice that, $H(e^{j\omega})$, that is Fourier transform of the impulse response also called frequency response is

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

If we compare this expression of frequency response with the general initial that we had earlierly write, let us say star then we find that, $a_0 = 1, a_1 = -\alpha, b_0 = 1, b_1 = 0$ and so on. And thus we can say that input and output are related as

$$y(n) - \alpha y(n-1) = x(n)$$

So, we see how we could use discrete time Fourier transform to get a relation between input and output of the system if the impulse response of the system is known. Now, let us take a step back, as we did in the case of continuous time case. And see what are the conditions, under which the infinite sum defining discrete time Fourier transform is well defined.

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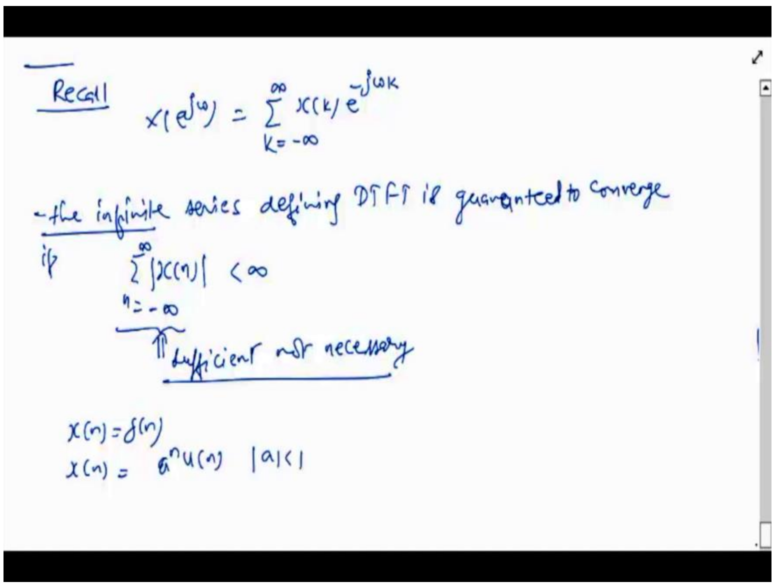

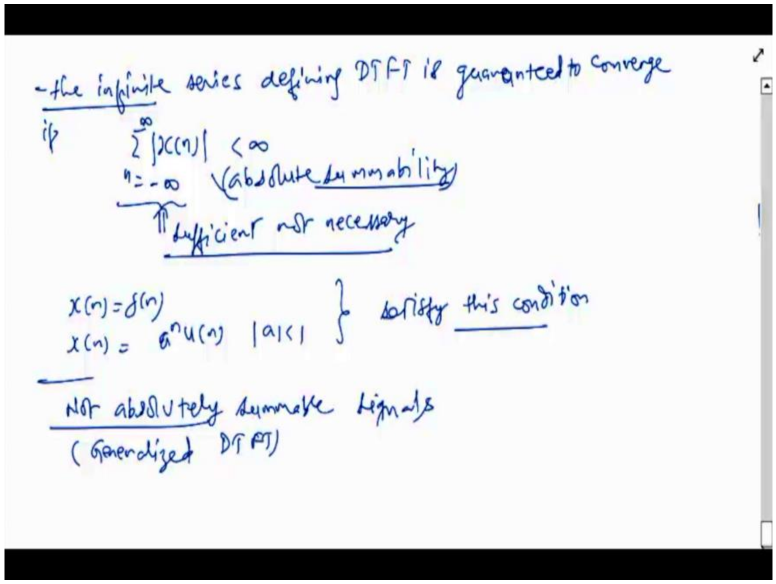

So, that is, let us recall that, discrete time Fourier transform of a signal

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}$$

Now, we are asking the question, is it that the sum on the right hand side and finite sum on the right hand side always defined? What are the conditions that we can put on x(t) to ensure that this sum is always defined? To recall that, we had a condition, we had a sufficient condition in the case of continuous time Fourier transforms, we get an similar condition here as well.

So, let us see what that condition is. The infinite series defining DTFT is guaranteed to converge, if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

However, notice that this is a sufficient condition not necessary. Several of the examples, that we saw earlier for instance, $x(n) = \delta(n)$ or $x(n) = \alpha^n u(n)$, where $|\alpha| < 1$ satisfy this condition, this condition.

But there are many other signals of interest, for which this condition is not satisfied. What do we do in that case? Notice that, recall that in the case of continuous time Fourier transforms, we devise a notion of, we introduced a notion of generalized Fourier transforms to handle that, such cases.

Even now, we can develop even, even in this case, we introduced the notion of generalized Fourier transforms to deal with the cases, where signals are not absolutely integrable.

So, this condition is called when the signals are not absolutely summable. So, this condition is called absolute summability, absolute summability. For instance, let us take a few cases, where the signals are not absolutely summable, not absolutely summable signals, as I said in this case we will talk of generalized fourier transform, generalized DTFT.

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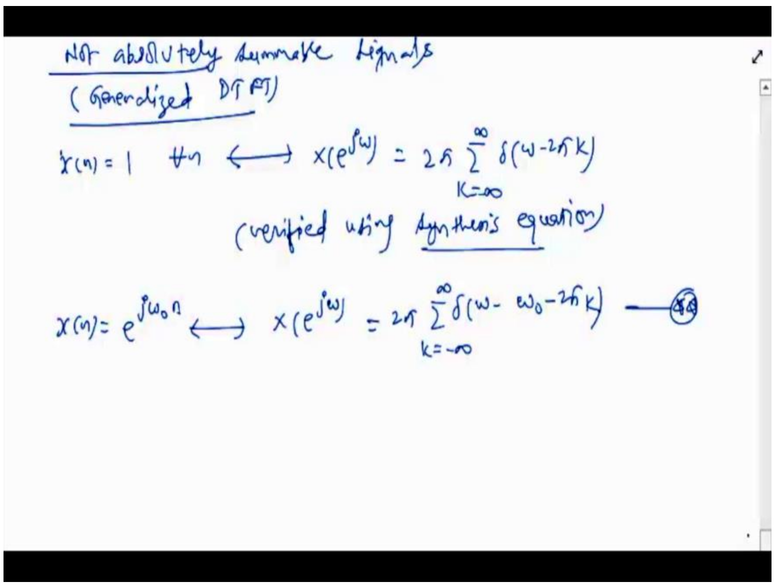

So, let us say x(n) is equal to constant, that is 1 ∀n. In this case, it turns out that x, the fourier, generalized fourier transform of

$$x(n) \leftrightarrow x(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

It this equation can be easily verified using synthesis equation. So, if I just substitute $x(e^{j\omega})$ in the synthesis equation, I get $x(n) = 1$ ∀n, this is the equation.

Similarly, if x(n) is raised to is, similarly, if $x(n) = e^{j\omega_0 n}$, now since I know the Fourier transform of x =1, I can use the time shift property to infer that. Now, since I know the fourier transform of x(n) = 1, I can use the frequency shift property to infer that fourier transform of this

$$x(n) \leftrightarrow x(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$$

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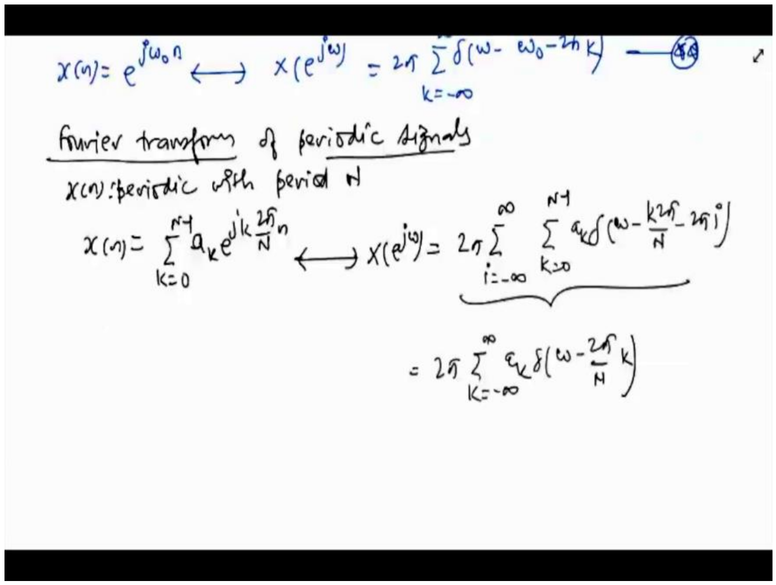

We can use this property, this star, let us say double star we can use this property double star along with linearity property to compute the Fourier transform of periodic signals. Let us see how it is done. So, Fourier transform of periodic signals. So, let us say x(n) is a discrete time signal, which is periodic with period N such that, I can write the discrete time Fourier series for

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

Then the fourier transform of xn can be computed using linearity as follows,

$$x(n) \leftrightarrow x(e^{j\omega}) = 2\pi \sum_{i=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k \delta(\omega - k\frac{2\pi}{N} - 2\pi i)$$

And this can be simplified to be equal to

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi}{N}k\right)$$

This simplification is not straightforward and it evolves as fewer steps, I will skip those in the interest of time.

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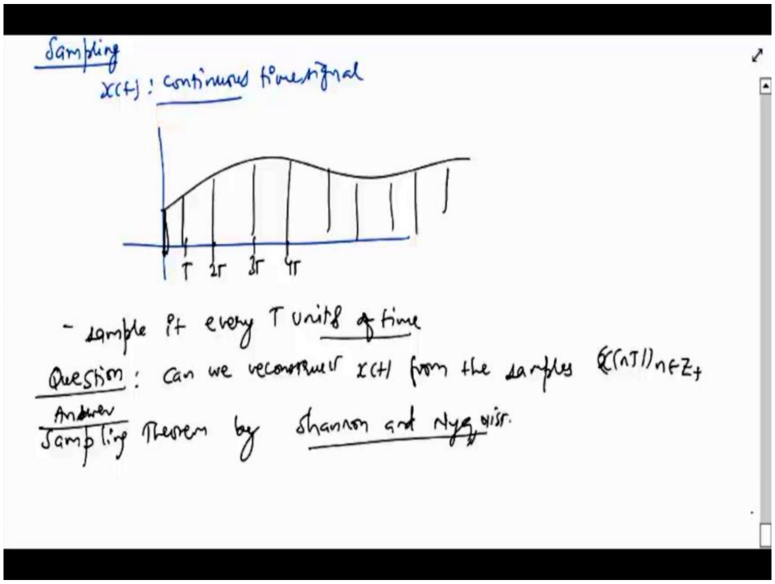

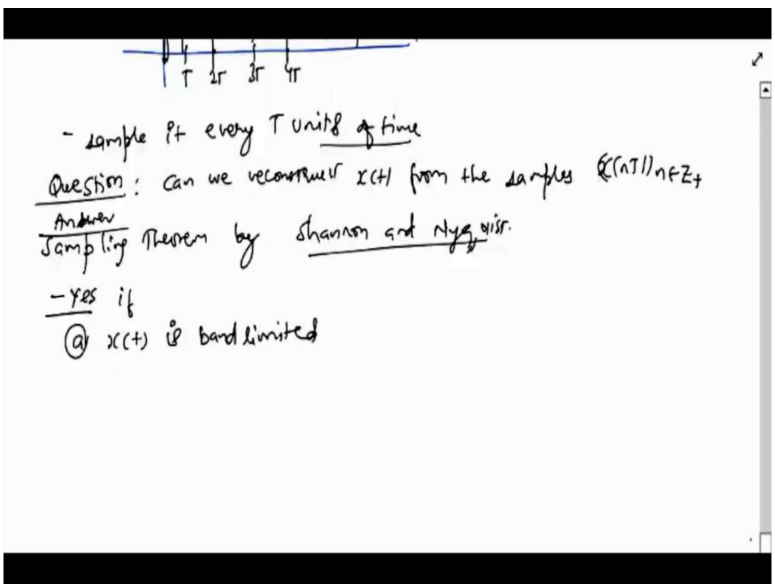

This completes our discussion of Fourier transform for discrete time signals. Next, we will look at the notion of sampling. Suppose, we have a continuous time signal x(t). For instance, say this is x(t), and say we sample it every T units of time, this T 2T 3T 4T we take these samples. So, we sample it every T units of time, sample it every T units of time. So, often question is, can we reconstruct x(t) from the samples, can we reconstruct x(t) from the samples which are x(nT) n∈z+?

That is n = 0 1 2 3 4 etcetera. This is the question, which answer is given by sampling theorem by Shannon and Nyquist. So, here is the answer, sampling theorem by Shannon and Nyquist. These two gentlemen have shown that the answer is affirmative. That is we can indeed retrieve x(t), if the following conditions are met. First condition is x(t) is band limited. That is, its Fourier transform x(ω)= 0 $\forall|\omega| > \omega_m$ for $\omega_m > 0$.

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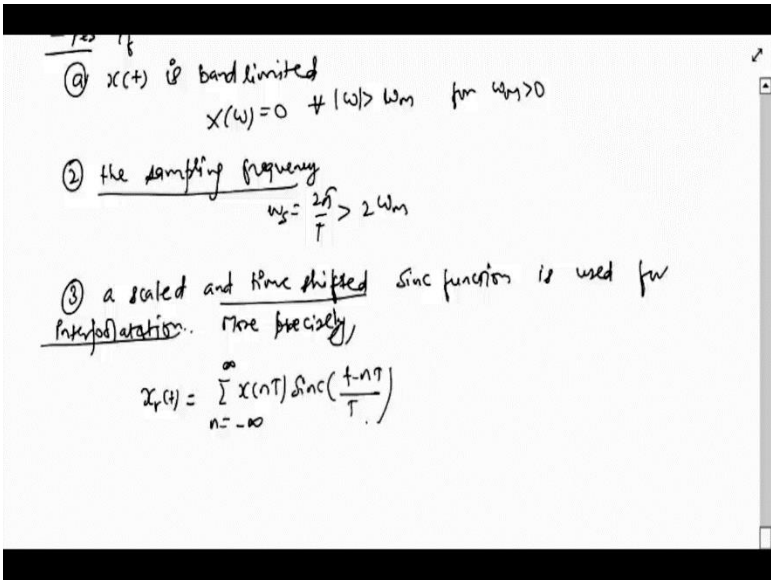

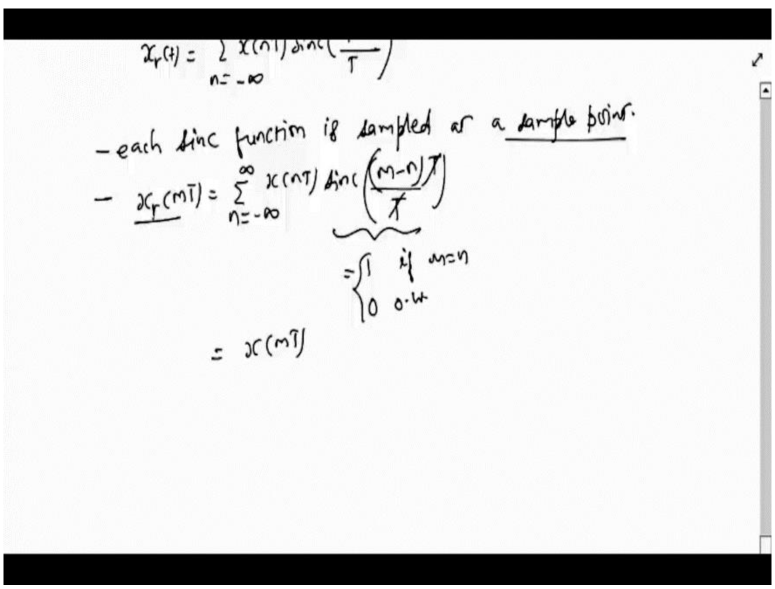

Second condition is, the sampling frequency, say $\omega_s = \frac{2\pi}{T} > 2\omega_m$. So, this is twice the band width of the signal that is being sampled. And the third condition is, use a scale and time shifted sinc function is being used for reconstruction for interpolation. A scaled and the time shifted sinc function is used for interpolation. More precisely, we obtain the interpolated signal say

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) sinc\left(\frac{t - nT}{T}\right)$$

So, this is shifted as well as scale sinc function.

Notice that each sinc function and the summation on the right hand side is centered at a sample point. Moreover, if we evaluate the right hand side at the sample points, then the value return is same as the value of the original function at the sample points. In other words, if I compute say

$$x_r(mT) = \sum_{n=-\infty}^{\infty} x(nT) sinc\left(\frac{(m - n)T}{T}\right) = \begin{cases} 1 \ \ if\ m = n \\ 0\ otherwise \end{cases}$$

So, the value of

$$x_r(mT) = x(mT)$$

So, clearly the original signal and reconstructed signal agree on the sample points. The interesting feature of sampling theorem ensures that the two signals match each other at all the points. So, here is the formal statement of the theorem.

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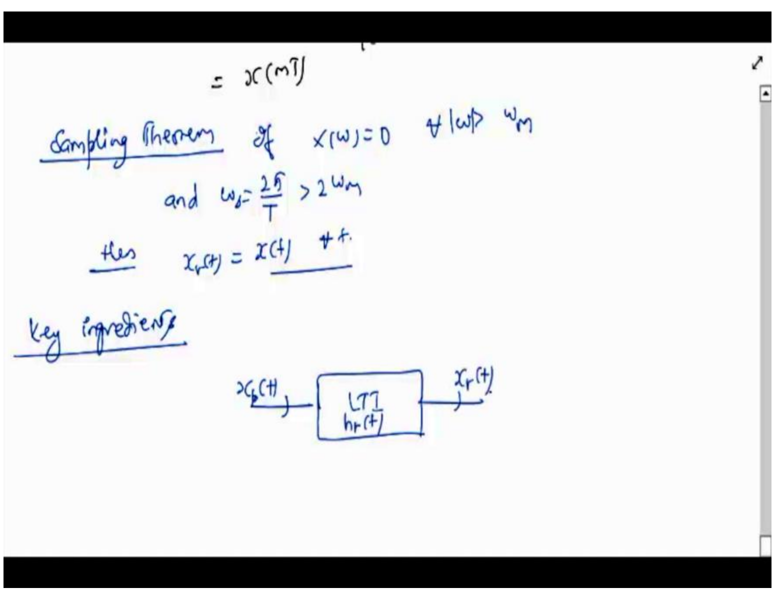

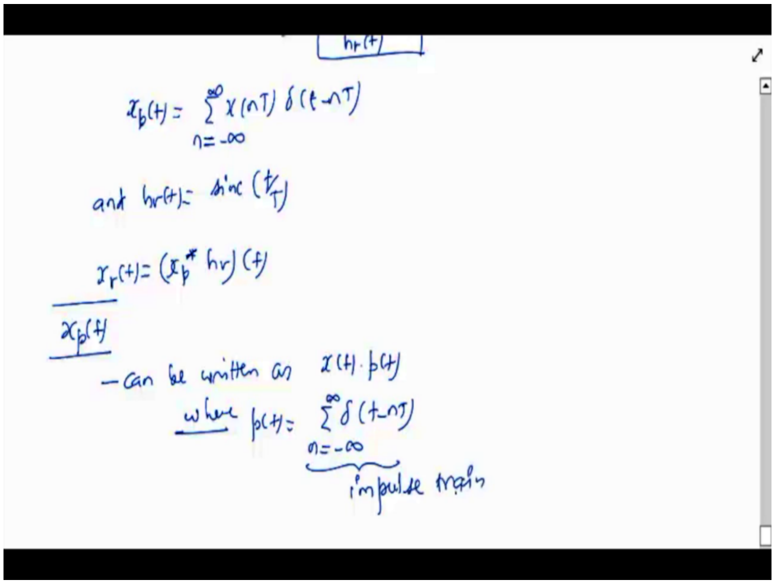

It says that if $x(\omega) = 0 \ \ \forall|\omega| > \omega_m$ and $\frac{2\pi}{T}$ is, which is equal to sampling frequency. So, if fs equals sorry, if $\omega_s = \frac{2\pi}{T} > 2\omega_m$ then x rt as given by the above equation. Let us say, let us call this star as we refer to it again and again that $x_r(t) = x(t) \ \forall t$. Why, we will, we will not see the formal proof of the sampling theorem in this lecture.

I will explain a few key ingredients that lead to this theorem. So, here are key ingredients. Let us first observe that the signal $x_r(t)$, reconstructed signal $x_r(t)$ can be viewed as output of an LTI system, with input $x_p(t)$ and output and impulse response $h_r(t)$. Where,

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

And

$$h_r(t) = sinc\left(\frac{t}{T}\right)$$

So, what we claim is that, $x_r(t)$ is convolution of $x_p(t)$ and $h_r(t)$.

To see the properties of $x_r(t)$, let us see these two signals individually. So, first we will see $x_p(t)$. It can be seen that $x_p(t)$ itself can be written as a product of two signals x(t) and p(t), where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Notice that, this p(t) is an impulse train, it is a periodic signal.

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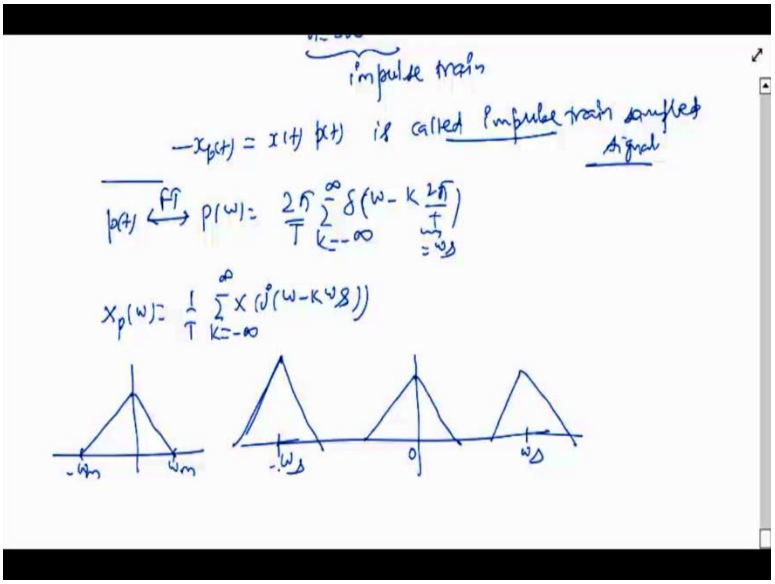

And so, the x(t) is called $x_p(t)$ is called impulse train sampled signal, called impulse train sampled signal. Further notice that, p(t) is a periodic signal and we can write its Fourier transform, using the theory that we have learned earlier to be the following. So, p(t) fourier transform is,

$$p(t) \leftrightarrow p(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$$

which is ω_s.

Moreover, I can use the convolution property of fourier transform to compute

$$x_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} x(j(\omega - k\omega_s))$$

Notice that, $x_p(\omega)$ is nothing but sum of, scale sum of shifted copies of $x(\omega)$. So, what I mean is that if my $x(\omega)$ is like this, so this is $-\omega_m, \omega_m$. And if $\omega_s > \omega_m$, then $x_p(\omega)$ would indeed look like this.

This is, ω_s, 0, $-\omega_s$ and this is $\frac{1}{T}$ whereas this was 1. And if the peak value of $x(\omega) = 1$, peak value of $x_p(\omega) = \frac{1}{T}$. Now, let us shift focus to $h_r(t)$.

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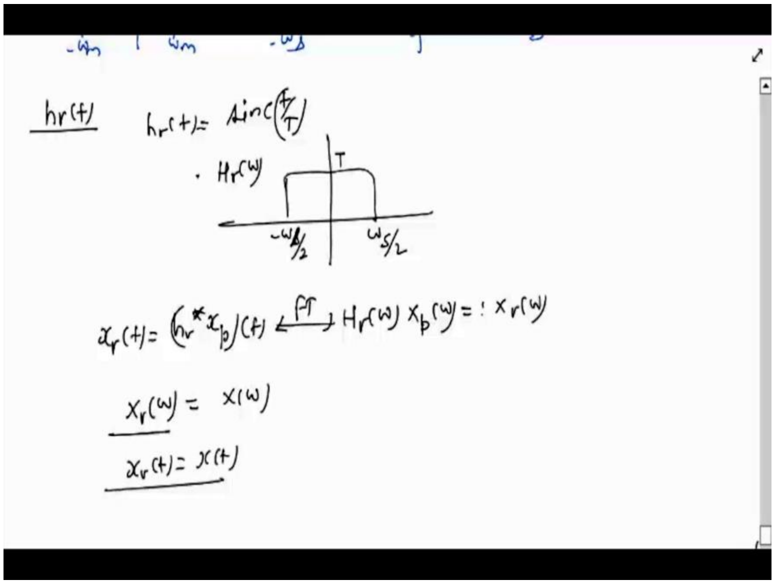

Recall that, $h_r(t) = sinc(\frac{t}{T})$ and so, its fourier transform will be simply the pulse signal with width the blue ω_s. This peak value is T, this is my $H_r(\omega)$. Now, since $x_r(t)$ was convolution of $h_r(t)$ and $x_p(t)$, so its fourier transform will be simply $H_r(\omega)X_p(\omega)$. If we observe the two Fourier transforms on the right hand side, you will notice that, $X_p(\omega)$ you will notice that $X_r(\omega) = X(\omega)$.

Which means that, if we reconstruct, which which means that the reconstructed signal $x_r(t) = x(t)$. Notice that, if $\omega_s < 2\omega_m$, then the copies of $x(\omega)$ in $x_p(\omega)$ they overlap. For instance, for the above example, we get a signal that looks something of this sort. When the copies overlap, the low pass filter outputs as signal that differs from the original signal.

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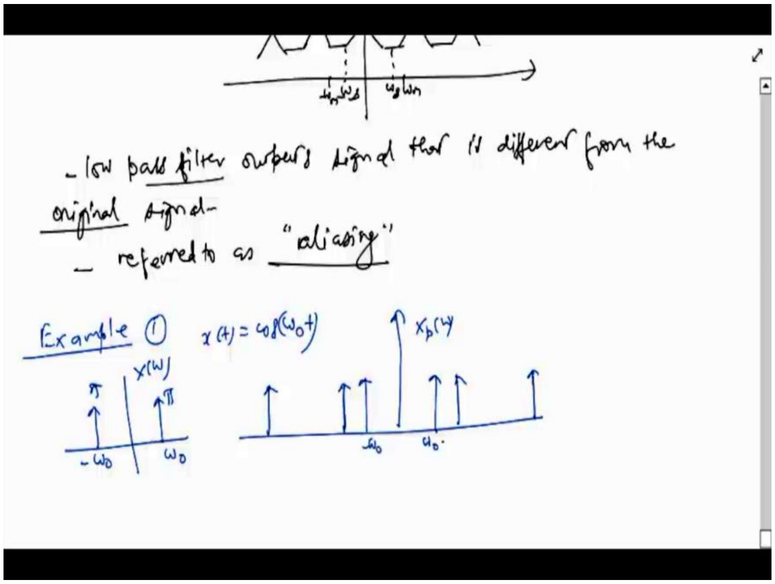

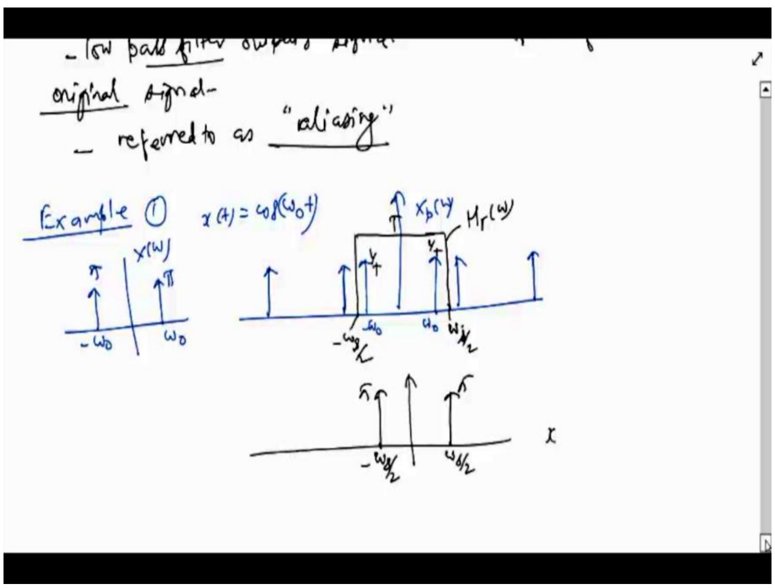

Low pass filter outputs a signal that is different from the original signal. This phenomena is referred to as aliasing. We now see a couple of examples that illustrate this phenomena of aliasing. So, here is example one, which is $x(t) = \cos(\omega_0 t)$. In this case, notice that $x(\omega)$ will be these two impulses at $-\omega_0$ and ω_0 each with amplitude π. Moreover, $x_p(\omega)$ will be sum of shifted version of these impulses.

So, I get this $x_p(\omega)$, $-\omega_0, \omega_0$. In this case, the reconstruction filter, that is $H_r(\omega)$ would look like this. So, this is between $-\frac{\omega_s}{2}$ to $\frac{\omega_s}{2}$. And we see that the reconstructed signal will have Fourier transform with to pulses exactly as the original signal, so which two impulses exactly as the original signal. And so, there is no aliasing in this case and $x_r(t)$ is equal to $x(t)$.

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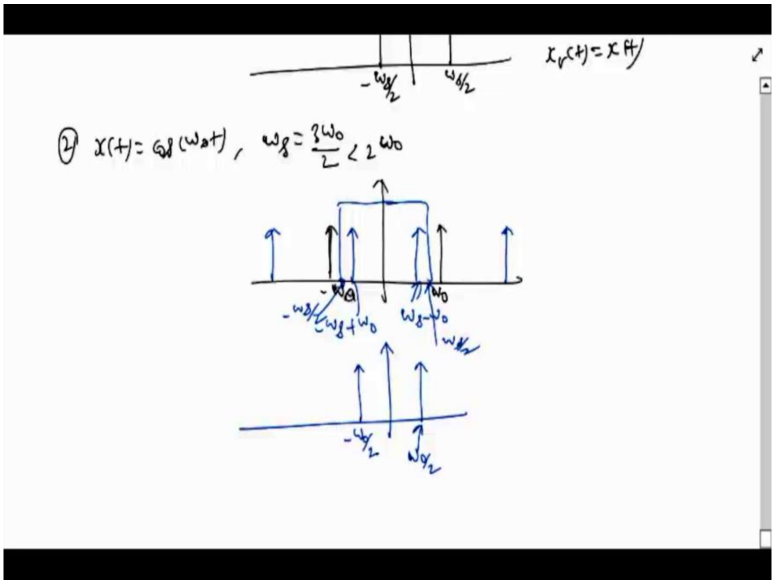

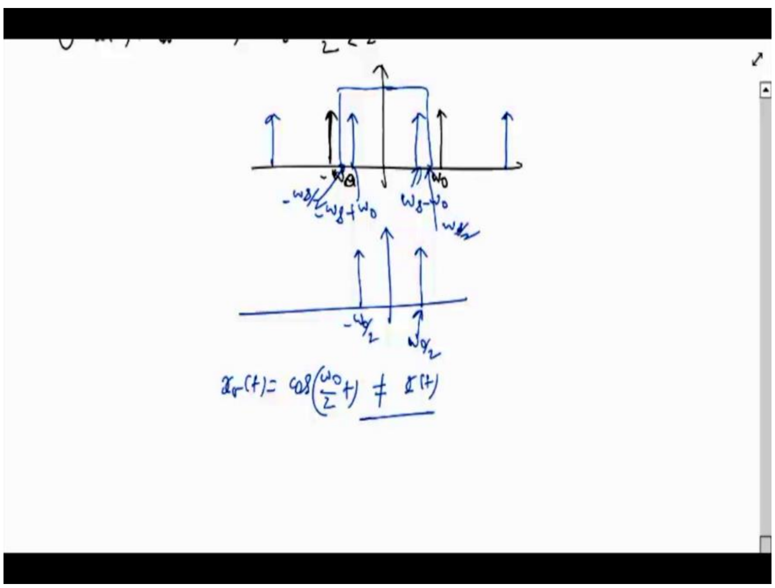

Let us now look at another example, where now $x(t) = \cos(\omega_0 t)$ as before, but $\omega_s = \frac{3\omega_0}{2} < 2\omega_0$. In this case, $x_p(t)$ will be as follows. So, we have $-\omega_0, \omega_0$ and then the shifted version of this. But those shifted versions will be like this. So, this is one shifted version. This is another shifted version. So, this value is $\omega_s - \omega_0$, this one is $-\omega_s + \omega_0$ and so on.

This will be $-\frac{\omega_s}{2}$, this will be $\frac{\omega_s}{2}$ and in this case if we employ a low pass filter, the reconstructed signals Fourier transform will not be same as the original signals transform. In fact, reconstructed signals Fourier transform will be this. So, this pulses will be at $\omega_s - \omega_0$, which will be, which is $\frac{\omega_0}{2}$. And similarly here at $-\frac{\omega_0}{2}$ respectively. And so, $x_r(t) = \cos\left(\frac{\omega_0}{2}t\right) \neq x(t)$. So, this illustrates the phenomena of aliasing.

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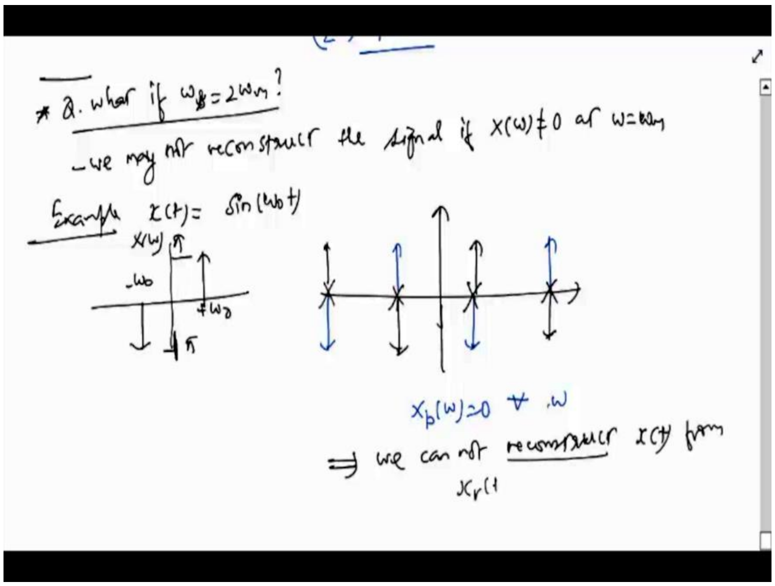

Now, there is one question that remains, what if $\omega_s = 2\omega_m$. We will give an example that will say that, setting $\omega_s = \omega_m$ does not suffice, that is we are not we will not be able to reconstruct the signal. In particular, when $x(\omega_m) \neq 0$. So, we may not reconstruct the signal spatially, if $x(\omega) \neq 0\ at\ \omega = \omega_m$.

Let us see, it with an example. So, let us consider x t to be equal to sine omega naught in this case. Notice that, x omega will be as follows. This is omega naught, this is plus omega naught, these amplitudes as before are pi and minus pi respectively. Now, in this case, if I use $\omega_s = 2\omega_0$, in this case $x_p(\omega)$ will be obtained as follows. I will draw copies of $x(\omega)$.

So, this is $x(\omega)$, this is the next copy. At each point $x_p(\omega)$ is nothing but sum of all these signals. See that, the two pulses at all the points neutralize each other because they have same amplitude but opposite signs. So, $x_p(\omega) = 0\ \forall\omega$. Clearly in this case, we cannot recover, we cannot reconstruct x(t) from $x_p(t)$. This brings us to the end of the module on signals and systems. In the next lecture, we will begin with few topics in linear algebra. Thank you.