Mathematical Aspects of Biomedical Electronic System Design
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Lecture 6
Continuous Time Fourier Transform

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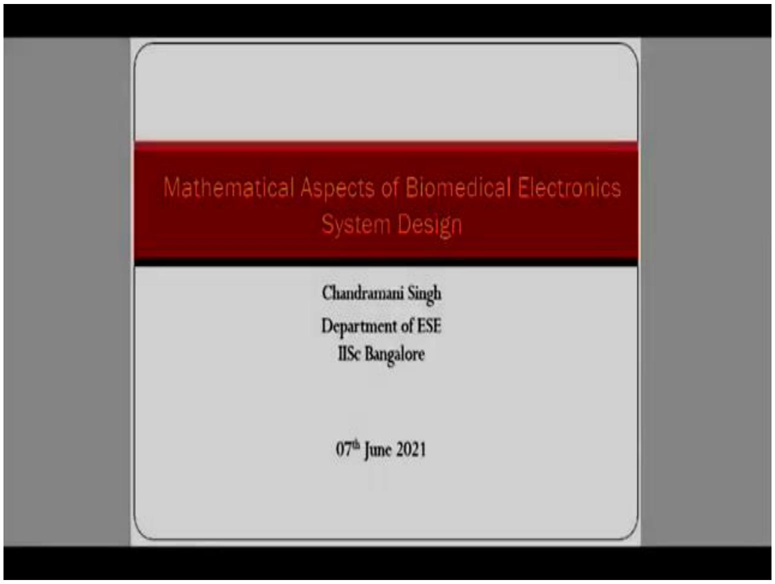

Hello everyone, welcome to the third lecture of the course mathematical aspects of biomedical electronic system design.

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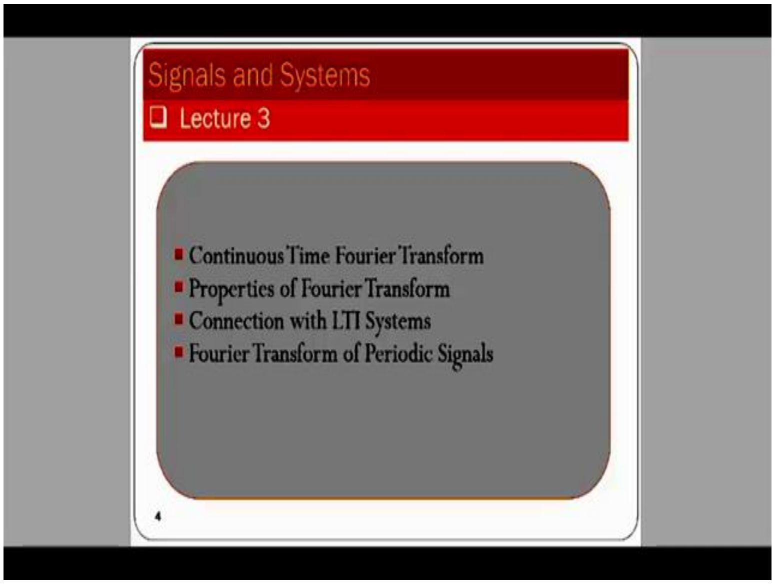

In today's lecture, we will learn about continuous time Fourier transforms, properties of Fourier transform, connection with linear time invariant systems and Fourier transform of periodic signals. Let us begin today's lecture.

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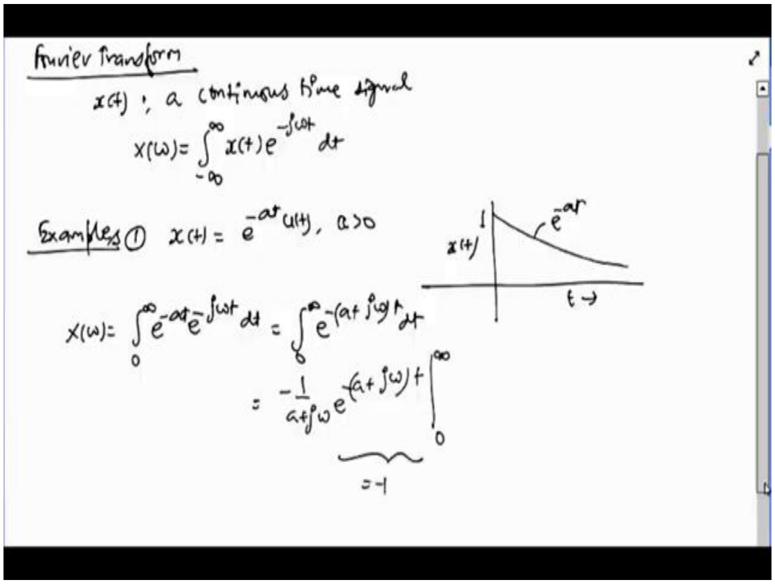

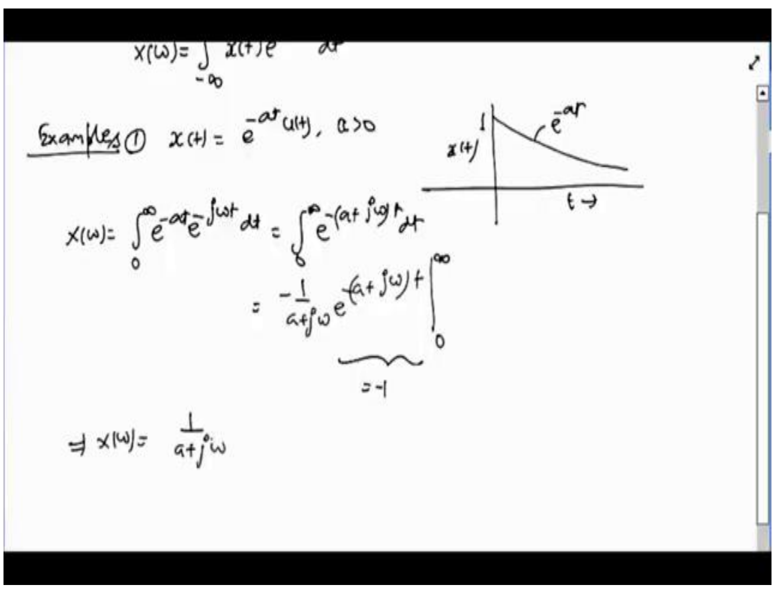

We will start with defining what a Fourier transform is? Fourier transform unlike Fourier series, applied to signals here continuous time signals that are a periodic. So, a signal does not have to be periodic for us to define its Fourier transform, Fourier transform have continuous time signal x(t) is defined as for so forth. So, this is a continuous time signal, Fourier transform we will use x(ω) to denote its Fourier transform and it is defined as

$$\mathrm{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Let us see an example, in fact, we will see several examples. So, here is your first example, say $\mathrm{x(t)} = e^{-at}u(t)$, t greater than a greater sorry, a>0, if you plot this signal looks good look like for it is real signal, this is 1, e^{-at}, this is E, this is 1 x(t). If we apply the above definition, we see that

$$\mathrm{x}(\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

which is

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\int_{0}^{\infty} dt$$

And this is equal to minus 1. So, we have that

$$\mathrm{x}(\omega) = \frac{1}{a+j\omega}$$

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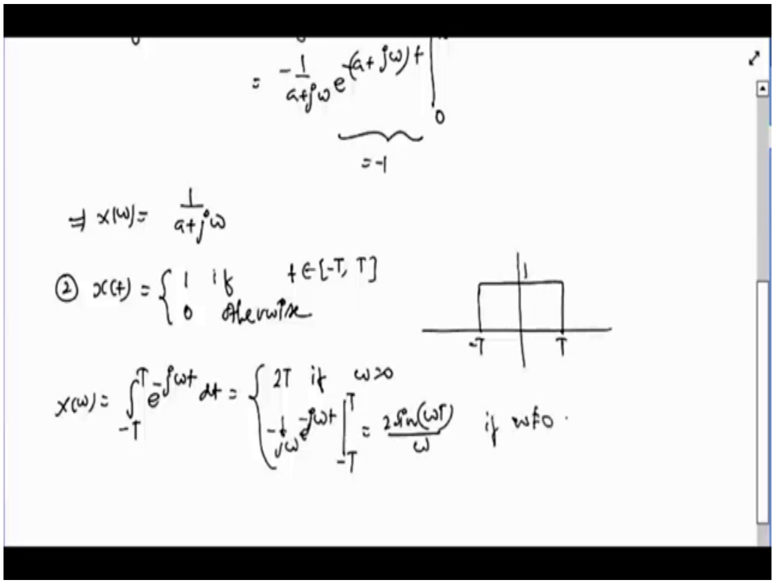

Let us see another example say x(t) is pulse signal, so x(t) is 1, if t is in the range - T to T and 0 otherwise, if you plot this, this looks like following, it is -T, T, 1. Now, if we apply the above definition,

$$\mathrm{x}(\omega) = \int_{-T}^{T} e^{-j\omega t} dt = \begin{cases} 2T \; if \; \omega = 0 \\ \dfrac{-1}{j\omega} e^{-j\omega t} \displaystyle\int_{-T}^{T} dt = \dfrac{2\sin(\omega t)}{\omega} \; if \; \omega \neq 0 \end{cases}$$

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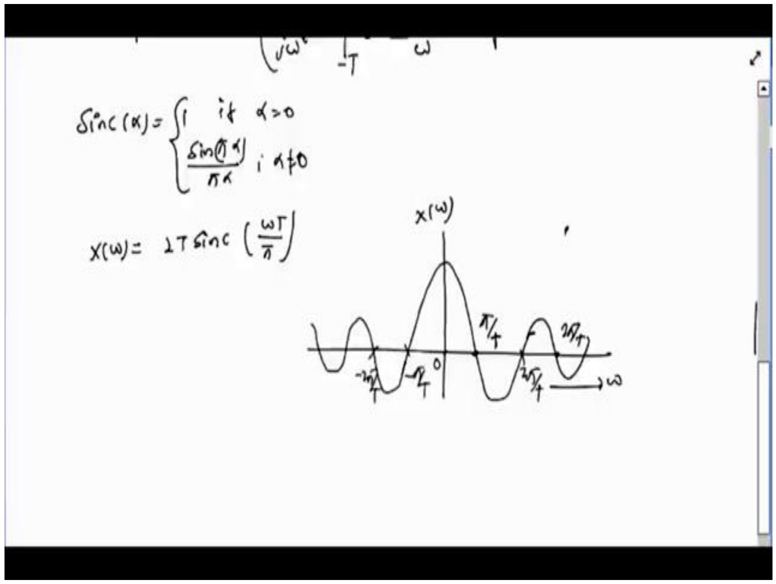

We can compactly express the, this Fourier transform in terms of the so called sinc function, sinc function is defined as follows

$$sinc(\alpha) = \begin{cases} 1 \; if \; \alpha = 0 \\ \dfrac{\sin(\pi\alpha)}{\pi\alpha} \; if \; \alpha \neq 0 \end{cases}$$

In terms of sinc function, we see that x(ω) can be expressed as follows,

$$x(\omega) = 2Tsinc\left(\frac{\omega T}{\pi}\right)$$

If we plot this function Fourier transform, it looks like as follows, it is a symmetric function. So, this is how it looks this is ω, so x(ω), these points are as follows. So, we have 0 here, this is $\frac{\pi}{T}, \frac{2\pi}{T}, \frac{3\pi}{T}$ and similarly, we have $\frac{-\pi}{T}, \frac{-2\pi}{T}, \frac{-3\pi}{T}$ and so on.

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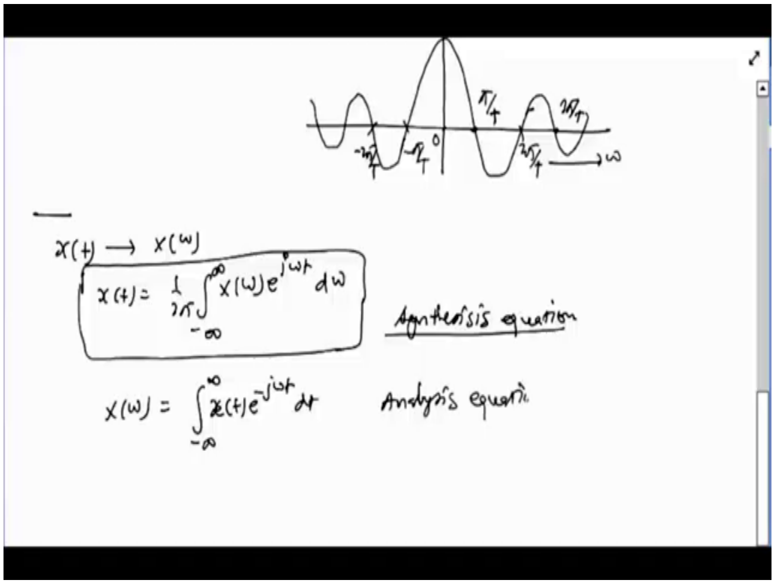

Now, let us see having seen Fourier transform of a signal, how can we retrieve the signal given its Fourier transform. So, this the inverse of Fourier transform namely on a, in other words, so here, so here is the relation x(t) if x(t) has Fourier transform x(ω), then I can retrieve x(t) from x(ω) by the following operation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{j\omega t}d\omega$$

Now, as in case of Fourier series, this equation is called synthesis equation, the previous equation that got x(ω) that is Fourier transform from the signal, that is called the analysis equation. So, this called analysis equation.

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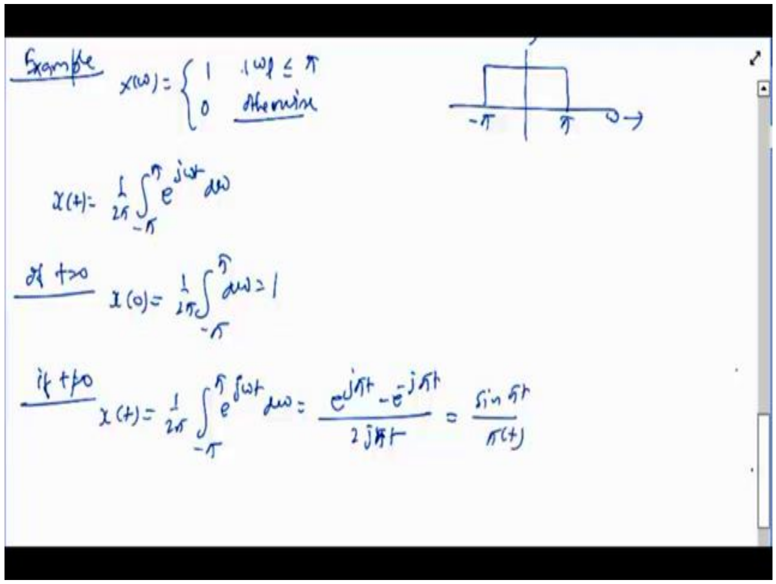

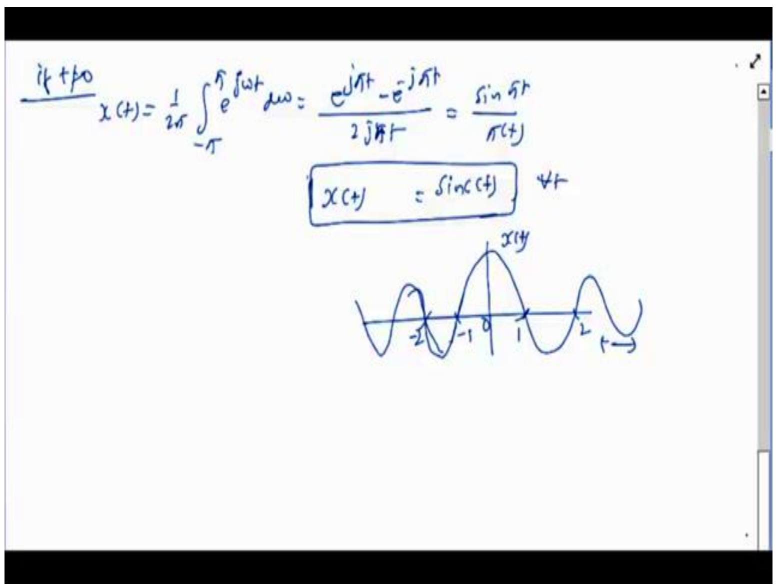

Let us see an example x(ω) is Fourier transform of the signal be 1, or $|\omega| \leq \pi$ and 0, otherwise. So, the Fourier transform itself is a plot signal, this is -π, π this is x(ω).

So question is what is the inverse Fourier transform of the signal?

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} d\omega$$

from definition we see that if T =0 and

$$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1$$

and if T ≠0, then

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} d\omega$$

which on simplifying becomes sin (πt), well first we get it

$$= \frac{e^{j\pi t} - e^{-j\pi t}}{2j\pi t}$$

which is nothing but $= \frac{\sin(\pi t)}{\pi t}$ which in turn if expressed in terms of sinc function is simply a sinc(t).

So, we have

$$x(t) = sinc(t) \quad \forall t$$

If you plot this x(t) looks like for , here is the rough sketch of x(t), T, these points are 1, 2, 0, -1, -2 etc. So, we see that the Fourier transform of sinc function is a pulse signal.

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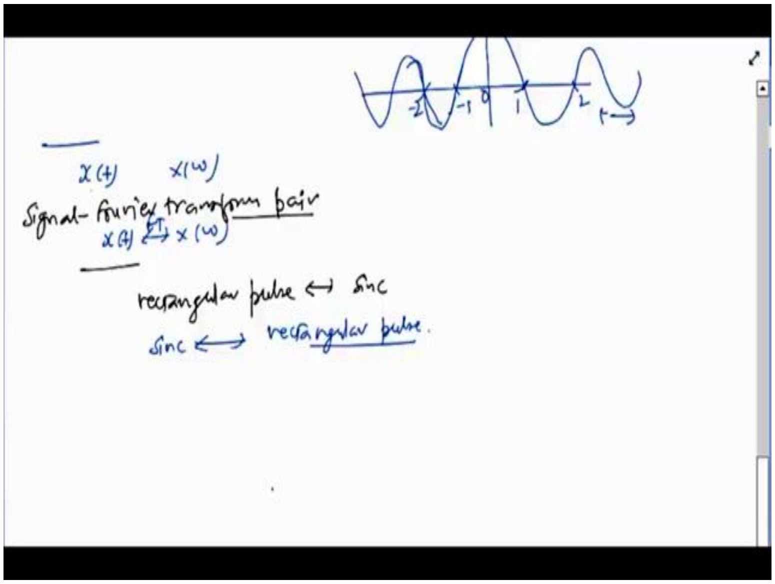

Now, in below we will use the following convention if x(t) and X(ω) are signal Fourier transform pair that is x(ω) is a Fourier transform of x(t), and x(t) is inverse Fourier transform of x(ω), we will denote as, we will denote it as for follows, it is a Fourier, this is a signal Fourier transform pair, or sometimes also called Fourier transform inverse transform pair, signal Fourier transform pair.

So, what we just observed that, for a rectangular pulse its Fourier transform is sinc and for a sinc function, it is Fourier transform is a rectangular pulse. So, we see a duality between Fourier transform and inverse Fourier transform, we will see such duality several such dualities as we go along.

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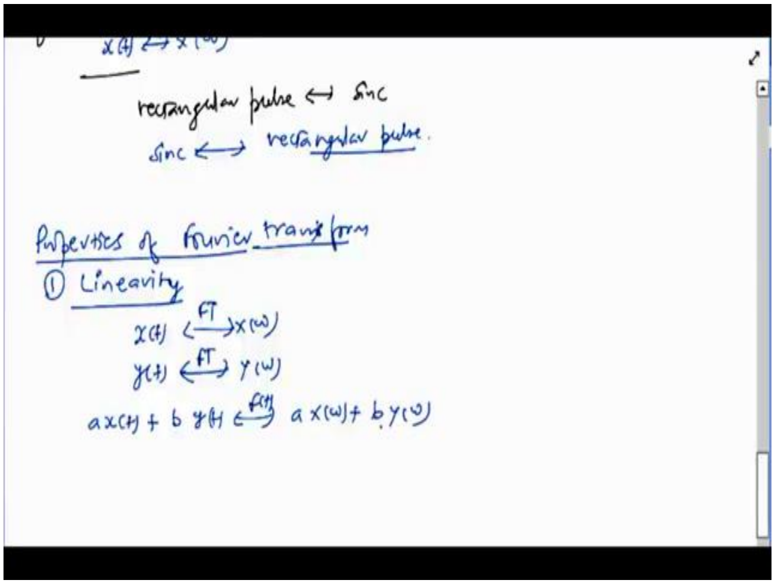

So, now we come to the properties of Fourier transform, first property is called linearity, this property says that if x(t) has Fourier transform, x(ω) and y(t) has Fourier transform sorry, y(ω), then ax(t) + by(t) will have Fourier transform ax(ω) + by(ω). This property can be easily seen following the definition of Fourier transform.

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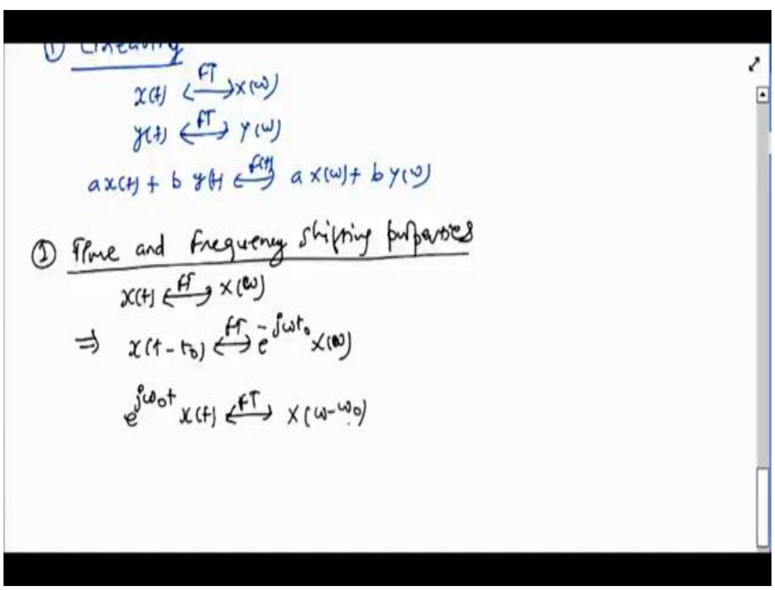

Let us see next property and this property is called time and frequency shifting property. So, actually these are two properties, that we combined together, time and frequency shifting properties. Let us see what these properties are; it says that, if x(t) has Fourier transform x(ω) then a shifted version of x(t), that is x(t-t_0), will have Fourier transform

$$e^{-j\omega t_0} x(\omega)$$

So, this is time shifting property similarly, if we multiply x(t) with $e^{j\omega_0 t}$.So, this new signal

$$e^{j\omega_0 t} x(t)$$

will have Fourier transform, which has shifted version of the Fourier transform of x(t). So, the new Fourier transform is this, so we see a duality between time and frequency shifting.

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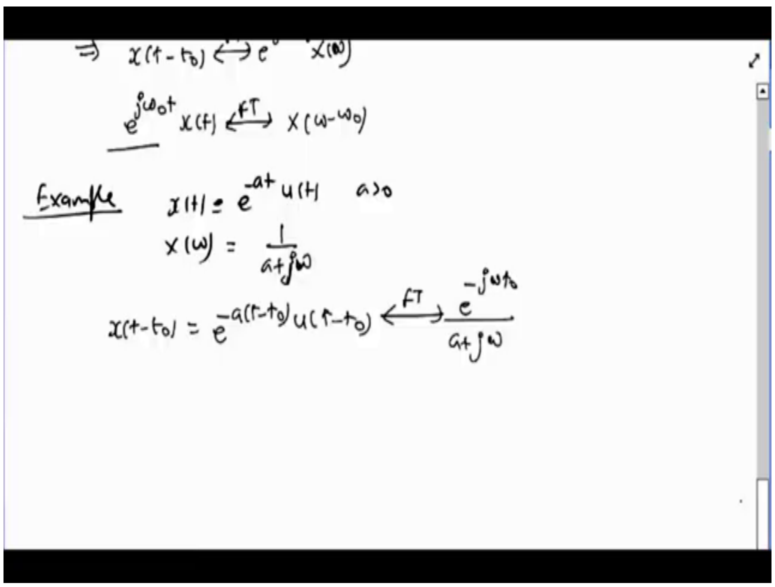

Let us see an example, let us consider the first example. Recall that there

$$\mathrm{x(t)} = e^{-at} u(t), \qquad a > 0$$

and we saw that it is Fourier transform is, its Fourier transform, its Fourier transform was

$$x(\omega) = \frac{1}{a + j\omega}$$

Now, this says that the Fourier transform of

$$\mathrm{x(t} - t_0) = e^{-a(t-t_0)} u(t - t_0)$$

Fourier transform of this signal will be

$$\frac{e^{-j\omega t_0}}{a + j\omega}$$

So, this is an illustration of time and frequency.

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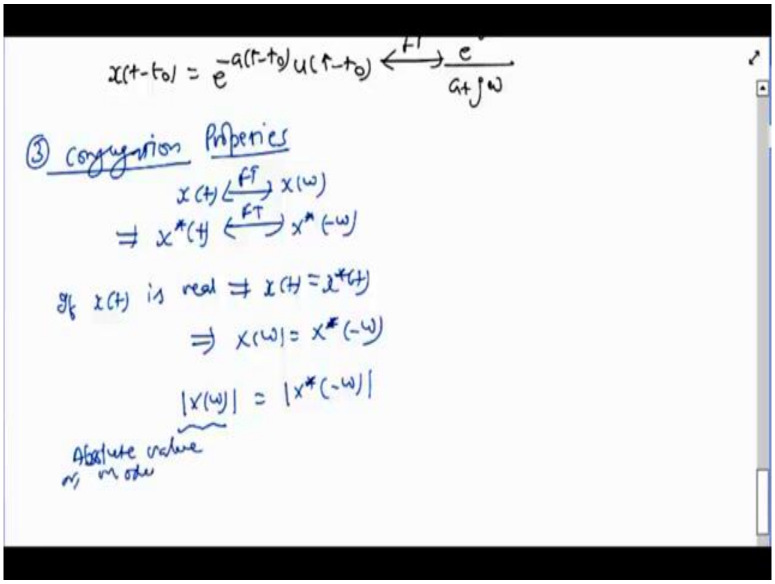

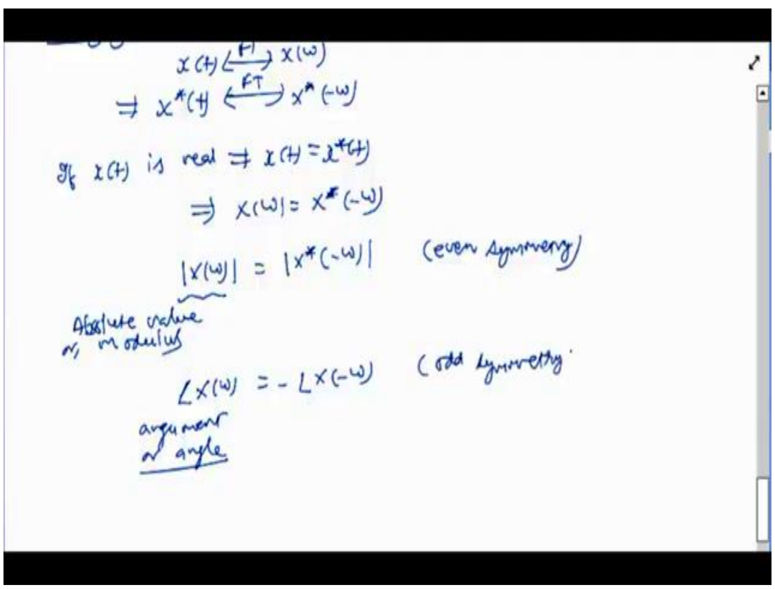

Let us move on to the next property, which is conjugation, conjugation property, it says that if x(t) has Fourier transform x(ω) then x*t which is complex conjugate of x(t) will have Fourier transform x*(-ω). So, in particular if x(t) is real, then $x(t) = x*(t)$, it is complex conjugate, this says that $x(\omega) = x*(-\omega)$

In particular, if I take modulus on both the sides, we will see that $|x(\omega)|$ which is also called absolute value of x(ω),

$$|x(\omega)| = |x * (-\omega)|$$

So, this is called absolute value or modulus or sometimes it is called magnitude or amplitude, these are all same things. Similarly, ∠x(ω) often also called argument of

$$\angle x(\omega) = -\angle x(-\omega)$$

So, this is called argument. So, you see that modulus, or absolute values have even symmetry, whereas angles have odd symmetry.

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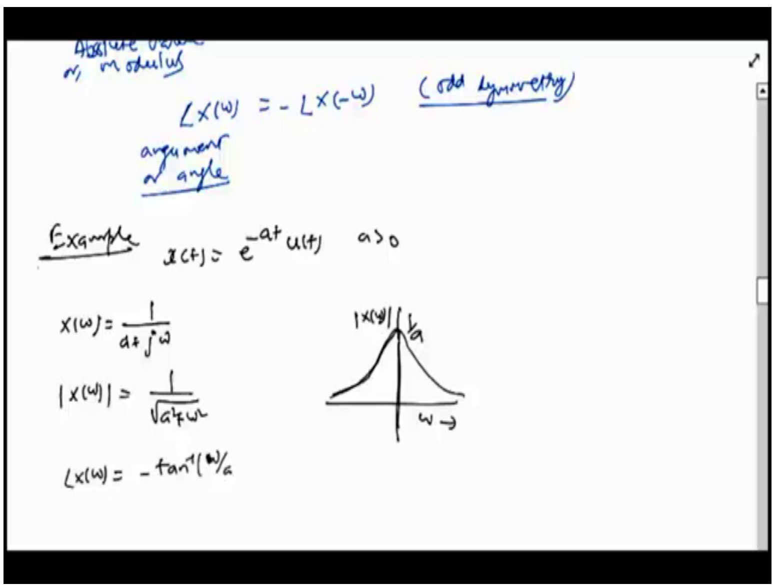

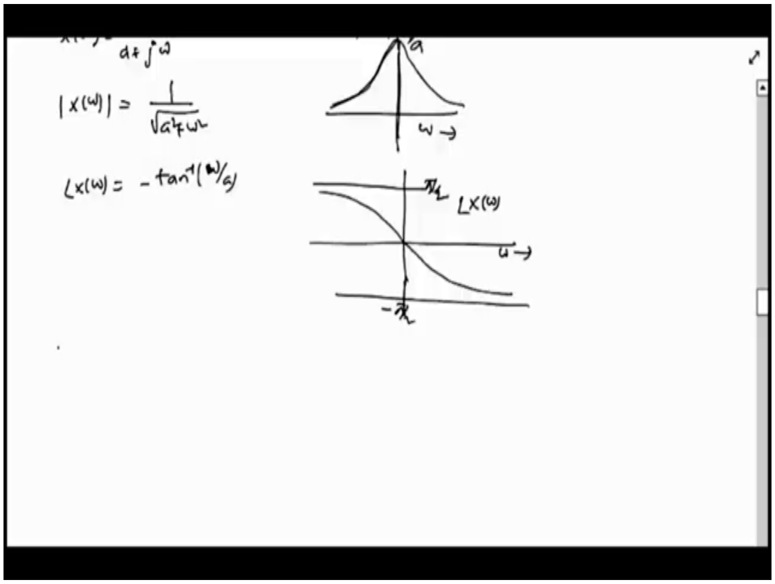

Let us see an example of this property, again recall the first example that we have taken that is

$$\mathrm{x(t)} = e^{-at}u(t), \qquad a > 0$$

recall that the Fourier transform of x(t)

$$x(t) = x(\omega) = \frac{1}{a + j\omega}$$

$|x(\omega)|$ in this case is $\frac{1}{\sqrt{a^2+\omega^2}}$, which is plotted looks as follows, its peak value is $\frac{1}{a}$. So, here is ω and this is $|x(\omega)|$,

$$\angle x(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

If you plot this angle, it looks as follows, these values are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ respectively, so here it is ω, this is $\angle x(\omega)$, in this example, x(t) is a real signal. So, we would expect x $|x(\omega)|$ to be real symmetric and $\angle x(\omega)$ to be odd symmetric, which is how they are in this picture. So, this is the conjugation property, this is an instance of the implication of conjugation property.

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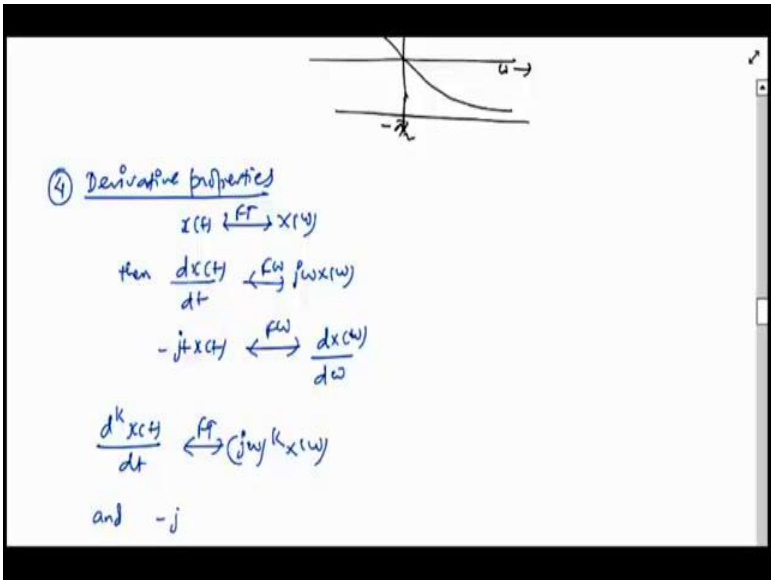

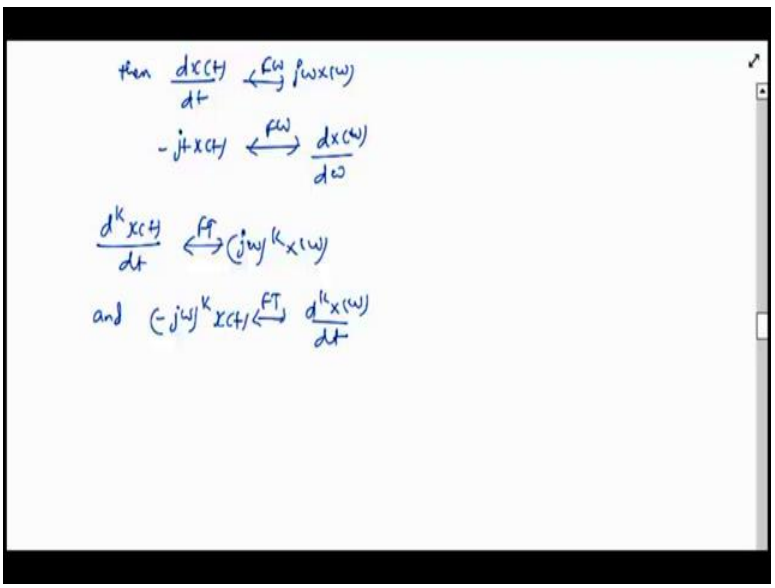

Next property that, we will see are called derivative properties. Let us say that x(t) is a signal, that has Fourier transform x(ω), if we differentiate the synthesis equation with respect to t, then we get as $\frac{dx(t)}{dt}$, will have Fourier transform $j\omega x(\omega)$. Similarly, if you differentiate the analysis equation, we see that $-jtx(t)$ will have Fourier transform that is derivative of Fourier transform of x(t), this is $\frac{dx(\omega)}{d\omega}$.

These are called derivative properties, we see that there is certain type of duality between these two relations, we will see in uses of this property in a while. Likewise, if we differentiate the synthesis and analysis equations, k times we get the following properties, we see that the k^{th}

derivative of x(t) has Fourier transform to $(j\omega)^k x(\omega)$ and the Fourier transform of $(-j\omega)^k x(t)$ is k[th] derivative of Fourier transform of x(t). You will see several uses of this derivative properties in a while.

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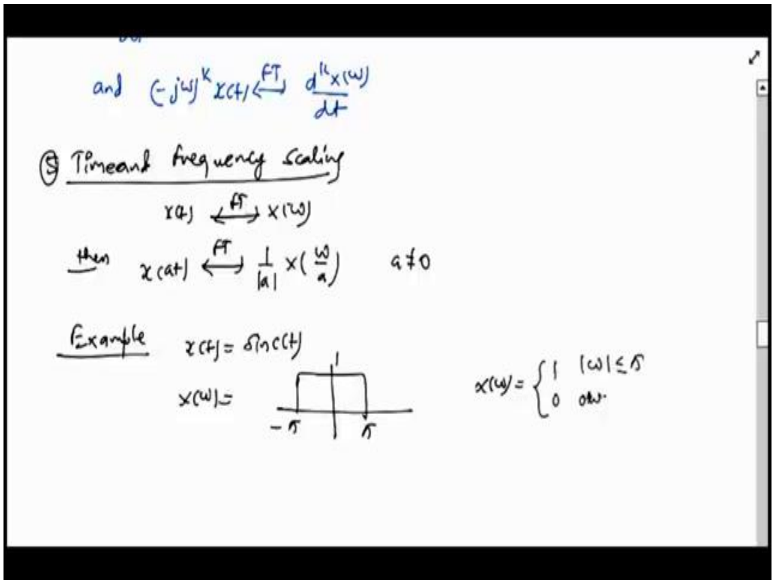

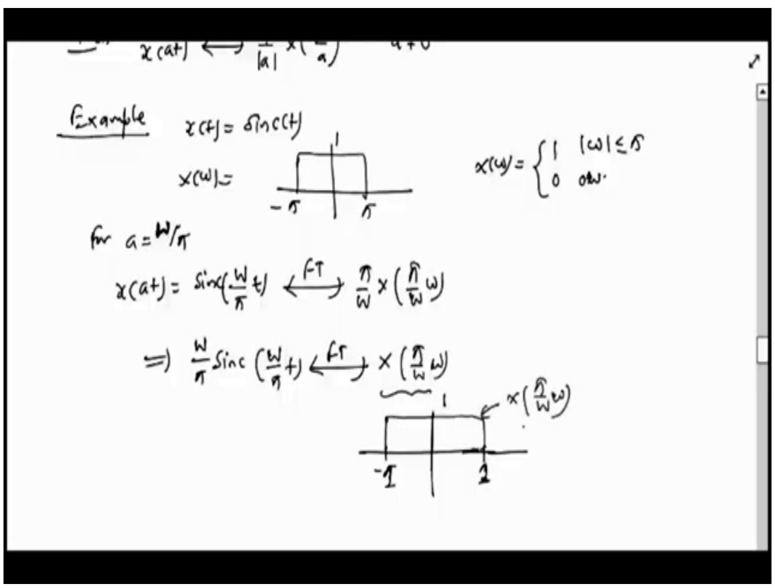

But for now, we will move to the next property that is, you will see several usage of derivative properties in a while, but for now, let us move to the next property, which is time and frequency scaling, time and frequency scaling properties. So, this says that if x(t) has Fourier transform x(ω), then x(at), where a≠0, will have Fourier transform

$$\frac{1}{|a|}x\left(\frac{\omega}{a}\right)$$

it is clear a is a nonzero number.

Let us see an example recall that x(t) = sinc(t) has Fourier transform x(ω) which is a pulse signal as follows. So, this is 1 between -π and π, n j to otherwise. So, x(ω) = 1 for ω between -π and π and 0 otherwise.

Now, if I choose a = $\frac{W}{\pi}$ and consider x(at), which is $sinc(\frac{W}{\pi}t)$, then Fourier transform of this signal will be

$$\frac{\pi}{W}x\left(\frac{\pi}{W}\omega\right)$$

which in turn implies that Fourier transform of W by

$$\frac{W}{\pi}sinc\left(\frac{W}{\pi}t\right) \overset{ft}{\leftrightarrow} x\left(\frac{\pi}{W}\omega\right)$$

If you see this signal, this is again a pulse signal, but between -1 and 1, this is a signal that takes value 1 between -1 and 1 and it is 0 otherwise, this is $x\left(\frac{\pi}{W}\omega\right)$.

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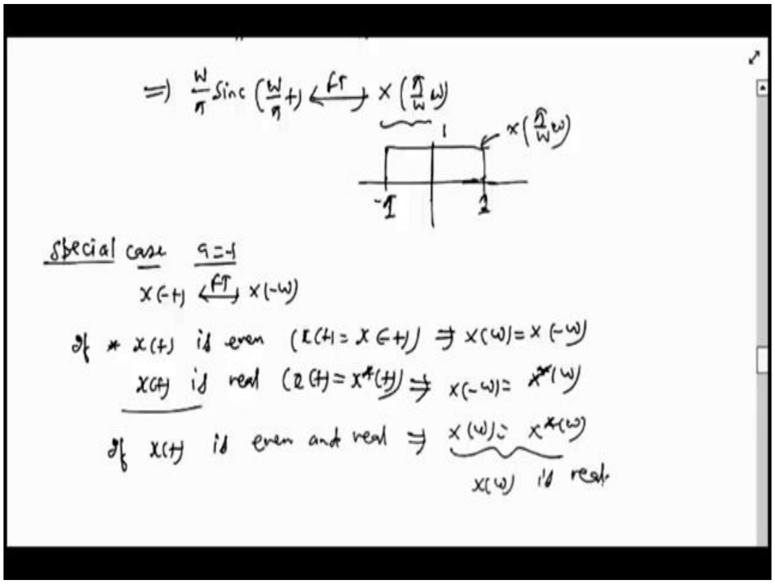

Let us now see a special uses of this time and frequency scaling properties. This is a special case, that quite important case and this is when a = -1, then the above property implies that the Fourier transform of $x(-t) \leftrightarrow X(-\omega)$.

Now, we can combine this property along with conjugation properties to get some interesting fact, in particular, if the x is real, sorry, if x is even which says that $x(t) = x(-t)$, then $x(-\omega) = x(\omega)$, this follows from the above equation.

If x(t) is moreover, if x(t) is a real, that is $x(t) = x^*(t)$, then the conjugation properties is that $x(-\omega) = x^*(\omega)$, that is conjugate of x*(omega), they combine these two properties, then we infer that if x(t) is real and even, even and real, then $x(\omega) = x^*(\omega)$, in other words, $x(\omega)$ is real.

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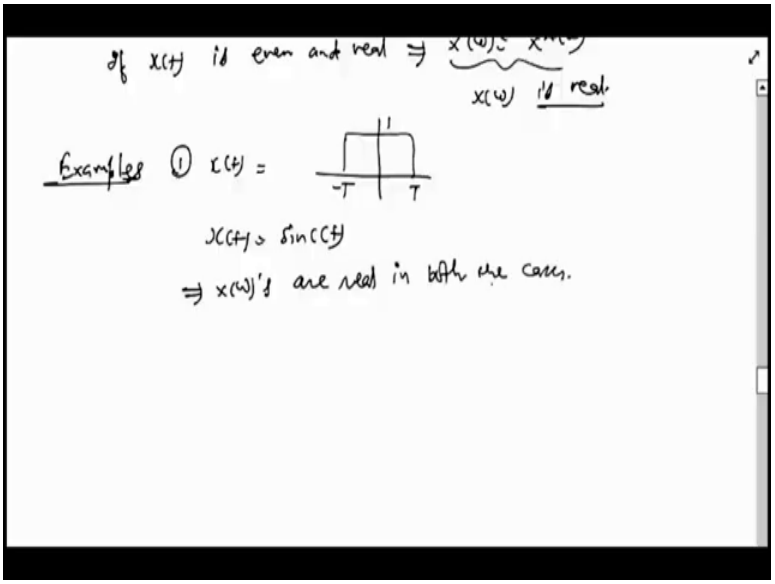

We have already seen several illustrations of this example for instance, let us consider just recall the two examples that we saw earlier, in the one x(t) was a pulse signal, this was our first example, the other one was where x(t) was sinc function, in both these examples, x(t) were real and so, as expected x(t) was real and even so as expected x(ω) also turned out to be a real in both the cases, x(ω) which are real in both the cases.

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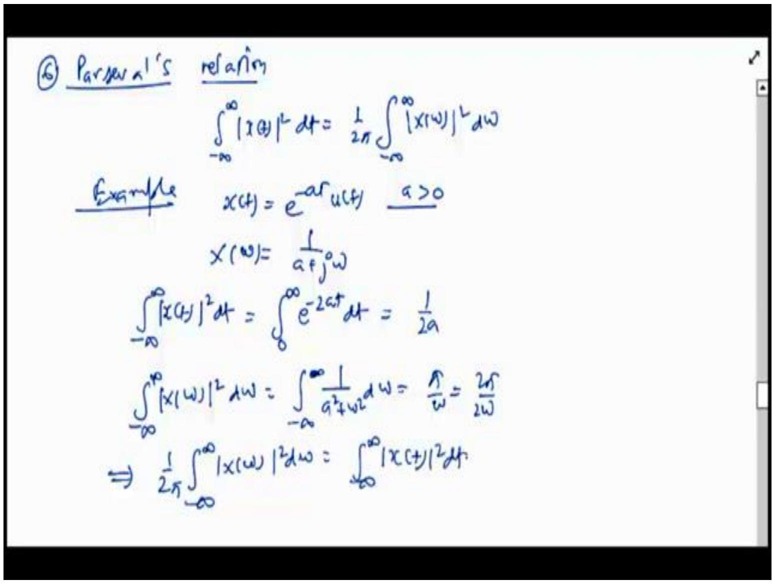

Let us now move to the next property, which is called Parseval's relation, this relation is as follows,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt$$

this can be expressed in terms of integral of mod of x, $|x(\omega)|^2$ and integrate. So, this relation is called Parseval's relation.

Let us see an example that illustrates this, again go back to the example where

$$\mathrm{x(t)} = e^{-at}u(t), \qquad a > 0$$

In this case, we have seen readily seen that

$$x(\omega) = \frac{1}{a + j\omega}$$

let us compute the following

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{e} e^{-2at} dt = \frac{1}{2a}$$

On the other hand if I compute the following

$$\int_{-\infty}^{\infty} |x(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega = \frac{\pi}{\omega} = \frac{2\pi}{2\omega}$$

This last equality, you applies that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

as claimed by Parseval's relation. So, this is an illustration of Parseval's relation.

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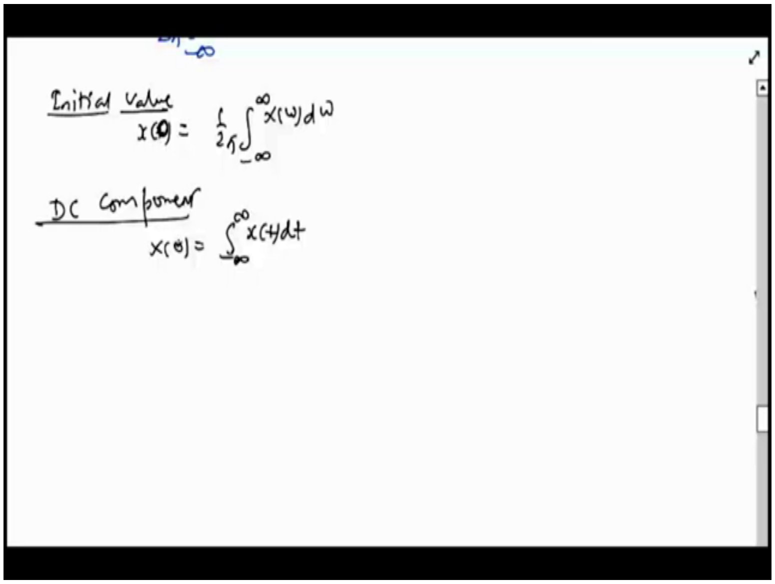

Next will move, you can see a few corollaries of this property that we have already seen. Namely, we see that the initial value of the signal n time domain can be that is

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) d\omega$$

This relation is obtained by simply setting t = 0 in the synthesis equation. Similarly, If I set ω = 0 in the analysis equation, I get the DC component of the signal namely,

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

See the right-hand side, it is taking to average of the time domain signal.

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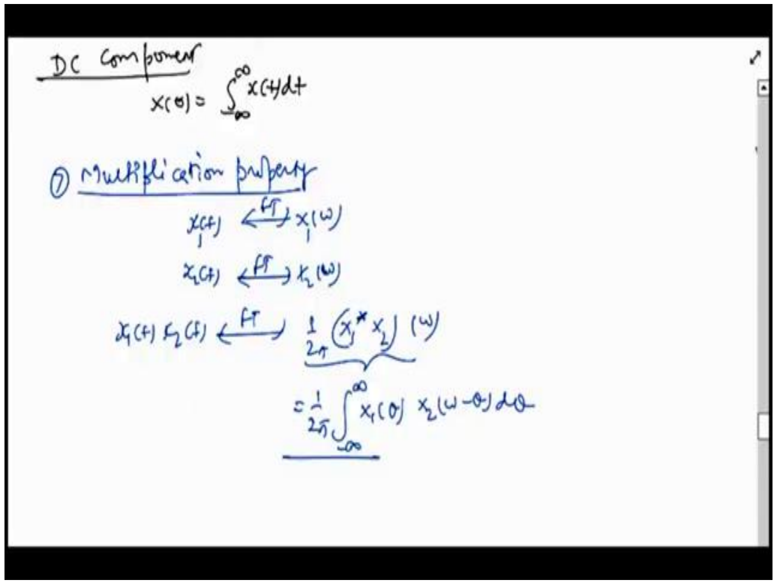

Next property is multiplication property, which says that, if x(t) has Fourier transform say $x_1(t)$ has Fourier transform $x_1(\omega)$, and $x_2(t)$ has Fourier transform $x_1(\omega)$ then product of the two time domain signals that is $x_1(t)x_2(t)$ will have Fourier transform, which will be

$$\frac{1}{2\pi}(x_1^* x_2)(\omega)$$

If we expand this we get

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(\theta)x_2(\omega - \theta)d\theta$$

So, this is Fourier transform product of two time domain signals.

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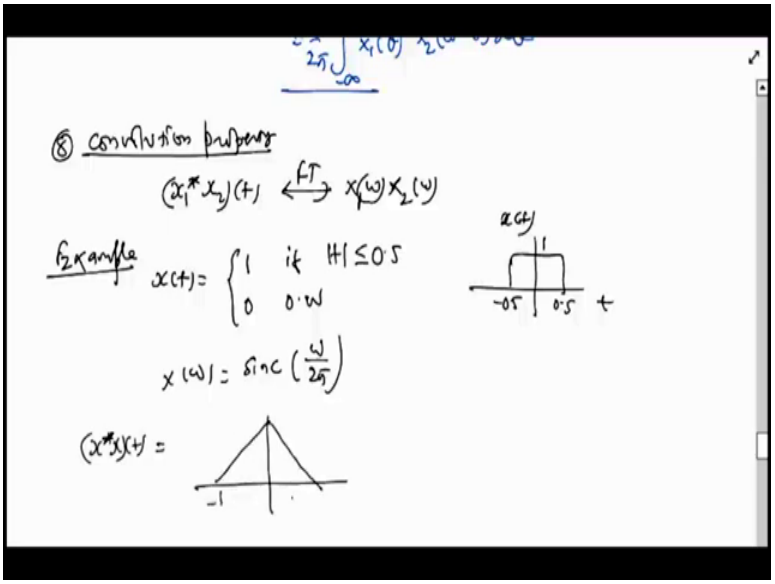

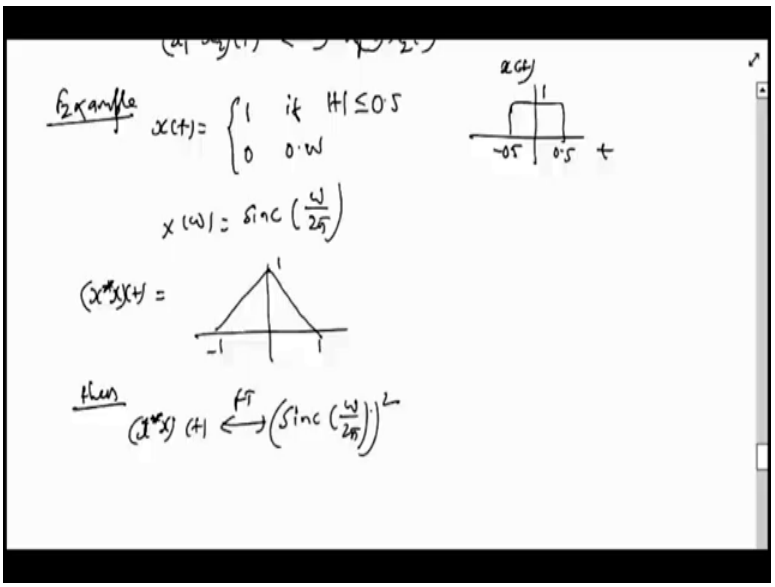

The dual of this region is what is called convolution property, is very important property, we will see many uses of this. So, convolution property, it says that if $x_1(t)$ has Fourier transform $x_1(\omega)$ and $x_2(t)$ has Fourier transform $x_2(\omega)$, then convolution of x_1 and x_2 that is $(x_1 * x_2)(t)$, this will have Fourier transform that will be simply the product of $x_1(\omega)\ x_2(\omega)$.

Let us see an example illustrating this property, recall that if a set x(t) to be equal to pulse signal that is 1, $|t| \leq 0.5$ and a 0 otherwise, here is my x(t), this is -0.5, 0.5, 1 t, x(t) then the Fourier transform of x(t),

$$x(\omega) = sinc\left(\frac{\omega}{2\pi}\right)$$

This is obtained by simply setting T = 0.5 in the example that we had seen earlier.

Now, in this case the convolution of x with itself that is (x*x)(t) turns out to be a triangular pulse, namely it is this function and the above convolution property states that, the Fourier transform of this pulse is $\left(sinc\left(\frac{\omega}{2\pi}\right)\right)^2$. So, in this way we get Fourier transform of a triangular pulse from Fourier transform of rectangular pulse.

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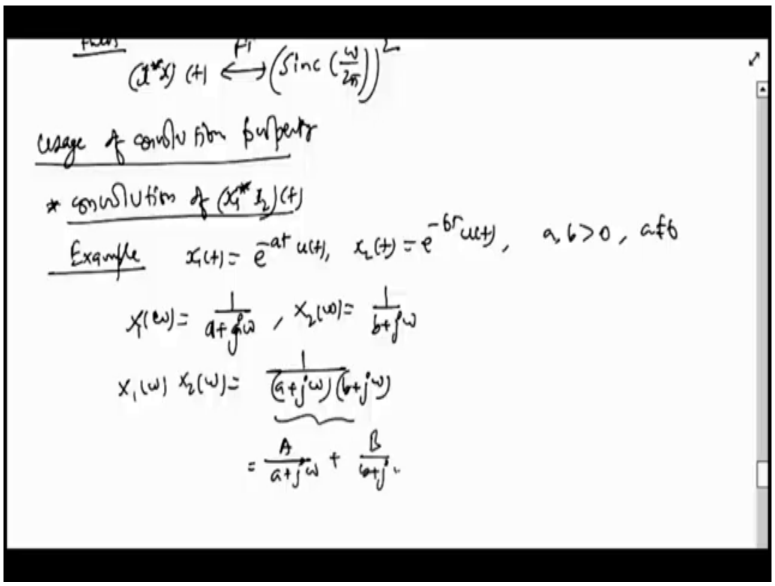

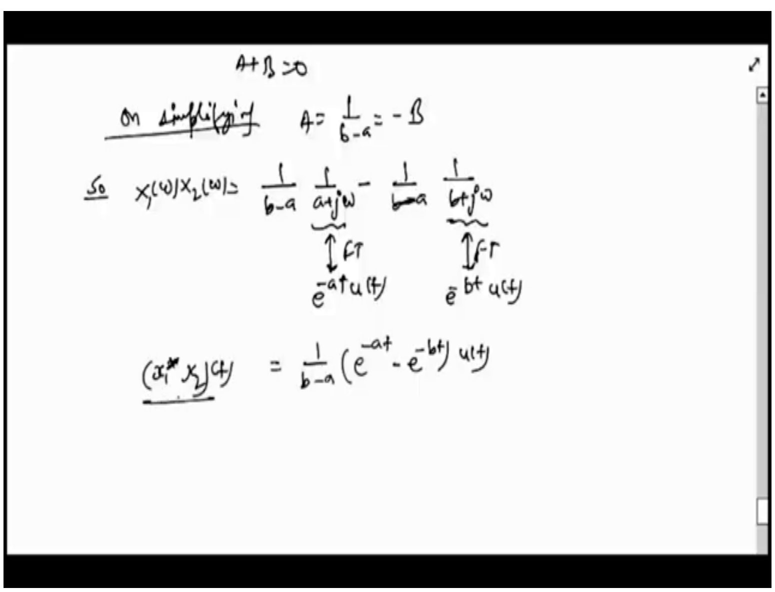

Let us now see such few uses of convolution property, as I indicated earlier this property has many uses and the first use is in computation of convolution of functions. The convolution of x_i, suppose, I have two signals $x_1(t)$ and $x_2(t)$, I can use this property to compute $(x_1 * x_2)(t)$, not always but in several interesting problems.

So, here is an example, let us say

$$x_1(\mathrm{t}) = e^{-at}u(t)$$

And

$$x_2(\mathrm{t}) = e^{-bt}u(t)$$

here a b are both positive and a and b are unequal, they are not same. We readily know that, the Fourier transforms

$$x_1(\omega) = \frac{1}{a + j\omega}$$

$$x_2(\omega) = \frac{1}{b + j\omega}$$

In this case, the product of the two Fourier transforms is,

$$x_1(\omega)x_2(\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

We know that the inverse of this Fourier transform, inverse of this product would give convolution of $x_1(t)$ and $x_2(t)$.

So, let us try to find out the inverse of this, towards this we will use the technique of partial fraction expansion, which says that we can write this right hand side is

$$= \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$

for two numbers a and b. This can further be worked out to the Ab + Ba + j,

$$\frac{A + B}{(a + j\omega)(b + j\omega)}$$

If we equate the constant and, and multipliers of ω, then we get that we see that Ab + Ba = 1 whereas A + B = 0, on simplifying this gives

$$A = \frac{1}{b - a} = -B$$

So, we see that

$$x_1(\omega)x_2(\omega) = \frac{1}{b - a}.\frac{1}{a + j\omega} - \frac{1}{b - a}.\frac{1}{b + j\omega}$$

We readily know that the Fourier transform of this signal inverse, rather inverse Fourier transform of this signal is $e^{-at}u(t)$ and for this function it is your inverse Fourier transform is $e^{-bt}u(t)$.

This says that inverse, inverse Fourier transform of

$$x_1(\omega)x_2(\omega) = \frac{1}{b - a}(e^{-at} - e^{-bt})u(t)$$

and from (convo) from the above property, this is equal to convolution of x_1* and x_2 sorry x_1 and x_2. So, we see how convolution property allows us to compute convolution of two functions.

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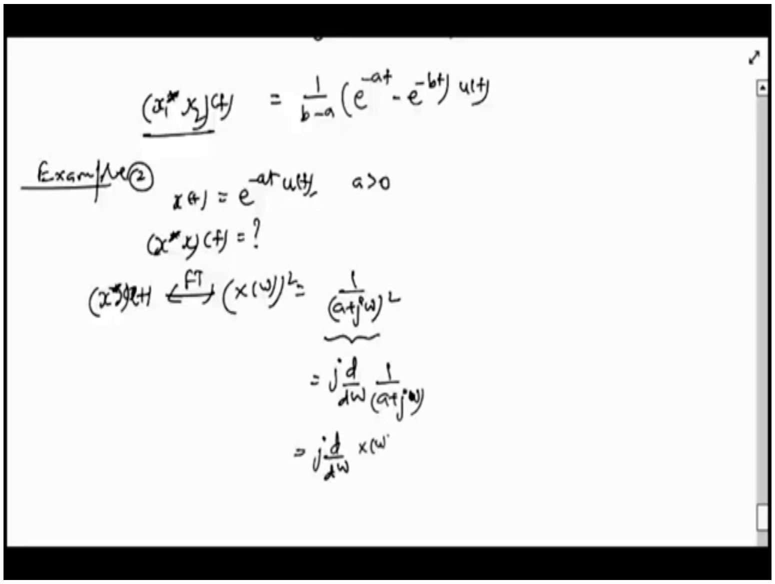

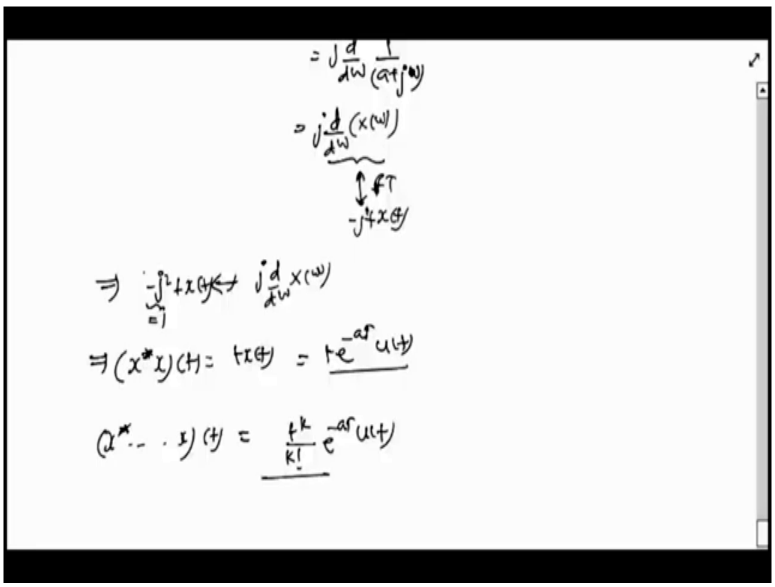

Let us see another example. I again take

$$x(t) = e^{-at}u(t), \qquad a > 0$$

Now, I am interested in convolution of x with itself, we see that we cannot use the partial fraction expansion property used ever, however, I can use convolution property along with derivative property to compute this convolution, to see that observe that (x*x)(t) will have Fourier transform which is

$$x(\omega)^2 = \frac{1}{(a + j\omega)^2}$$

$$= j\frac{d}{d\omega}.\frac{1}{(a + j\omega)}$$

We can compute the derivative and can verify this assertion.

So, this in turn is

$$= j\frac{d}{d\omega}x(\omega)$$

We know that $\frac{d}{d\omega}x(\omega)$ has inverse Fourier transform, which is

$$= -jtx(t)$$

this have inverse Fourier transform $-jtx(t)$. This says that this whole thing $j\frac{d}{d\omega}x(\omega)$ will have inverse Fourier transform, which will be $-j^2tx(t)$, this says that convolution of x with itself is nothing but $tx(t)$, notice that $-j^2 = 1$, which in this case is $te^{-at}u(t)$.

So, we see how we could use these properties to get convolution of x which is, in fact we can repeat this process k times to get k times convolution of x(t) and the result turns out to be as follows. So, Fourier transform k times convolution of x with itself t turns out to be

$$= \frac{t^k}{k!}e^{-at}u(t)$$

So, we see that this quite useful property.

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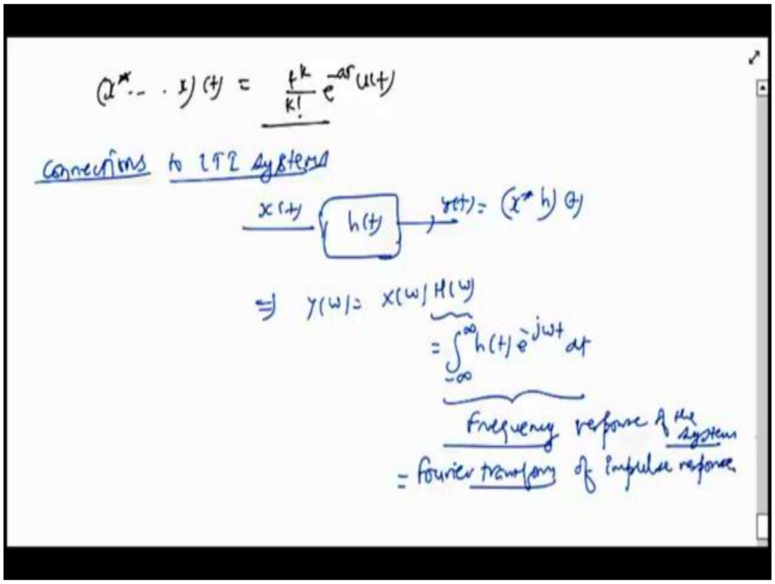

Now, we will see uses of these properties in analyzing LTI systems. So, we will see connections to LTI systems, to LTI systems. Recall that in LTI system whose impulse response is h(t), if we give input x(t), the output y(t) is convolution of x(t) and h(t), it is convolution of x(t) and h(t). So, from the convolution property mentioned above, we know that Fourier transform of y(t), that is

$$y(\omega) = x(\omega)H(\omega)$$

there $H(\omega)$ is given by, where $H(\omega)$ is a Fourier transform of h(t), so that is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

If we recall the first lecture, this is what we call, we defined as frequency response of the system, response of the system. So, we see that frequency response of the system is nothing but Fourier transform of impulse response, form of impulse response.

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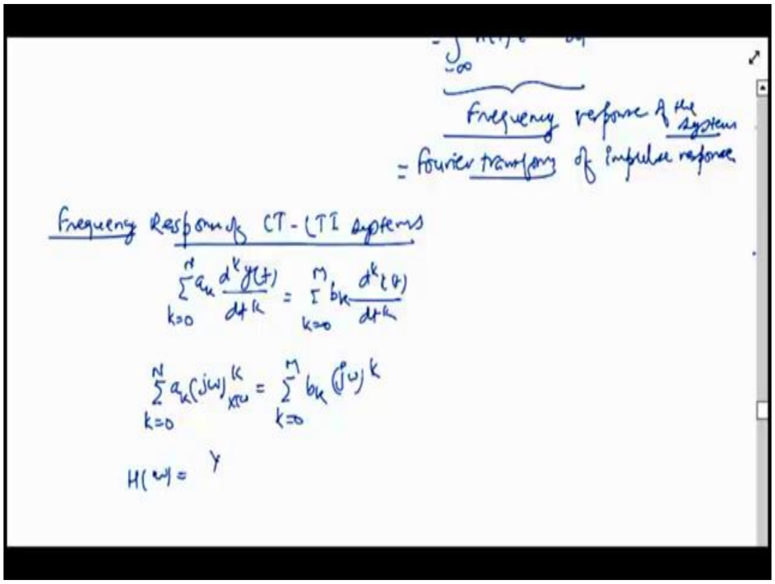

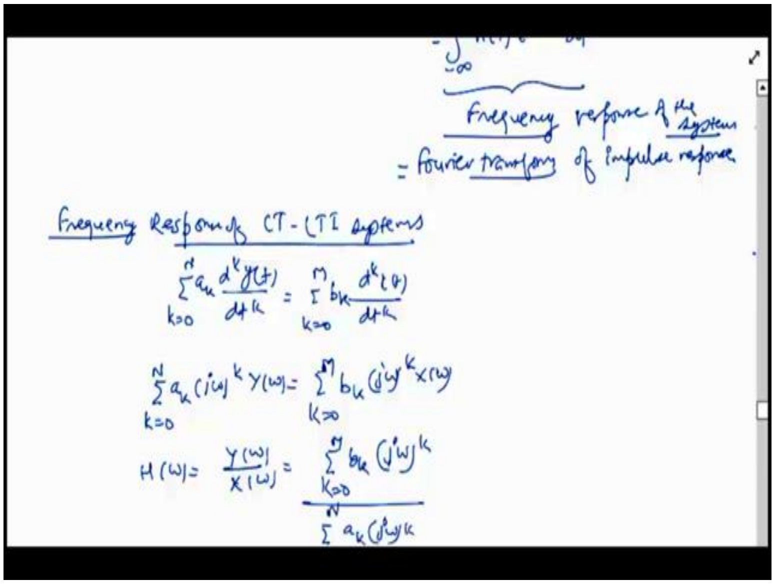

Let us now use the relation that we have just derived to get frequency response of continuous time LTI systems, frequency response of continuous time LTI systems, recall that systems, for systems are characterized by following differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

If we use time, if use derivative property we can write a Fourier transform of both the sides as follows

$$\sum_{k=0}^{N} a_k (j\omega)^k y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k x(\omega)$$

which in turn implies that from the definition of frequency response

$$H(\omega) = Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k x(\omega)$$

which from the definition of frequency response further implies that frequency response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

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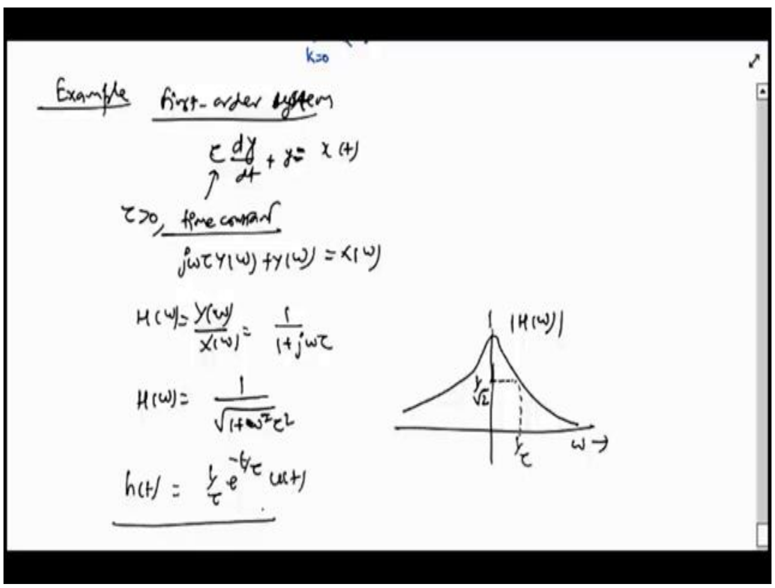

Let us see an application of it. So, an example, consider the following system called first order, a first order system given by differential equation

$$\tau \frac{dy}{dt} + y = x(t)$$

τ here is often referred to as time constant. If we compute the Fourier transform of both the sides using derivative property, we see that

$$j\omega\tau y(\omega) + y(\omega) = x(\omega)$$

which in turn gives the frequency response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega\tau}$$

Notice that the modulus or the absolute value of this function

$$H(\omega) = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

which if we plot looks as follows. So, this is absolute value of $H(\omega)$, peak value is 1 and at $\omega = \frac{1}{\tau}$ the value is $\frac{1}{\sqrt{2}}$. So, this value $\frac{1}{\tau}$ has special significance.

In fact, we can compute the inverse Fourier transform of $H(\omega)$ to get an impulse response of the system which turns out to be

$$= \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$$

Here we have used scaling property of Fourier transform and earlier result on Fourier transform of such functions. So, this was an example of Fourier transform to LTI systems.

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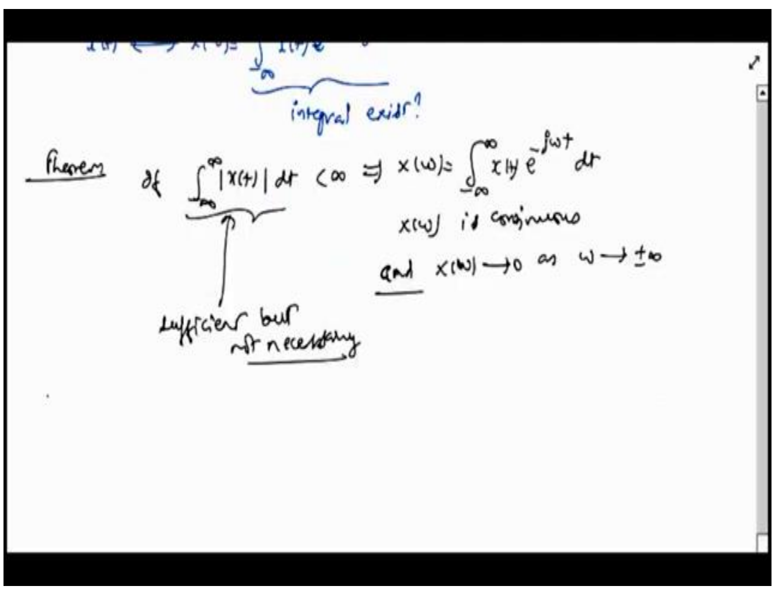

Let us now take a step back and look at the definition of Fourier transform. Notice that, recall that Fourier transform for a signal

$$x(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

this very definitions raises a few questions for instance, does this integral, when does this integral exist? Are there any sufficient condition for existence of this integral? What happens for x(t) for which this integral does not exist? So, let us try to answer a few of these questions.

First, we will state a result that ensures existence of this integral. So, here is a theorem that says that, if

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

then this integral exists, not only it exists, X(ω) is continuous and moreover it approaches 0 as ω goes to ∞ or -∞, so X(ω) approaches 0 as ω approaches ±∞. However, notice that this is a sufficient condition as we will see soon it is not a necessary condition for existence of Fourier transform in the sense discussed here. So, this is sufficient, but not necessary.

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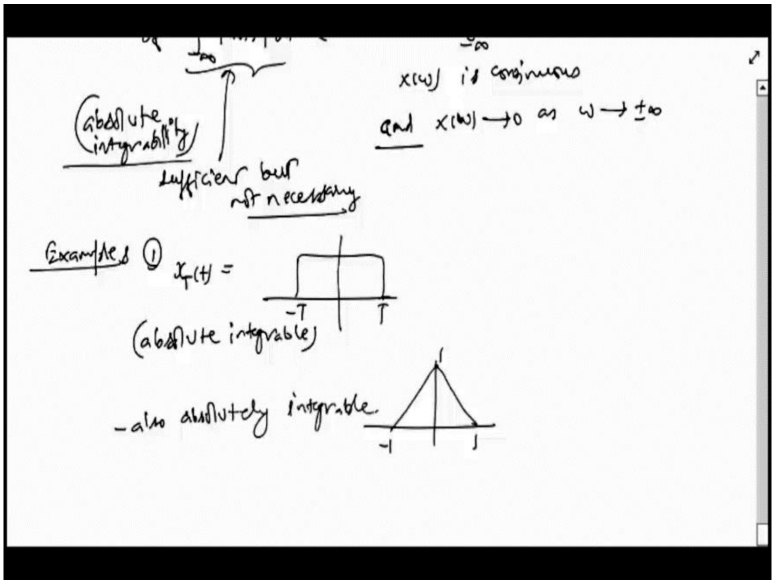

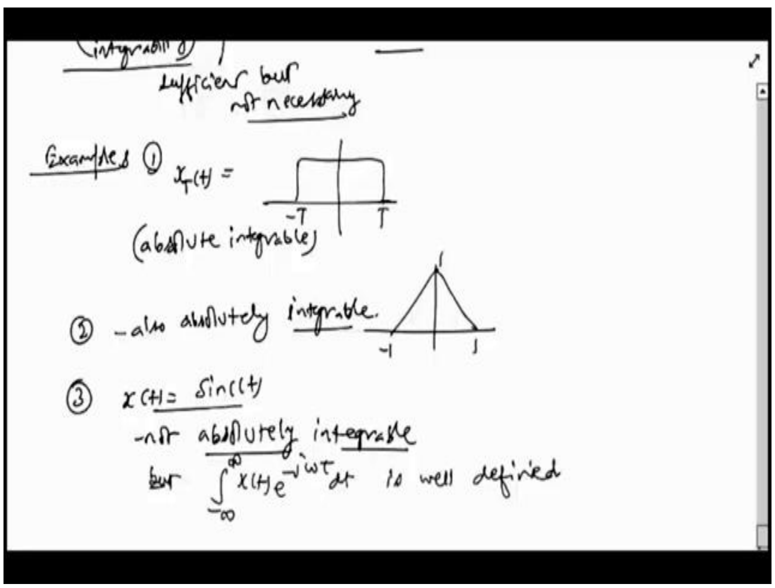

Let us revisit a few of the examples that we saw earlier to clarify this point, one of the examples as x(t) to be equal to pulse signal -T to T, let us call this signal $x_T(t)$. So, this is $x_T(t)$, clearly this is absolutely, so this notion of, this notion, this in the left hand side, this is called absolute integrability.

So, absolute integrability is enough to ensure existence of Fourier transform. So, this $x_T(t)$ is absolutely integral. So, its Fourier transform is guaranteed to exist, the same goes for the signal, let us say a triangular signal. Recall this signal which was convolution of two rectangular pulses. This is also absolutely integral, also absolutely integral. So, above theorem, implies existence of

Fourier transform for this absolute, this is absolute, implies existence of Fourier transform of the signals.

On the other hand, if we see the sinc, yeah, if we consider the sinc function where $x(t) = sinc(t)$, this is not absolutely integrable. I will not get into details of why it is not absolutely integrable. But it is important to notice that even though it is not absolutely integrable, it is Fourier transform exists, that is the integration that defines Fourier transform is well defined,

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

is well defined.

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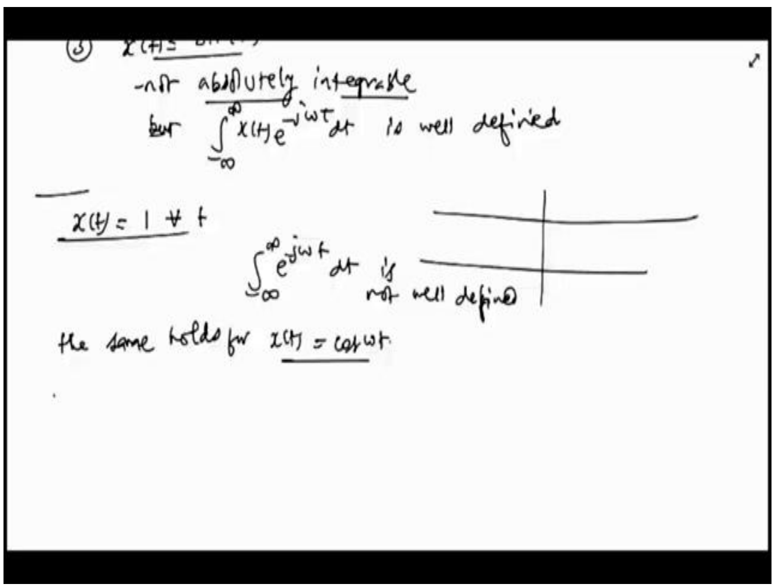

But what about the signals for which this integral, Fourier transform integral is not well defined? Let us see, for instance, x(t) =1 $\forall t$. So, this is a constant signal, for this signal,

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

is not well defined. The same holds for x(t) equals to, so the same holds for x(t) equals to sinusoidal signal. That is, that is the same holds for $x(t) = cos\omega t$. So, what do we do in these cases? It turns out that in this case we can appeal to a generalized, more generalized notion of Fourier transform. And that is what I have explained, now via an example.

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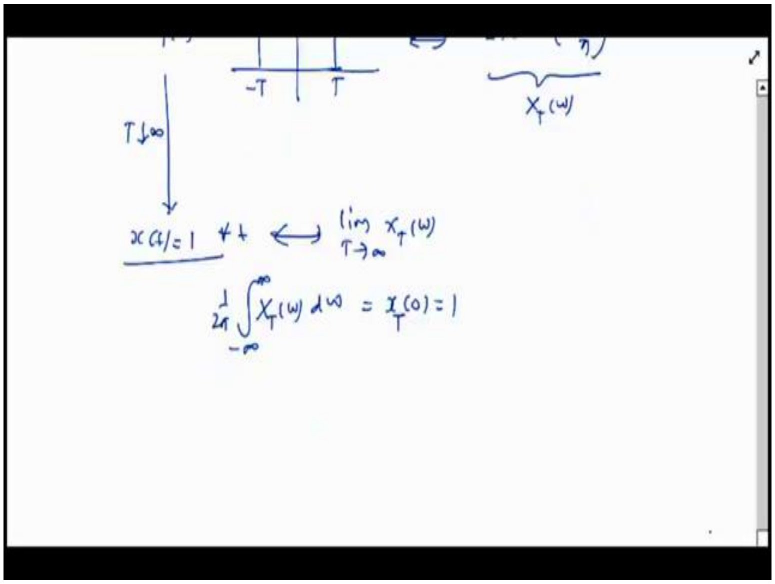

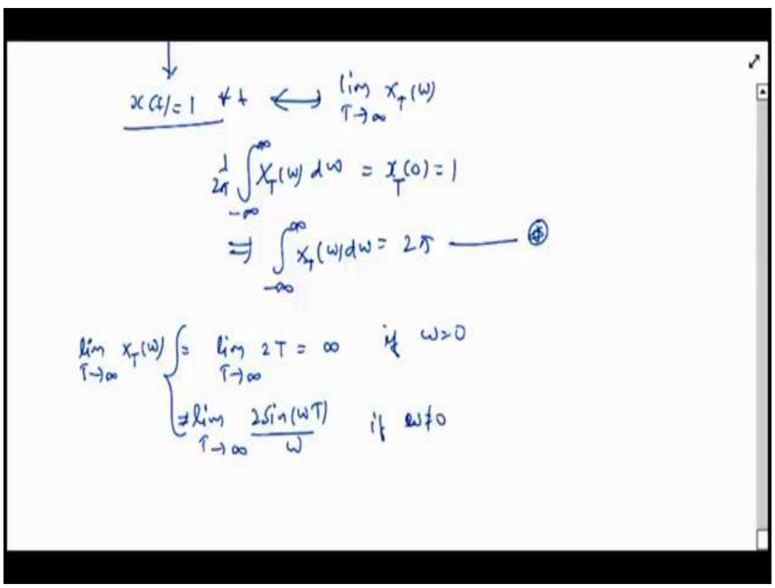

So, let us consider, so this is what we are going to see is a generalized notion of Fourier transforms that does not necessarily require the Fourier transform integral to exist, generalized notion of Fourier transform. Let us recall the signal $x_T(t)$ just mentioned above, so $x_T(t)$ is -T to T, so 1 and it is 0 everywhere else.

Recall that its Fourier transform is $2Tsinc(\frac{T\omega}{\pi})$. Let us call this $x_T(\omega)$, I am writing subscripts capital T just to show the dependence on capital T. Now, as T goes to ∞, as T goes to ∞, this $x_T(t)$ converges to the constant function, why, its period becomes longer and longer. So, it converges to constant function, where.

So, we would expect that the Fourier transform of the constant function will be given by

$\lim_{T\to\infty} X_T(\omega)$. Now, let us see what this right hand side is. So, this $X_T(\omega)$ has a couple of interesting properties, first property says that if I integrate

$$\int_{-\infty}^{\infty} X_T(\omega)\, d\omega$$

I know from the initial value property that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(\omega) d\omega = x_T(0) = 1$$

So, one of the properties of

$$\int_{-\infty}^{\infty} X_T(\omega)\, d\omega = 2\pi$$

Let us call this property star and observe another quite interesting property of $X_T(\omega)$ which is following. If a compute

$$\lim_{T\to\infty} X_T(\omega) = \begin{cases} \lim_{T\to\infty} 2T = \infty, & \omega = 0 \\ \lim_{T\to\infty} \dfrac{2\sin(\omega T)}{\omega}, & \omega \neq 0 \end{cases}$$

Now, if I plot this function $X_T(\omega)$ for larger and larger values of capital T, I see the following, I observe the following.

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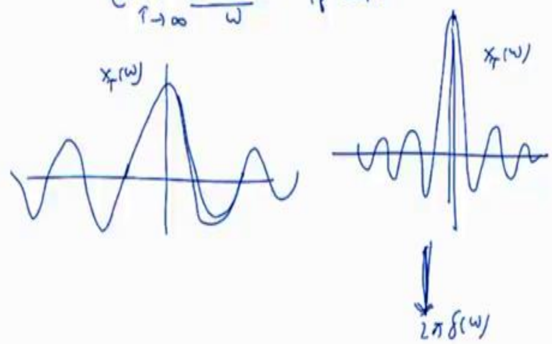
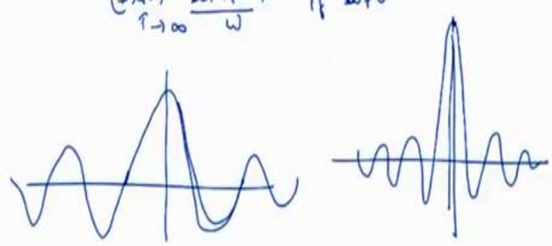
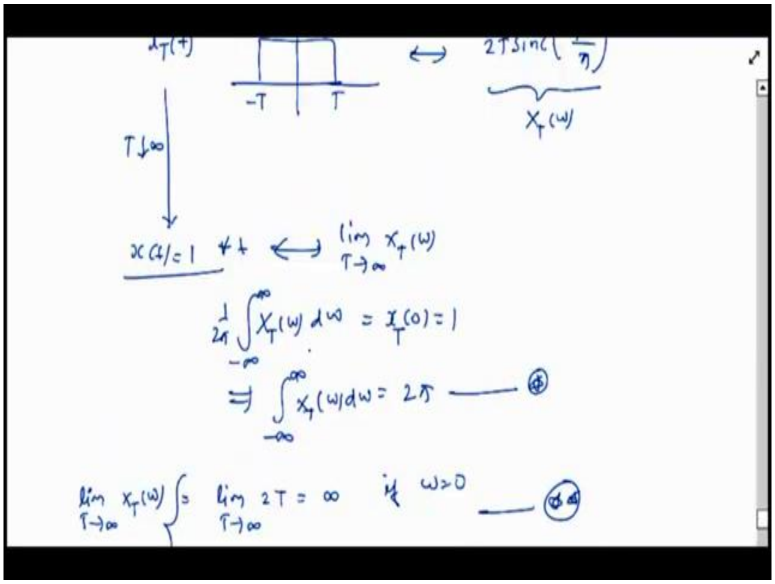
$$\lim_{T\to\infty} X_T(\omega) \begin{cases} = \lim_{T\to\infty} 2T = \infty \\ = \lim_{T\to\infty} \dfrac{2\sin(\omega T)}{\omega} & \text{if } \omega \neq 0 \end{cases}$$



$$\Rightarrow \int_{-\infty}^{\infty} X_T(\omega)\,d\omega = 2\pi$$

$$\lim_{T\to\infty} X_T(\omega) \begin{cases} = \lim_{T\to\infty} 2T = \infty & \text{if } \omega = 0 \\ = \lim_{T\to\infty} \dfrac{2\sin(\omega T)}{\omega} & \text{if } \omega \neq 0 \end{cases}$$





Suppose for some T, I have the following plot, if I increase T the amplitude at ω=0 increases and at the same time the plot gets compressed. So, I see something of the following sort. So, we can imagine that this picture along with these two properties namely star and double star will imply that as we increase T to ∞, this $X_T(\omega)$ converges to Dirac delta function, more precisely it converges to $2\pi\delta(\omega)$. See 2π here coming from, is coming from the fact that

$$\int_{-\infty}^{\infty} X_T(\omega)\, d\omega = 2\pi$$

for all values of capital T that gives rise to this

$$2\pi\delta(\omega)$$

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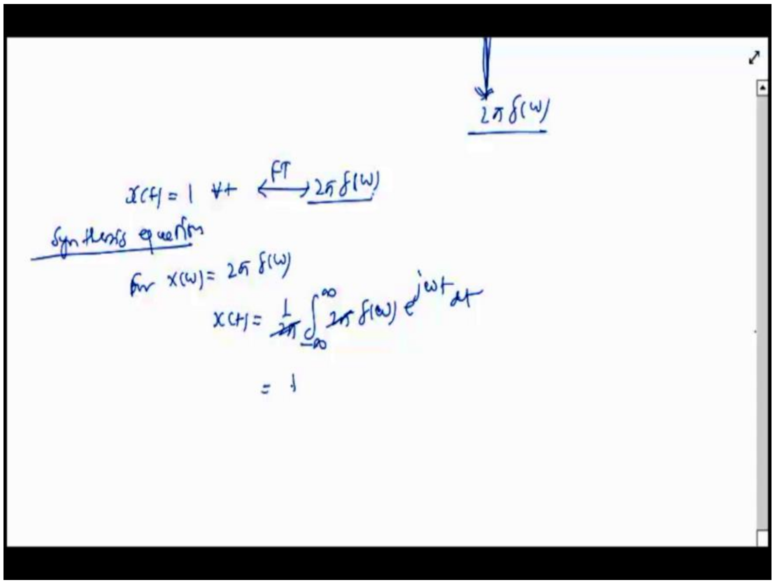

So, from the above we can conclude that this constant function has Fourier transform $2\pi\delta(\omega)$, even though the above argument looks a little complicated, the same can be verified from the following observation. If we use synthesis equation for Dirac delta function, then we see that for

$$X(\omega) = 2\pi\delta(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}dt = 1$$

So, we can follow the similar logic to get dual results as follows.

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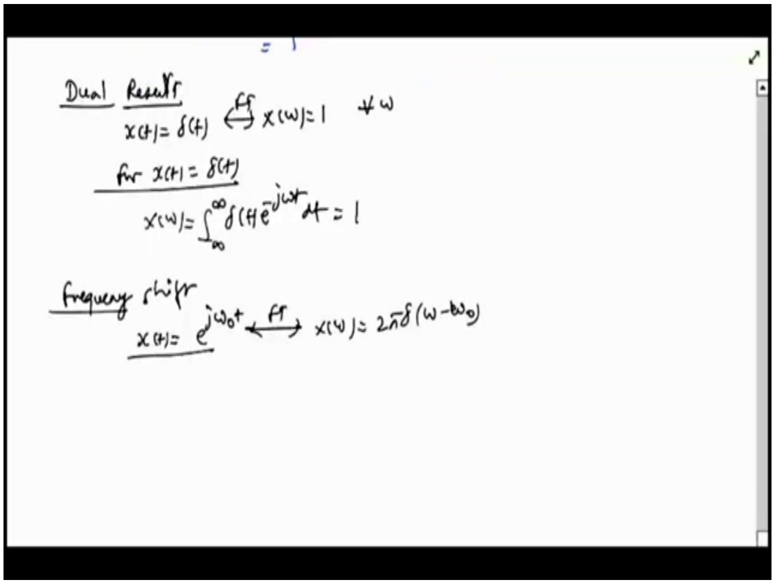

So, the dual to the above pair is if x(t) is Dirac delta function, then its Fourier transform is 1 for all ω, it is constant function for all ω. Again, this can be justified with analysis equation which says that for $x(t) = \delta(t)$ analysis equation implies that

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Now, we can use frequency shift property to derive a generalization of the above relation, shift, if we do a frequency shift to get $x(t) = e^{j\omega_0 t}$ then its Fourier transform turns out to be

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

So, this is a more general result.

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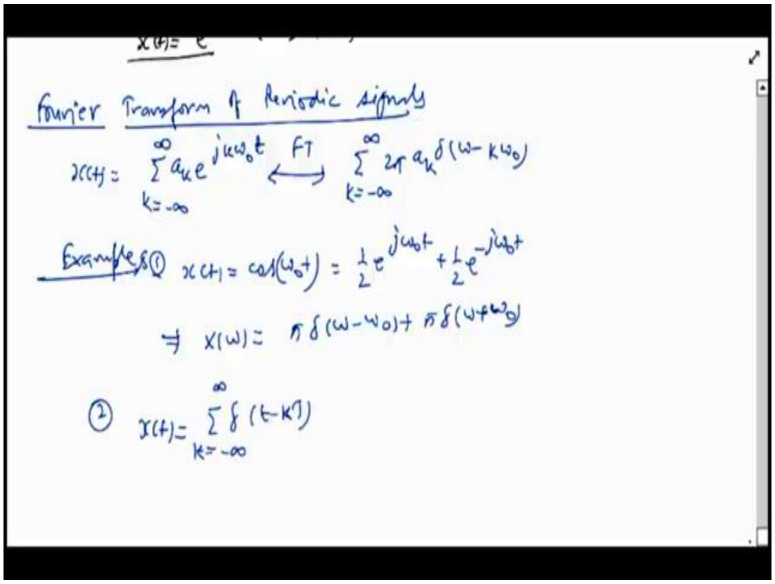

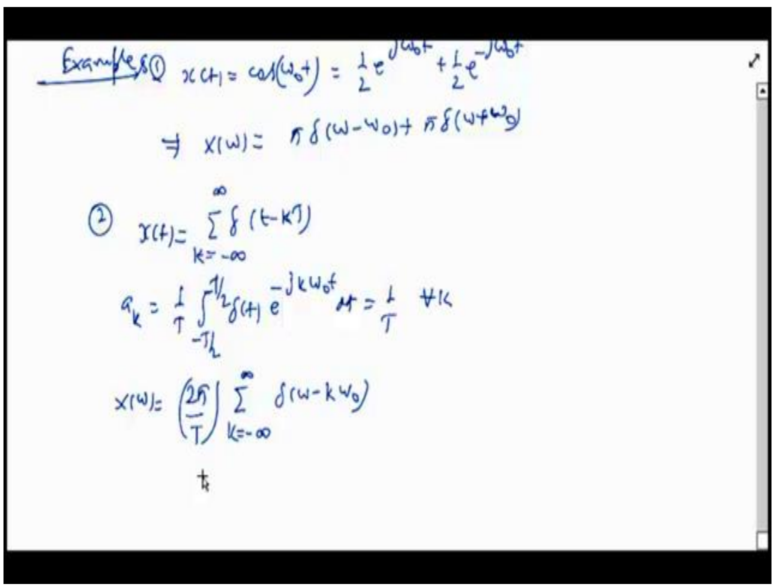

Now, let us see a few of the implications of this result, few of the uses of this result until now, we have seen Fourier transform for aperiodic signals, but what if we are faced with periodic signals, Fourier transform of periodic signals. Let us say we have a periodic signal x(t) for which we can write the Fourier series. So,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

We can then use linearity along with the above property to write the Fourier transform of x(t) to be

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Let us see an example, if

$$x(t) = cos\omega_0 t = \frac{1}{2} e^{jk\omega_0 t} + \frac{1}{2} e^{-jk\omega_0 t}$$

And accordingly, the Fourier transform of x(t) can be written as

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Let us see another example, suppose, x(t) is an impulse stream that is

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

In this case, we can compute the Fourier series coefficients of x(t) as follows. So, xt as,

$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \ \ \forall k$$

So, now, I can write the Fourier transform of x(t)

$$X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_0)$$

Notice that $\frac{2\pi}{T}$ is constant so I can get out of the summation and Fourier transform can also be written as follows.

$$X(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

So, this brings us to the end of our discussion of Fourier transform for continuous time signals. In next lecture we will look at Fourier transform on discrete times.