Mathematical Aspects of Biomedical Electronic System Design

Professor. Chandramani Singh

Department of Electronic Systems Engineering

Indian Institute of Science, Bangalore

Week – 11

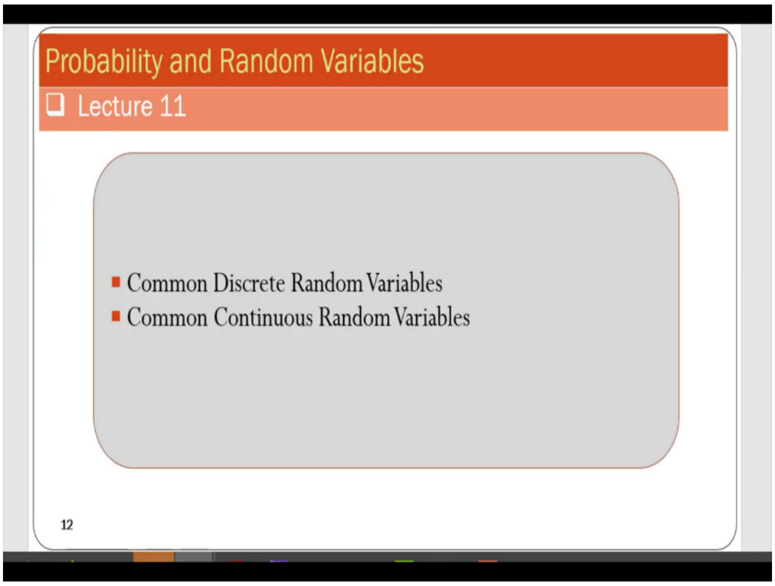
Random Variables and Signal Conditioning Circuits

Lecture – 34

Common Random Variables

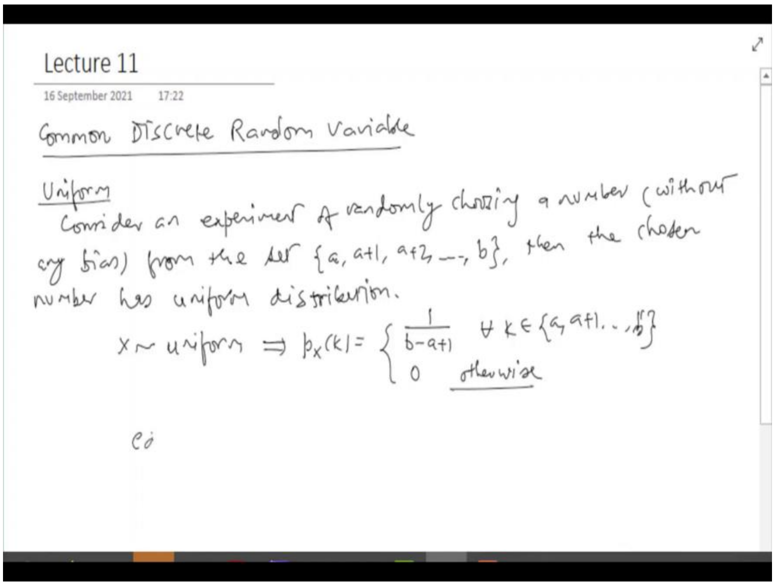
Hello everyone, welcome to another lecture of the course Mathematical Aspects of Biomedical Electronic System Design.

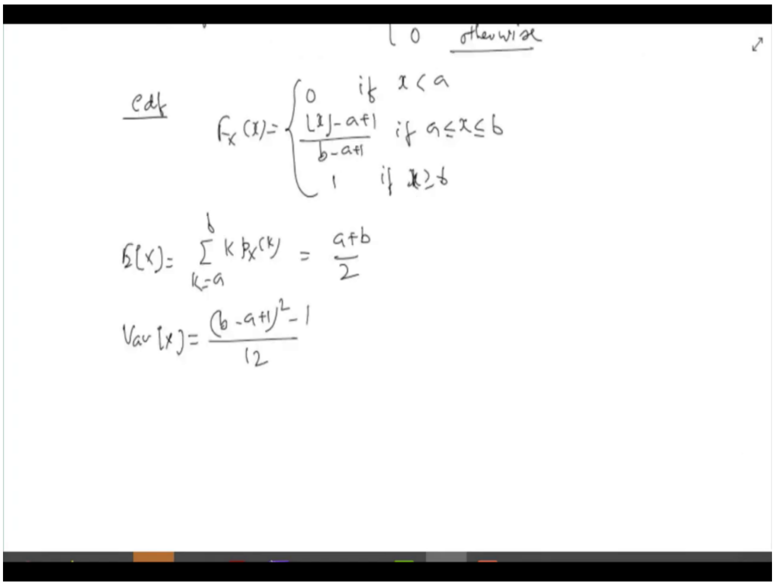
(Refer Slide Time: 0:36)



Today we will continue with our discussion on random variables. In today's lecture we will see a few common discrete and continuous random variables. We will see what their means and variances are. We will also see quite a few examples. So, let us start with today's lecture.

(Refer Slide Time: 0:55)





We will start with common discrete random variables. Let us begin with uniform random variable. Consider an experiment of randomly choosing a number. When I say randomly choosing I mean without any bias on the set, {a, a+1, a+2 … b}. Here a and b are two numbers, then the chosen number has uniform distribution. That is the probability that it will take any of this values a, a+1, a+2, etcetera is seen.

So, when we say that x is uniform what we mean that probability mass function of x is a constant. It does not depend on k, b-a+1 for all k, a, a+1 … b, notice that there are total b-a+1 elements in this set. So each one has probability $\frac{1}{b-a+1}$. It is 0 otherwise.

Clearly, the cdf of x can be written as

$$F_X(x) = \begin{cases} 0 \ \ if \ x < a \\ \dfrac{\lfloor x \rfloor - a + 1}{b - a + 1} \ \ if \ a \leq x \leq b. \\ 1 \ \ if \ x \geq b \end{cases}$$

Note that the floor of a number x is the largest integer that is less than or equal to x. For instance floor of 2.2 is 2, floor of 2.001 is 2, floor of 2.999 is also 2 but floor of 3 is 3.

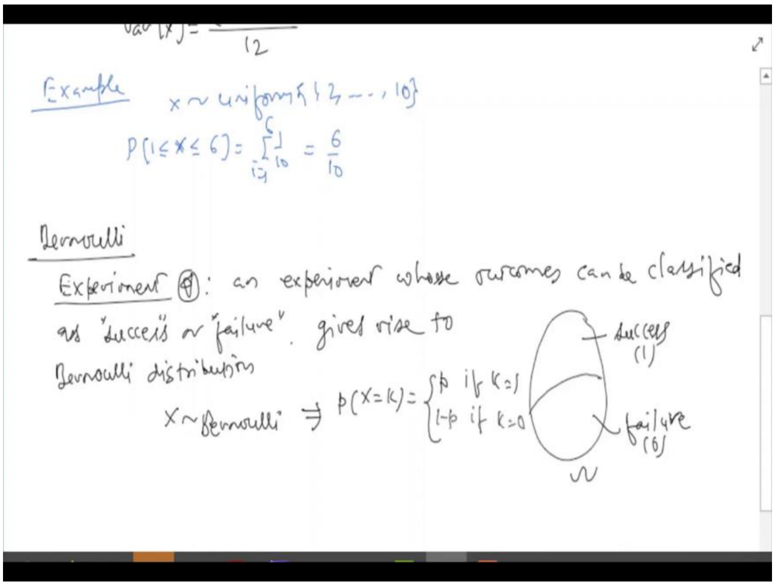
Next we can see that mean of the uniform random variable can be computed using the formula that we had seen earlier. So, it will be

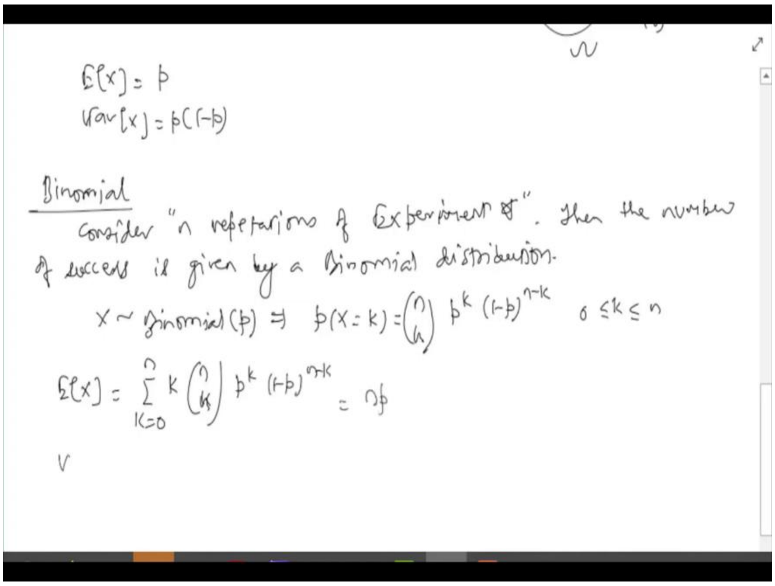
$$E[X] = \sum_{k=a}^{b} kP_X(k) = \frac{a+b}{2}.$$

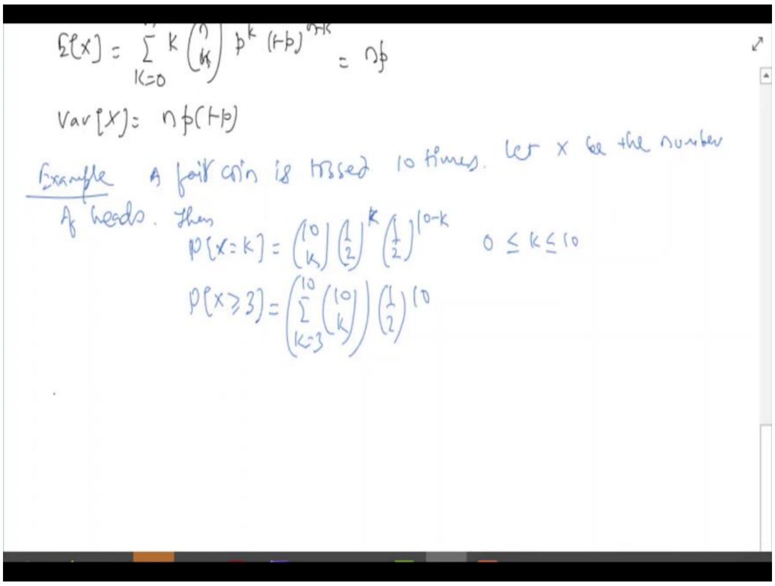
Moreover,

$$Var[X] = \frac{(b-a+1)^2 - 1}{12}.$$

(Refer Slide Time: 04:24)







Let us see an example. Suppose x is uniform over the set {1, 2 … 10}. Then what will be probability that X will be between 1 and 6. So,

$$P(1 \leq X \leq 6) = \sum_{i=1}^{6} \frac{1}{10} = \frac{6}{10}.$$

Let us now move to the next random variable and that is Bernoulli. Before we see Bernoulli random variables, let us consider an experiment and this experiment will refer to again and again in today’s lecture. So, let me call it experiment star. Let us see what it is. It is an experiment whose outcomes can be classified as success or failure.

For instance what we mean that if this is the set of all outcomes Ω, then we can divide the set of outcomes in 2 parts. These outcomes correspond to success and these outcomes correspond to failure. For instance we can think of tossing a die where outcomes are 1 to 6 but we may consider odd numbers as success and even numbers as failure. So, we can partition the set of outcomes in these 2 parts. Let us call success 1 and failure 0, then this kind of classification gives rise to Bernoulli random variable. So, this gives rise to Bernoulli random variable or Bernoulli distribution.

What is Bernoulli distribution? If X is Bernoulli, P this mean the collective probability of all the outcomes that lead to success is P. So,

$$P(X = k) = \begin{cases} P & if\ k = 1 \\ 1 - P & if\ k = 0 \end{cases}.$$

One can easily see that mean and variance of this random variables are P and P(1-P) respectively. In fact, we had seen this calculation in the previous lecture.

Now we look at the third random variable that we commonly see and it is binomial. To understand binomial random variable let us consider n repetitions of the above experiment that is experiment star and count the number of successes, then the number of successes is given by binomial distribution. So, when we say that X has binomial distribution with parameter P, what we mean that the probability mass function of this random variable is

$$P(X = K) = \binom{n}{K} P^K (1 - P)^{n-K}, \quad 0 \leq K \leq n.$$

We do n repetitions the number of successes could be anything between 0 and 8. To compute the mean of this random variable let us observe that expected value of X

$$E[X] = \sum_{K=0}^{n} K \binom{n}{K} P^K (1 - P)^{n-K} = nP.$$

Similarly,

$$Var[X] = nP(1 - P).$$

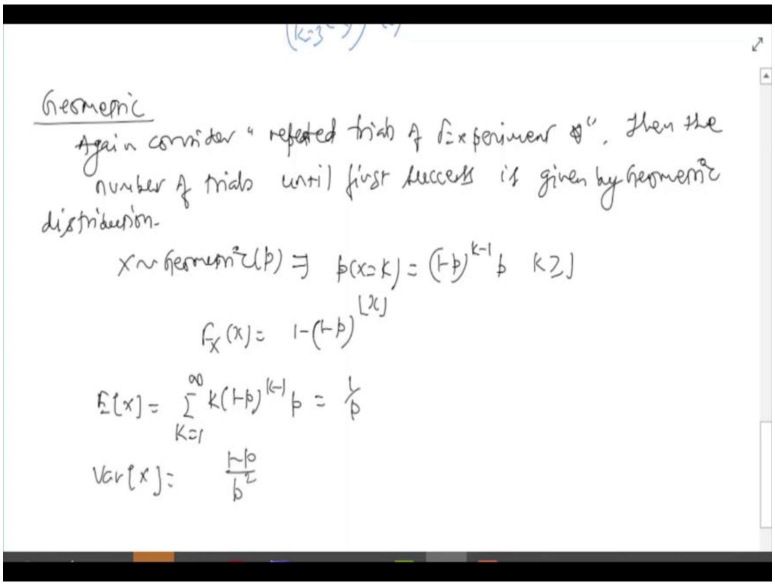
Let us consider an example. Suppose a fair coin is tossed 10 times. So tossing a fair coin is an instance of experiment star. Here success may correspond to head, failure may correspond to tail. A fair coin is tossed 10 times. Let X denote the number of heads, be the number of heads, then clearly from the above description, X follows a binomial distribution and

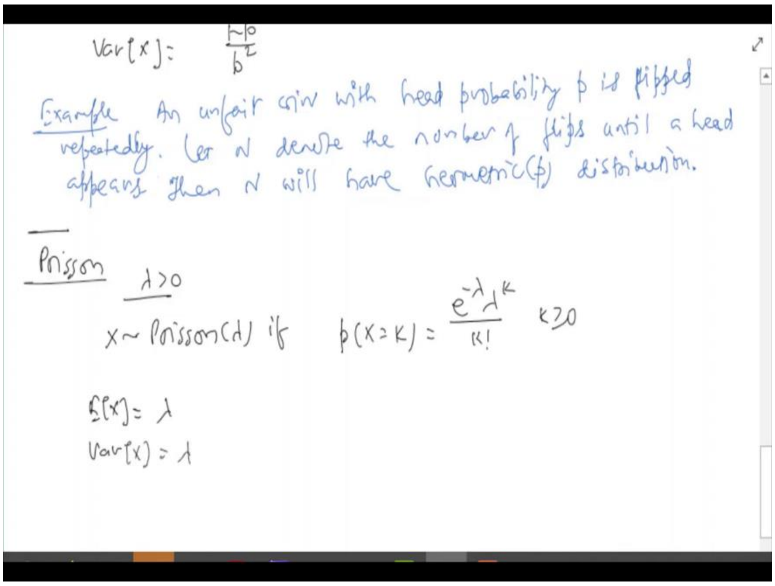
$$P(X = K) = \binom{10}{K} \left(\frac{1}{2}\right)^K \left(\frac{1}{2}\right)^{10-K}.$$

Since its fair point we have P = $\frac{1}{2}$ and this is for K between 0 and 10. Similarly,

$$P(X \geq 3) = \left(\sum_{K=3}^{10} \binom{10}{K}\right) \left(\frac{1}{2}\right)^{10}$$

(Refer Slide Time: 10:57)





Let us consider another distribution that is quite common and this is geometric distribution. To understand geometric distribution, let us again think of experiment star and in fact let us think of repeated trials of experiment star until we get a success. Again, consider repeated trials of experiment star then the number of trials until the first success, this has geometric distribution is given by geometric distribution.

So, if X is a geometric random variable with parameter P, then its pmf is given by

$$P(X = K) = (1-P)^{K-1}P, \;\; K \geq 1.$$

In this case the cdf of the random variable x,

$$F_X(x) = 1 - (1-P)^{\lfloor x \rfloor}.$$

We can compute the mean and variance of geometric random variables using the formulae that we saw in the last lecture. So, in particular

$$E[X] = \sum_{K=1}^{\infty} K(1-P)^{K-1}P = \frac{1}{P}.$$

Similarly,

$$Var[X] = \frac{1-P}{P^2}.$$

Let us see an example. Suppose an unfair coin with head probability P is flipped repeatedly. An unfair coin with head probability P is flipped or tossed repeatedly, let N denote the number of flips, number of flips until a head occurs, until a head appears, then N will be geometric random variable. N will have geometric P distribution.

Let us see one more example of discrete random variable that is very important and this is Poisson. For a positive number lambda, we say a discrete random variable X has Poisson distribution with parameter lambda if its pmf is given by the following expression probability that $P(X = K) = \frac{e^{-\lambda}\lambda^K}{K!}$, this was one of the random variables that we saw in the last lecture and there we saw that expectation of Poisson random variable is lambda and its variance is also lambda.

(Refer Slide Time: 15:18)

$Var[X] = \lambda$

Example: If number of accidents occurring on a highway each day, N, is a Poisson(3) random variable, then

$P(\text{no accident occurs on a particular day}) = P(N=0) = e^{-3} \approx 0.05$

$P(\text{at least one accident occurs on a particular day})$
$= P(N \geq 1) = 1 - P(N=0) = 1 - e^{-3} \approx 0.95$

Common Continuous Random Variables

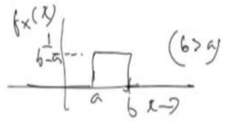
uniform

Common Continuous Random Variables

uniform: Consider an experiment of randomly choosing a number (without any bias) from the interval $[a, b]$, then the chosen number has continuous uniform distribution

$X \sim \text{uniform}([a, b])$

$$\ni f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



c.d.f

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } x \in [a, b] \\ 1 & \text{if } x > b \end{cases}$$

$F_X(x) = \frac{x-a}{b-a}$ if ... ; 1 if $x > b$

$E[X] = \frac{a+b}{2}$

$Var[X] = \frac{(b-a)^2}{12}$

Example $X \sim \text{uniform}([0, 10])$

$P(1 < X < 6) = \int_1^6 \frac{1}{10} dx = \frac{5}{10} = \frac{1}{2}$.

Let us see an example to illustrate Poisson random variables. If the number of accidents occurring on a highway each day and let us call this number N as a Poisson(3) random variable then probability that no accident occurs on a particular day probability that no accident occurs on a particular day, that is probability that N will be 0 according to the pmf given above. It would be $P(N = 0) = e^{-3} \approx 0.05$.

Similarly, probability that at least one accident occurs on a particular day that is $P(N \geq 1) = 1 - P(N = 0) = 1 - e^{-3} \approx 0.95$. So, these were a few common discrete random variables.

Now we turn to common continuous random variables. We see a few of those. The first one in our list would be again continuous uniform random variable. To understand continuous uniform random variables let us consider an experiment of randomly choosing a number again randomly means without any bias, from the interval [a, b]. Here b > a then this chosen number has uniform continuous uniform distribution.

Chosen number has continuous uniform distribution. So, if X is uniform over [a, b], we have already seen that X will have a pdf that will be constant over the interval [a, b]. We discussed this in the last lecture, and it will be 0 outside. So, in fact since the area under the pdf should be 1, this value will be $\frac{1}{b-a}$. So, X being uniform means, its probability density function

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & otherwise \end{cases}.$$

Clearly, its cdf can be written as

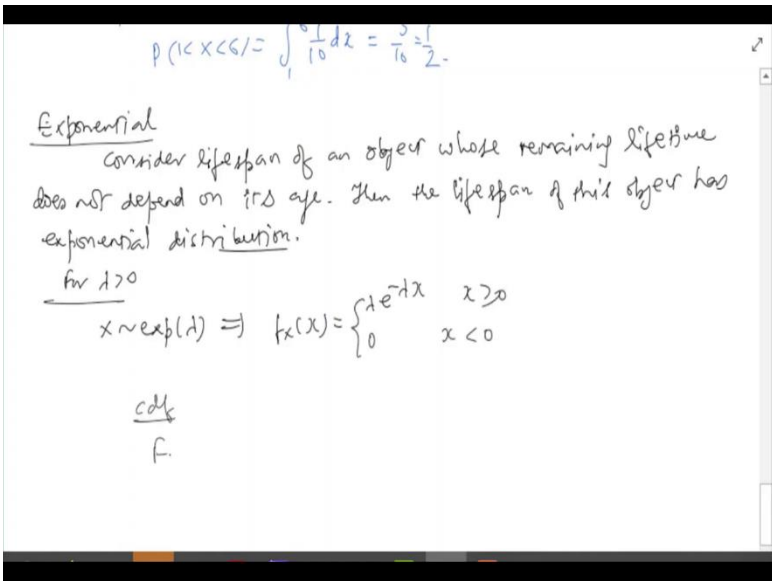
$$F_X(x) = \begin{cases} 0 & if\ x < a \\ \frac{x-a}{b-a} & if\ x \in [a, b] \\ 1 & if\ x > b \end{cases}.$$

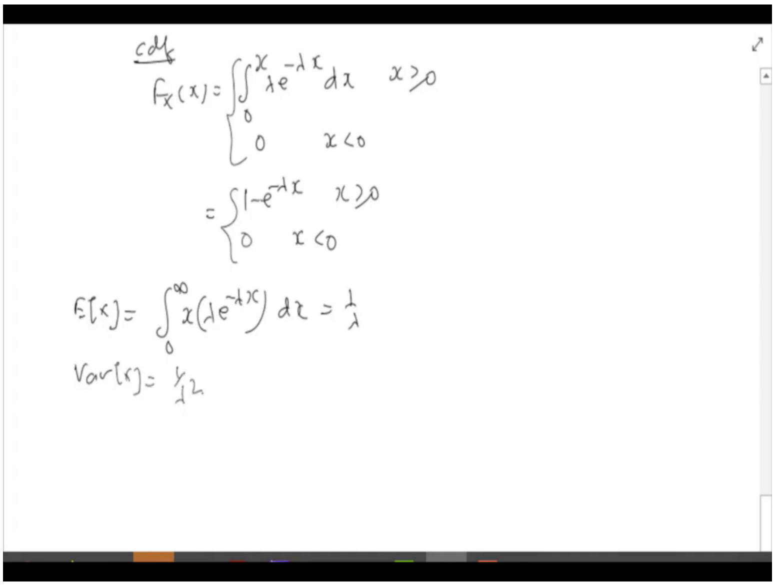
Moreover its expectation is $\frac{a+b}{2}$ same as in the discrete case and variance is $\frac{(b-a)^2}{12}$. This has slightly different expression than in the discrete case.

Let us see an example, suppose X is uniform over the interval [0, 10]. Then

$$P(1 < X < 6) = \int_1^6 \frac{1}{10} dx = \frac{5}{10} = \frac{1}{2}.$$

(Refer Slide Time: 21:14)





Next, we see a very common distribution called exponential distribution. To understand exponential distribution, consider lifespan of an object whose remaining lifetime does not depend on its age. Lifespan of an object whose remaining lifetime does not depend on its age. Then the lifespan of this object has exponential distribution. For instance, suppose we have a bulb whose remaining lifetime does not depend on its age, then the lifetime of the bulb is considered to have an exponential distribution.

For a number $\lambda > 0$ we say that X has exponential distribution with parameter λ if its density is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Since we are talking of lifespan so it is natural that density be 0 for all the negative values of x. The cdf of exponential random variable F_X is given as follows

$$F_X(x) = \begin{cases} \int_0^x \lambda e^{-\lambda x} dx & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

It can be easily checked that cdf is

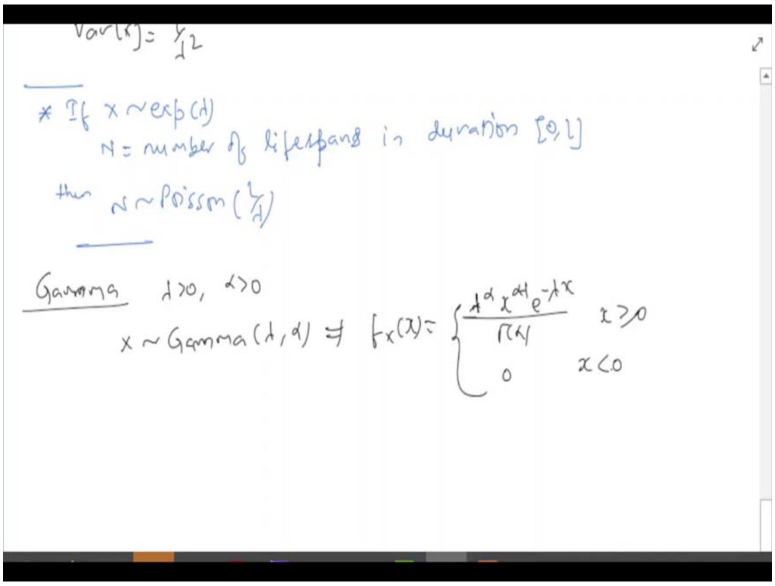
$$= \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

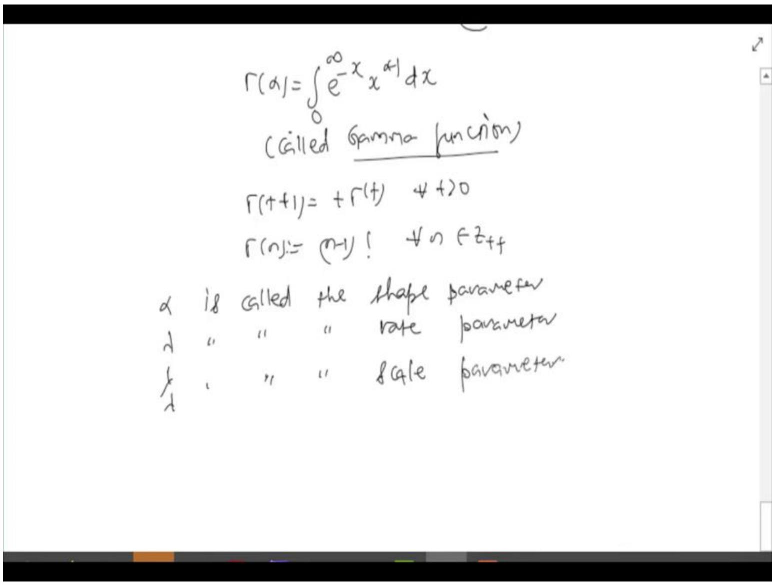
The mean of the exponential random variable X can be computed as

$$E[X] = \int_0^\infty x\left(\lambda e^{-\lambda x}\right) dx = \frac{1}{\lambda}.$$

Variance of exponential random variable turns out to be $\frac{1}{\lambda^2}$.

(Refer Slide Time: 24:05)





There is an interesting connection between exponential and Poisson random variables. The connection is as follows. If X is exponential random variable with parameter lambda, and N is number of lifespans in a duration [0, L] then N turns out to be Poisson with parameter $\frac{L}{\lambda}$. So, we have a connection between Poisson and exponential random variables

The next distribution that we will see is also one very commonly encountered and this is gamma distribution. For constants, λ and α that are both positive, x is said to be gamma distributed with parameters λ and α if it has density

$$f_X(x) = \begin{cases} \dfrac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

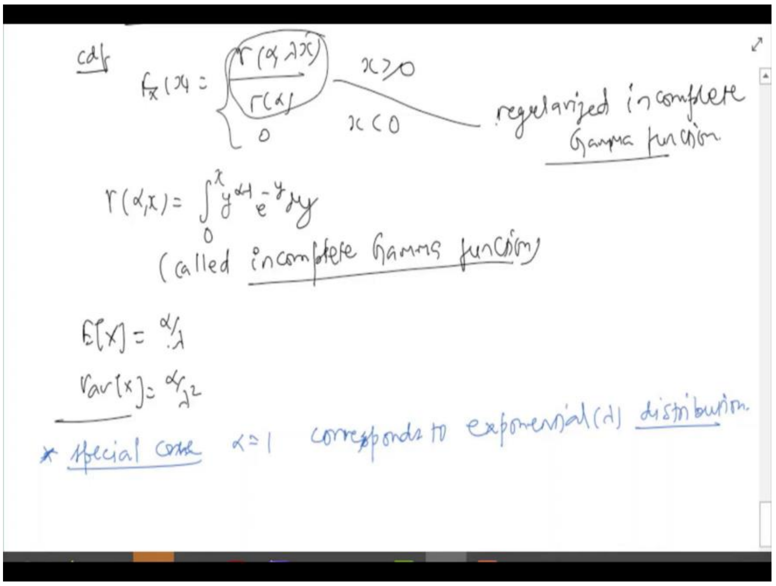
Here,

$$\gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

and in fact this is called gamma function. The gamma function has a few interesting properties. For instance, $\gamma(t+1) = t\gamma(t)\ \forall t > 0$. If n is a positive integer, then $\gamma(n) = (n-1)!\ \forall n \in Z_{++}$ that is for all positive integers.

In the context of gamma distribution the constant α is called the shape parameter. Parameter λ is called the rate parameter and often $\frac{1}{\lambda}$ is referred to as scale parameter.

(Refer Slide Time: 27:02)



The cdf of gamma distribution can be written as follows,

$$f_X(x) = \begin{cases} \dfrac{\gamma(\alpha, \lambda x)}{\gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Here the functions

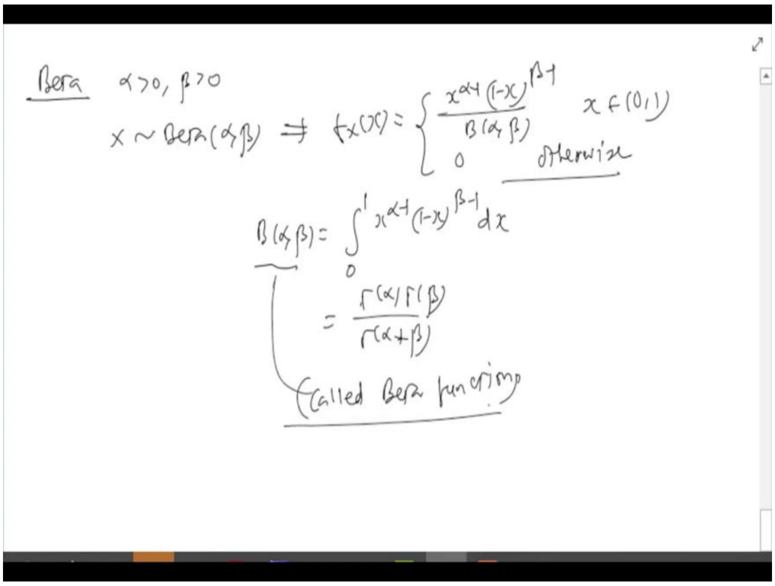
$$\gamma(\alpha, x) = \int_0^x y^{\alpha-1} e^{-y} dy$$

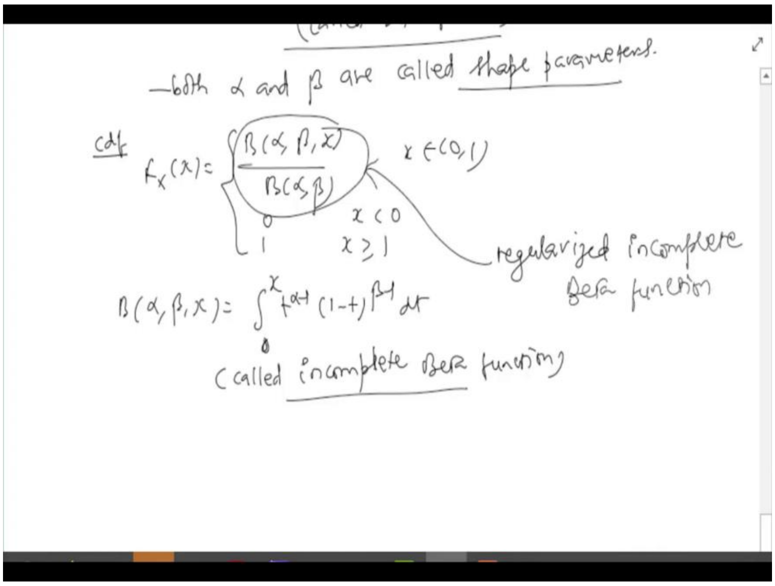
and this function is called incomplete gamma function. This is called incomplete gamma function. The expression of cdf for the non-negative values of x that is this expression, this itself is called regularized incomplete gamma function.

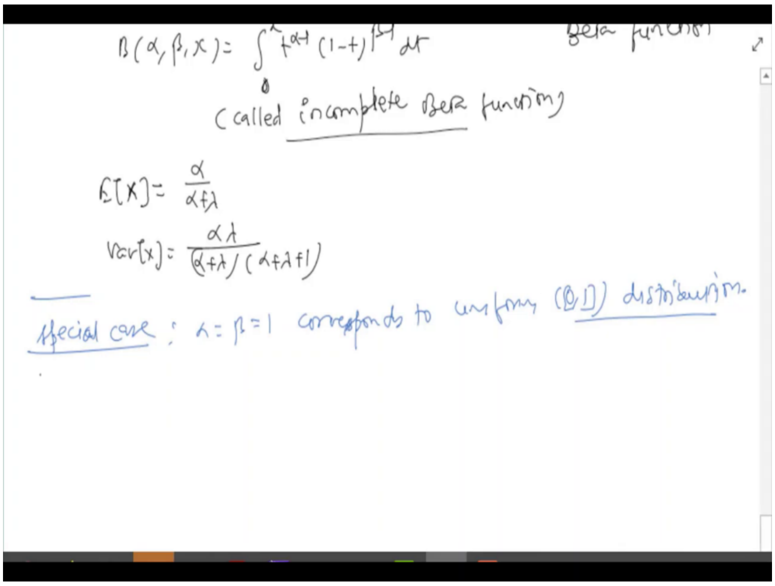
The mean of this gamma distributed random variable expected value of X turns out to be equal to $\frac{\alpha}{\lambda}$. Its variance turns out to be equal to $\frac{\alpha}{\lambda^2}$. There is a connection between gamma distribution and exponential distribution. In fact, if we consider the special case of gamma distribution, where $\alpha = 1$ this case corresponds to exponential distribution, so this corresponds to exponential lambda distribution.

Note that in the above expressions of mean and variance if I set $\alpha = 1$, I get $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ respectively and these are precisely mean and variances for exponential lambda distributions.

(Refer Slide Time: 29:22)







The next distribution that we will see is called beta distribution. As in case of gamma distribution, we consider 2 constants α and β both positive again. So for α and β positive constants, we say that a random variable X is beta distributed with parameters α, β if it has density

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} & x \in (0,1) \\ 0 \;\; otherwise \end{cases}.$$

$B(\alpha, \beta)$ is a normalizing constant and it is equal to

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

and in fact it turns out to be equal to

$$= \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)}$$

notice that here this gamma function is same gamma function that we saw in the context of gamma distribution. This B itself is called beta function.

In the context of beta distribution, both these parameters α and β are called shape parameters. The cdf of beta distribution F_X can be written as follows.

$$F_X(x) = \begin{cases} \dfrac{B(\alpha,\beta,x)}{B(\alpha,\beta)} & x \in (0,1) \\ 0 & x < 0 \\ 1 & x \geq 1 \end{cases}.$$

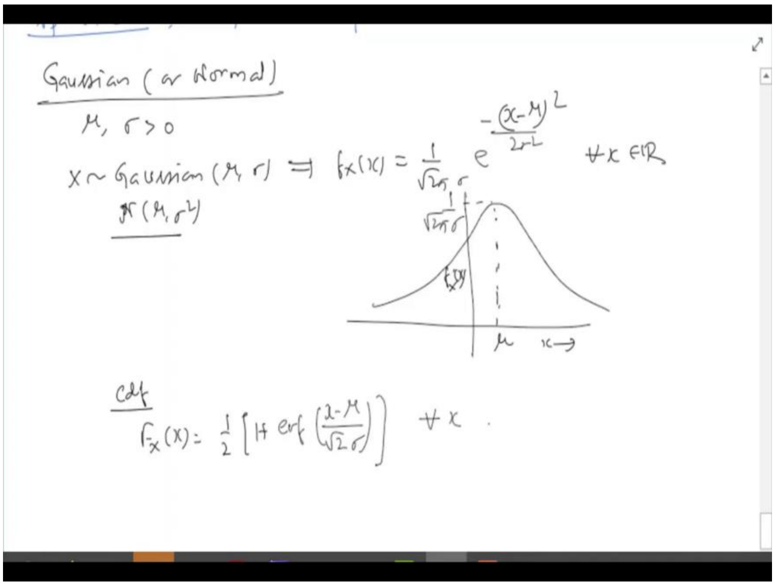
Here the function

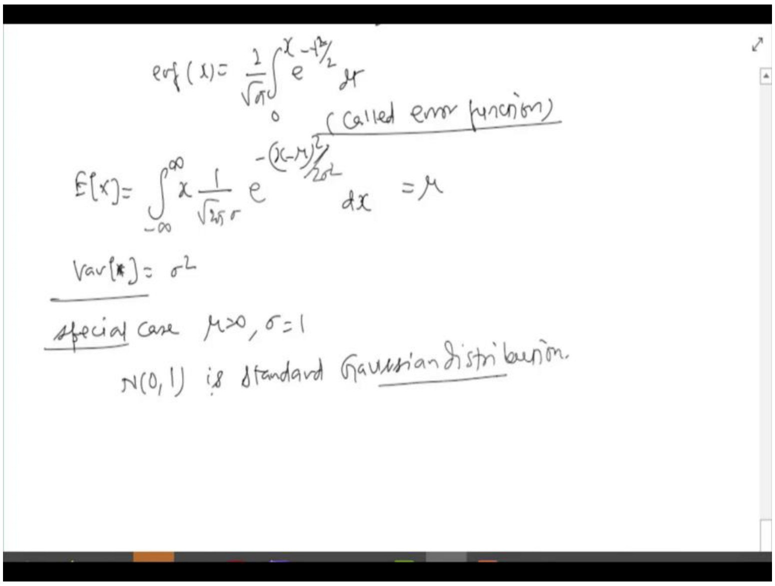
$$B(\alpha, \beta, x) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

and this is called incomplete beta function.

The expression of cdf for the values of x between 0 and 1 that is this expression. This itself is called regularized incomplete beta function. Expectation of the above beta distributed random variable that is ex turns out to be $\frac{\alpha}{\alpha+\lambda}$ and its variance turns out to be $\frac{\alpha\lambda}{(\alpha+\lambda)(\alpha+\lambda+1)}$. There is a connection between beta distribution and uniform distribution that is special case of beta distribution and $\alpha = \beta = 1$. This corresponds to uniform [0, 1] distribution. So, these were a few facts about beta distribution.

(Refer Slide Time: 33:38)





We end this lecture with another very important continuous distribution, and this is Gaussian distribution also known as normal distribution. We again consider 2 parameters μ and σ. μ is a real number and σ is a positive real number. We say that X is Gaussian distributed or normal distributed with parameters μ and σ if it has probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \;\; \forall x \in \mathbb{R}.$$

If we plot this pdf it looks like a bell shaped curve with its peak at μ and the value of the pdf at the peak $\frac{1}{\sqrt{2\pi\sigma}}$. This normal distribution is also denoted as $N(\mu,\sigma^2)$. The cumulative distribution function of Gaussian distribution turns out to be

$$F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] \;\; \forall x.$$

Here, the error function is given as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{\frac{-t^2}{2}} \, dt.$$

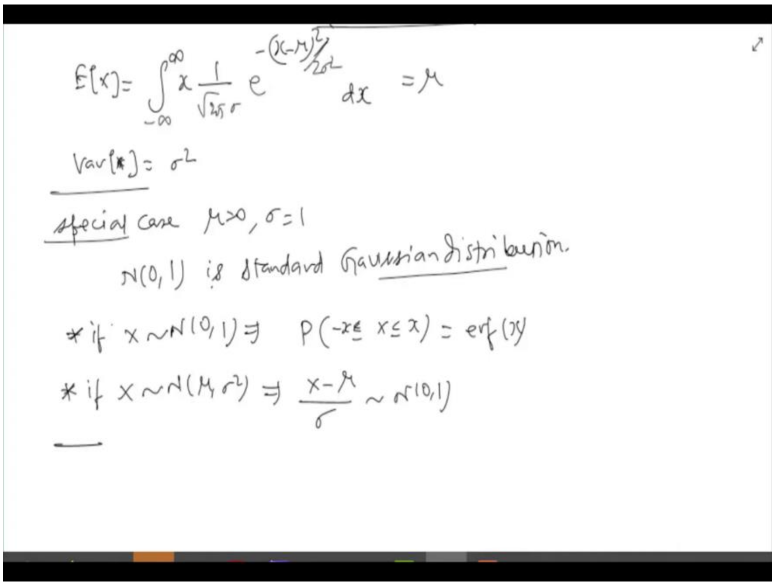
This is called error function.

The mean of the exponential random variable described above expected value of X can be computed as

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \mu$$

Var[x] = σ^2. Of special interest is the case, the special case where mu is 0 and sigma is 1. The special Gaussian distribution with parameters 0 and 1 is called standard Gaussian distribution or standard normal distribution.

(Refer Slide Time: 36:50)



Here are a couple of facts about Gaussian distributions. If X is standard Gaussian, N probability that X will take values between - x and + x, it turns out to be equal to error function of x. Further, if X is Gaussian with parameters μ and σ^2, then $\frac{X-\mu}{\sigma}$, it turns out to be Gaussian with parameter 0, 1 that is $\frac{X-\mu}{\sigma}$ is a standard Gaussian random variable. These were a few facts about Gaussian distributions. This also brings us to the end of this lecture. Thank you.