Mathematical Aspects of Biomedical Electronic System Design

Welcome to another tier session of the NPTEL course, Mathematical Aspects of Biomedical Electronic System Design. So in this session, we will be discussing about the mechanical modeling or mathematical modeling of mechanical properties of tissues and its applications in biomedical engineering. So why characterize or why mechanically model the tissues and what is its importance?

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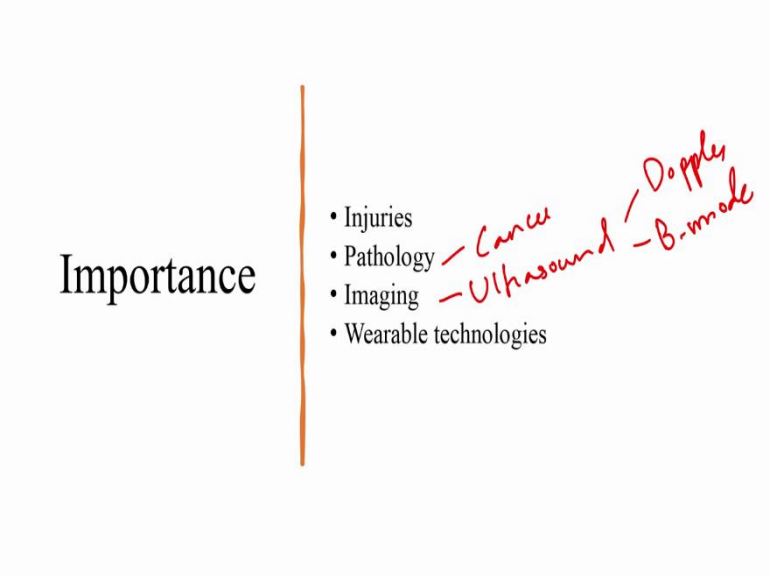

So, that is a key question that you might be having. So, why it is important? So we know that human beings or even animals can have different diseases and these diseases will cause the change in the human body or the tissues as such. So from one pathology to another pathology, from a normal condition to an abnormal or cancerous condition, actually the properties of tissue changes and the changes in properties can be mechanical, electrical or several other different properties altogether.

So in today's session we try to understand the changes in mechanical properties and how we can actually model them, so that we can use these models to understand the tissue properties better, so as to invent or discover new treatment opportunities or the therapeutic applications or even for diagnosing different tissue conditions.

So as a simple example say a person riding on a motorcycle and hits a car or meets with an accident and undergoes a brain injury or a head injury. So to prevent this we have been implementing several methods. In case of a bike accident, we implement helmets, but in case

of a car accident, we enforce seat belts, so to prevent several injuries happening to human body during an accident.

So, how do we come up with solutions like helmets? What should be the property of helmet so that it will be comfortable to a human to wear or what is the properties of a helmet that will enable it to absorb the energy once the person hits the floor or what should be the force at which an airbag opens up to prevent damage to the chest when an accident happens in a car.

So all these things actually takes into consideration, the mechanical properties of our human body, how much stress our bones can survive or how much the force that our tissues can survive without getting damages. Similarly, when it comes to cancer, the necrosis and all other different properties or phenomena actually causes changes in different properties including mechanical properties of the tissue and similarly we also use mechanical properties or mechanical characterization when it comes to ultrasound imaging.

So, there is a lot of forms of ultrasound imaging now like Doppler imaging, B-mode imaging. So, all these types of imaging that have been used currently all are based on mechanical properties of the tissue in one way or the other. So also when we come to wearable technologies, your designing is not band.

So what should be the appropriate comfort, I mean the material property of that band like a rubbery material or a metallic band, which actually gives you the most comfort. So for all this we actually analyze the mechanical properties of the tissues and it is essential to mathematically model them when it comes to simulating the actual conditions in a FEM tool or when it comes to designing a product. So that is why we in this course discuss about the mathematical modeling of mechanical characteristic of tissues.

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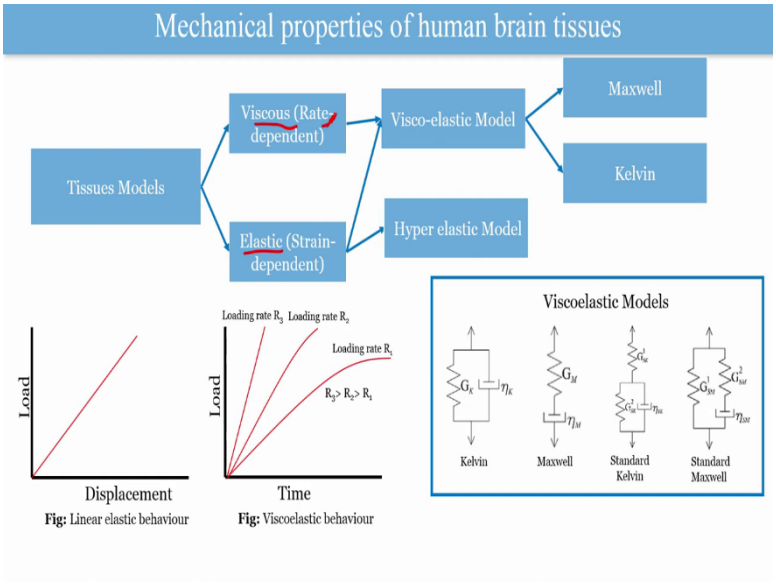

So say we have different models that have been put forth by different scientists over the period of time maybe after the mechanical properties of tissues might have been explored for last 50 years, 60 years extensively and we have a lot of models that have been proposed, mathematical models that have been proposed to categorize the tissues.

And before going deep into which are the specific models that have been used for tissues, we have to go through some basic understanding of different mechanical properties that we might discuss across this lecture. So as you can see the tissue models that have been proposed usually are viscous or elastic. We can broadly characterize them into viscous properties as well as elastic properties.

Now what is the difference between both of this is that one this rate dependent properties are basically nothing but if you apply a force at some specific rate, the issue will behave like this. And the elastic properties is more like if you apply this much displacement or you compress the tissue this much or you pull the tissue this much, then what is the behavior of the tissue.

So these are two different things that we have to understand towards developing these models. And the combination of several of these models have also been there such as viscoelastic model where the tissue have been modeled to compensate for both these behaviors together as well as hyper-elastic models and various scientists have devised their own models.

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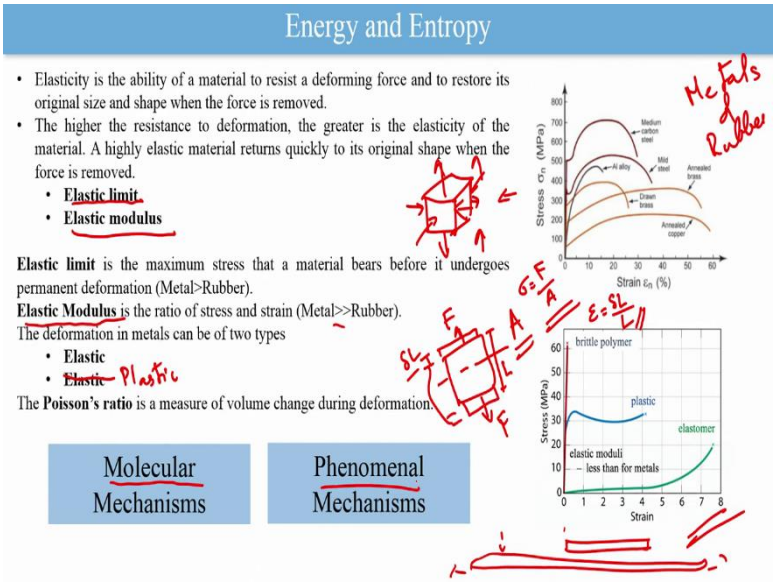

So now coming to basic understanding of the material properties before going towards tissues, we can actually understand what are the basic terms that we might use during this session. Now before going to even tissues, we will consider the properties of metals and polymers such as rubber in order to actually understand how these are similar to tissues and why we can use this polymers or other materials as analogous to tissues.

So rather than explaining everything based on tissues, we can go for materials that are very similar to tissues such as rubber in order to characterize the tissues. Now you might already know what elasticity means. So elasticity is basically like you say, we have an elastic band or a rubber band, which we can pull and it goes back to its normal shape.

So if we pull it like this with a force F, it stretches to a longer length and when you release this force, it goes back to its initial size. So you say this material is elastic. So basically elasticity is the ability of material to resist a deforming force and once this deforming force is removed its ability to come back to its original shape and size that is what elasticity is referred to as.

Now to this as – even though we might term this as elasticity or elastic properties of the tissue, we need to quantify it somehow or for any material we need to quantify what exactly elasticity means. That is why people have devised or brought up terms such as elastic limit and elastic modulus. And what exactly this means?

So this means that elastic limit is a maximum stress that the material can bear before it undergoes permanent deformation. That means if we apply some force F, the material stretches

and it comes back to its stretches like this and it comes back to its initial form like this when you remove the force. But beyond a point, if beyond a force say F-max, what happens is, it will deform such that it will not come back to its previous shape.

So it actually kind of gets damaged. So this is what actually is meant by the elastic limit and we through different material characterization steps, we have found this elastics in a limit for several materials. And say as you might already know, the elastic limits of metal is significantly higher than that of materials like rubber and other polymers, most polymers.

Now the next parameter, so elastic limit is basically nothing but the maximum stress that you can apply so that the material does not get permanently damaged, you can say it like that. Now there is another term that we say is the elastic modulus. So, elastic modulus is nothing but the ratio stress and strain. So when you apply a force F, so when you apply a force F on a block and of cross-sectional area of A, it experiences a stress.

This we might have studied in school itself so this stress $\sigma = F/A$. Now what is strain? This basically, when we apply this force, an elongation happens by say a mound ΔL. This is total displacing on both sides together. If it is ΔL and the original length of L then the strain ε can be expressed as $\Delta L/L$. So this is the strain basically. Now so these two parameters, the stress and strain and the ratio of it is known as elastic modulus.

So you can see the graph over here. So you can see a elastomer or something like a rubber when you stretch it, if it was – the length was like this and you stretch it this much, the stress developed in this rubber will be comparatively less than if you are stretching a same metal. So if you stretch a metal to the same extension as this rubber, the stress developed in the metal or a material of that much high stiffness will be way higher.

So the elastic modulus of metal is very much higher than rubber. Now the deformation of material can be of 2 types and elastic and plastic. So the elastic deformation is basically what happens within this elastic limit and the permanent deformation when you cross the elastic limit is the first towards plastic deformation. Now the materials when we apply the force, the deformation happens.

So if I draw a cube, consider this as a cube and you apply a force. Now when the cube get stretched in this direction, if it is a metal, what happens is, there will be a reduction in volume or reduction in width as well as breadth in these 2 directions is because the volume of the metal

does not change much that is like when it gets stretched, the compensation happens from the other 2 sides.

So, that the quantification of that effect is often referred to as Poisson's ratio. So there is a change in volume associated with the stretching and this is referred to as a Poisson's ratio of the material. Now how the deformation happens can be explained using 2 theories. And one is the molecular mechanics. It is not 2 theories, it is 2 principles that cause may be same metals and rubber or the elastic materials to show different behaviors and this can be classified as molecular mechanisms and phenomenal mechanisms.

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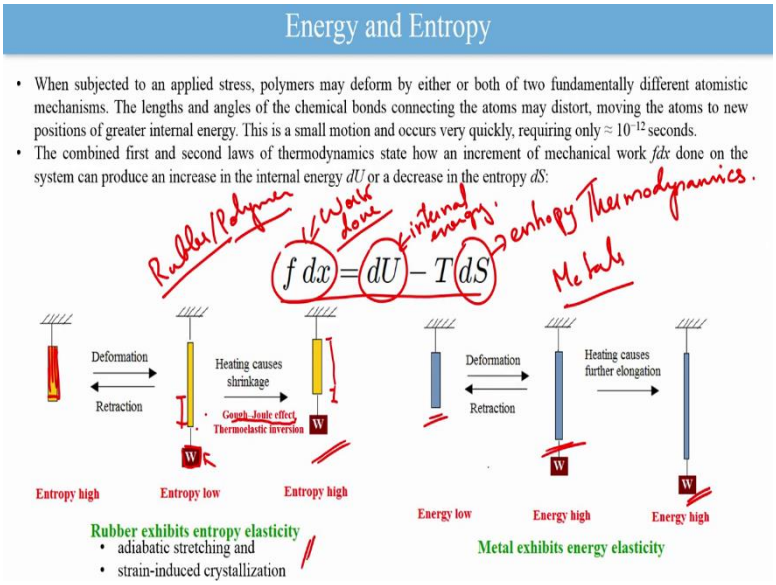

Now say for this I will just show 2 – take 2 different cases, one being the behavior of metals and one being the behavior of rubber. Now, rubber or polymer or polymers; a few of the polymers, now, how these 2 materials are different? So for this we are just doing going through a simple experiment where we have a rubber maybe a rubber band or rubber material here and then you hang a weight to it, of weight W.

So you will very intuitively understand that when you apply this weight W, it stretches, say it stretches by some distance length. Now if you apply some heat to this rubber when it is already under weight W you can notice that the rubber will shrink back may be not to its initial position but at least it will shrink back to a reduced length and this is called as Gough-Joule effect.

So in case of metal, if we do the same experiment, you can say that the metal was there then you applied a weight W and when you heat it, it will again elongate and how exactly does this

happen. Both are materials, both were subjected to the same weight and then you apply just temperature and the behavior totally change and how does this actually happen? And this can be explained based on thermodynamics.

So this fdx or you might already know that it is the work or the work done. This work done can be represented by the changes in both internal energy dU as well as change in entropy of the system, entropy and internal energy. Now said this how but still how does it actually cause this material behavior in rubber as well as metal. How does this change happen?

That is because of a specific effect that happens in polymers or rubber that is adiabatic stretching and strain induced crystallization.

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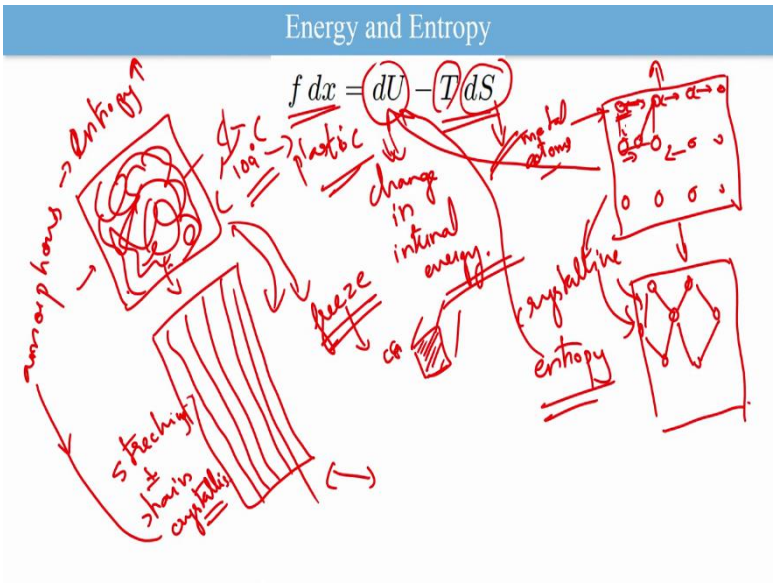

Now let us come here. So the internal structure of a metal might be like this. Say, there is a metal lattice like this where these are same metal atoms. Now this is very microscopic view so if this was a metal block and you were applying force like this, so what actually happens is these atoms which are in ionic interaction with the other atoms here undergoes deformation. So if it was say like a crystal structure like this, it might deform to something like this.

So as you can see if this atom deforms to something like this, and this atom moves in. So you can see that there is a change in volume altogether. So this change in volume is caused by this force and this change in volume also causes change in internal energy of this metal, change in – now what the changes actually happens is in – the ionic bonds get distorted.

So what happens? When you release the force these ionic forces are strong enough to bring it back to its original shape. So whatever force we applied here comes back to its initial state. This is what happens in case of metals. Now in case of polymers like rubber what happens is it is a polymer. So it is a combination of monomers that is within it and all these monomers are held together by covalent bonds and this monomers can be like this.

Each monomer can be like oriented in different fashions. And maybe some carbon based, carbon-carbon bond or something like that will be there. And the angle between this carbon-carbon bonds might be say restricted to say 109 degree. So when you apply the force since the covalent bond is strong enough to maintain this 109 degrees.

The covalent bond will maintain the same angle and if you apply more force the covalent bond just breaks. So it will cause say plastic deformation, permanent deformation. So in most of the cases what just happens is, if you stretch the rubber the coiling of the monomers that you have seen here will just stretch out to become straight.

That is the change what actually happens in rubber and when you leave it, so this state can be referred to as amorphous state. And here it is more like a crystalline state and the amorphous state are associated with huge entropy and here the entropy is lower. So most of the change when we stretch a metal, it contributes from the change in volume which is therein internal energy change.

When it comes to rubber the stretching actually reduces to the – or the coiling of the monomers reduced to stretching and causes strain induced unbending or uncoiling of this thing. So basically what happens, the stretching causes strain crystallization. So when it becomes crystalline from an amorphous state basically the entropy reduces.

So, most of the effect or the predominant effect that comes into the work done of a rubber material or a polymer when we apply and release is basically due to the change in entropy. Of course there is some change in volume but still most of the effect comes from the entropy change. That is why when we apply the temperature the rubber since the entropy term or the change in the entropy term is significant.

And if the temperature we increase it, it actually reduces the work done and in the case of metal since the dU is the predominant one and the volume change is significant, it will dominate over the work done and which will cause the elongation of the material.

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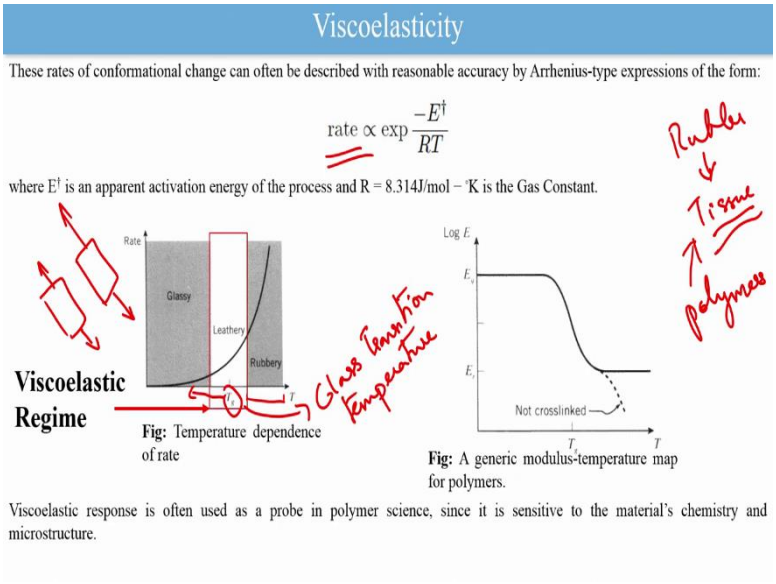

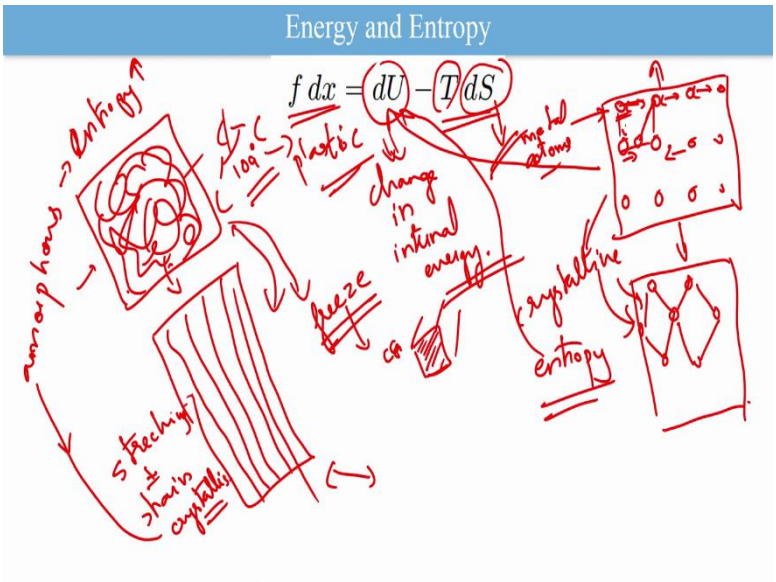

Now considering all these things, now we come to the viscoelastic behavior of the tissue. So why did I discuss all these things of the rubber as well as metal? To tell you that material like you could have a basic understanding like the rubber is very much similar to tissue when you feel it. And it is true that the mechanical properties of a rubber or a polymer is very much similar to that of a tissue in many ways.

So these properties of the tissue can – always are most commonly represented by the characteristics of polymers itself. So whenever scientists try to explore the mechanical properties of tissue it is always very – I mean tightly aligned with the properties or characteristics of polymers in general. So the methods used to characterize the tissue are very much similar to the methods used to characterize the polymers.

And this is why we discussed all these behaviors of rubber so far. Now so this is the basic idea that I wanted to give. And the conformation change that we discussed previously in the materials can – the rate of conformation change can be expressed using this or expression shown in here. Now coming back to the viscoelastic regime or what are the viscoelastic properties of tissue? You can see here.

So now consider we have a rubber and we are pulling apart very slowly or a tissue that we are pulling it up very slowly. And there is another same tissue we are pulling it fast. And you can see that its behavior totally changes depending on the rate at which you pulled it. And similarly when it comes to the temperature that we discussed previously, the change in temperature causes different behaviors in the tissue or the polymer based on the thermodynamic principle that we discussed earlier.

You can see here there is a plot of – the rate of deformation versus temperature and you can see that as temperature decreases the slope or the rate of reaction increases quickly. Now if you think about it, you take a rubber band or something, you can try this simple experiment in your house even the other I have told here the stretching and un-stretching of rubber, you can actually see this with a simple experiment at your house.

You can take a rubber band, a usual rubber band or a rubber block, you can freeze it, you can freeze it in a refrigerator or something and you will find that the rubber actually becomes brittle and you can just break it. So take just a rubber band, keep it in a freezer and freeze it at some minus some degrees Celsius and you will see that when you break it, you will see a coiled kind of behavior in the cross-section of the rubber band.

And which is actually feasibly visible. And in the another rubber band you take it, you stretch it, stretch it using something and again freeze it and when you break it, you will see very aligned structure. So this basically shows the orientation change or the uncoiling of rubber or the these kind of materials.

Now talking about this, so the temperature actually plays a significant role in the mechanical behavior of polymers like rubber as well as even for tissues and there is a temperature that is referred to as glass transition temperature, and this glass transition temperature is a very important temperature. And if you heat the material or the polymer beyond even this applies true for most materials even in glass or materials.

What happens is when you heat it, above its glass transition temperature, it becomes rubbery or it becomes more elastic and if the temperature is reduced well below the glass transition temperature it becomes brittle and you might be already able to understand why this happens based on the previous thermodynamic explanation that we have already gone through.

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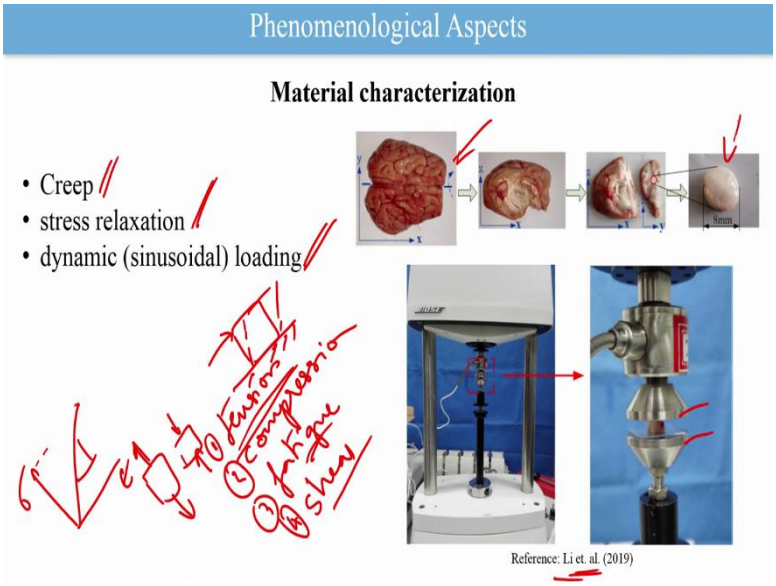

Now let us come back to material characterization or material characterization of human tissues. I have shown here an example of a characterization of brain tissues. So a brain tissue was cut into a diameter of 8 mm in this research work that I have cited here. And they are keeping it between 2 plates as you can see here. So what are they trying to do? So the materials have been characterized using different methods like the tension test, compression, then maybe a fatigue test, so a shear test.

A lot of tests have been done. In tension test, a material will be pulled to its – to understand its and once we pull it, we plot the stress versus strain and to see what is the elastic modulus, elastic limit, what is the elastic modulus, all these things and same for the compression. You apply a force to compress the tissue or the material to know the behavior. You load it and unload it at very high rates to understand the fatigue.

You apply the load in an angular way to understand the sheer behavior of the material. So all these tests have done a lot but when it comes to tissues the most important characterization that people do is the creep test, relaxation test and the dynamic or sinusoidal loading test. So why these tests are important to all these tests that we have already discussed like tension, compression, all these can be done, all these can be done always.

And these are very much similar to any other material but when it comes to specific case of tissues this is - that is when it actually becomes important to do this creep test, stress relaxation test and dynamic loading and you will understand over the course of a few slides that comes down that why it is actually important.

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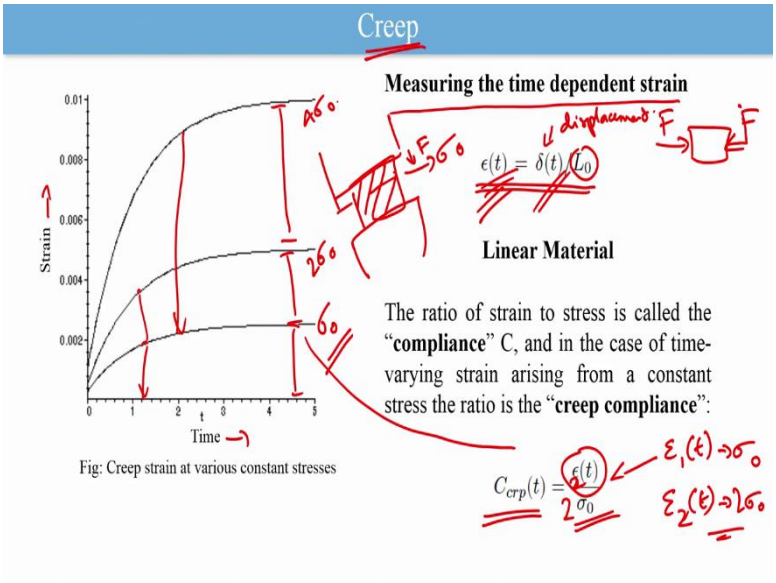

Fig: Creep strain at various constant stresses

First we will talk about the creep test. So now we will talk about the creep test or the creep behavior of the tissue. So you can see a graph over here. When you apply a strain and you plot it against time. So as you can see there are 1, 2, 3 plots here and I am naming them as – so what basically creep that we will explain first.

Now you apply stress on a material, you apply some force on a material. Initially you can apply a force on the tissue at a slower rate, slowly you pull it or you apply a force slowly then you increase the speed at which you apply the force, the speed at which or the rate at which you apply the load or the F. If you change this that is like you apply force faster.

If you apply the force slower, you will actually get different strains when it comes to this kind of material such as tissues or polymers. And this is known as creep of the material or it is basically the time dependent strain or that rate dependent strain of the material and hence the strain ε can be represented as a function of time and this δ here is the tissue displacement since also displacement changes with time and this is the initial length L_0.

So basically the strain can be expressed as a function of time in this case. And you can see here say if the stress in this case. So what basically what we can – what we usually do to do a creep

test is you keep a tissue, you almost instantaneously apply a force F. So the stress in this tissue will become say something like a σ_0 and this is what happens in a creep test basically.

Now if you apply double the load that is $2\sigma_0$ and the $4\sigma_0$, you can see that the strain observed will also get doubled like from σ_0 to $2\sigma_0$, the strain rate at every instant of time will be almost double. If you make it 4 times the initial value it will become 4 times so this is what creep means that is when you apply different – if you apply different loadings rates to the tissue its behavior changes and this is called as creep.

So, now creep is understandable. There is a strain that is a function of time and since the displacement is a function of time. Now you need to say quantify it or normalize it with respect to the stress that you apply. For example, the strain correspond to, the strain that correspond to σ_0 force, and the strain that correspond to $2\sigma_0$ force.

We need to actually compare them somehow. What is the actual effect of this? So that is why we have brought in a term known as creep compliance and this is basically nothing but the ratio of the strain that is a function of time with respect to the stress that has been applied.

And you could just see that if you apply or if you double the stress what actually happens is from the basic say if this was $\varepsilon_1(t)$ and if it was $\varepsilon_1(t)$ for σ_0 when you make it $2\sigma_0$, the stress when you double it or the initial stress, the stress that you initially applied gets doubled, what happens is the strain also gets doubled. So that is the basic understanding.

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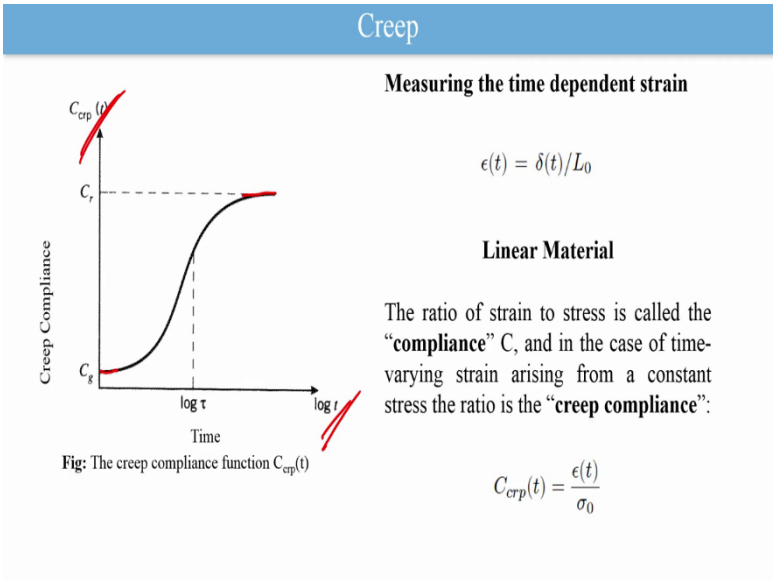

Now we can also do what is like to take the plot of this creep, creep compliance against logarithmic of time. You can see the variation from the glassy state to the rubbery state of the material when we change – over the period of time.

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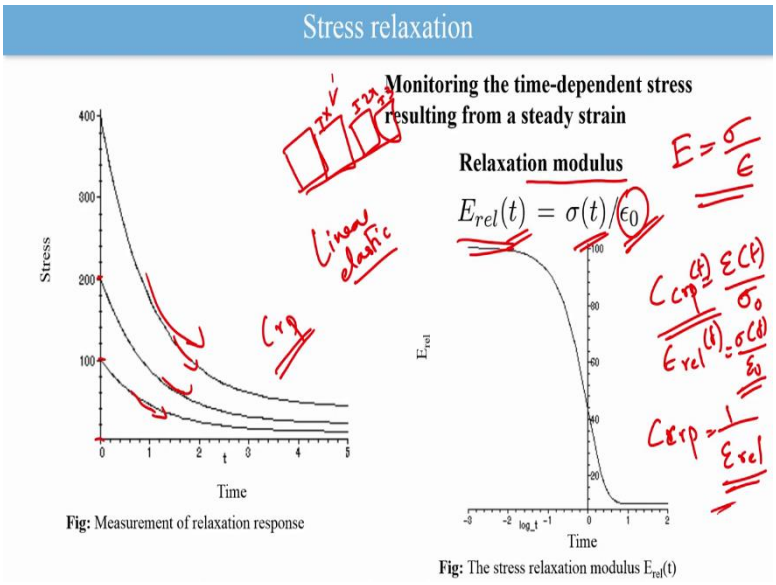

Now let us talk about the next characterization method that is known as the stress relaxation. Now we discuss that you apply a specific amount of stress to a material at different rates and you will get different strain rates. And similarly the vice versa of this is basically, you apply different strain or initially, this was the tissue, you compressed it to say x distance and in the next case, you compressed it to 2x distance and you apply it to 3x distance.

Now what happens is each different, different displacements also causes different, different stress in the materials. So each time if the material is linear, linear elastic, what happens is when you double the strain that you have given or if you double the displacement that you have given, the stress also just gets doubled when it comes to a linear elastic material or linear behavior of the material.

So what happens is, now after applying this displacement you just keep on measuring what is the stress in the material, you will understand that over the period of time the stress in the material comes down. That means the tissue or the material relaxes or the stress reduces so this is called as the stress relaxation.

And this is referred to as – you say we have quantified this using the term relaxation modulus and this you are already familiar. The elastic modulus of the material can be given by

$$Elastic\ modulus(E) = \frac{Stress(\sigma)}{Strain(\epsilon)}.$$

So it is the same thing but just that the variation with respect to time is also important here. So the initial strain that you apply is ε_0 and the relaxation modulus as a function of time can be plotted.

So this is the basic understanding of 2 different processes. Stress relaxation as well as creep. Now both creep and relaxation are both manifestations of the same molecular mechanisms but you will feel that even these two are related. Basically that

$$C_{crp}(t) = \frac{\varepsilon(t)}{\sigma_0}$$

and

$$E_{rel}(t) = \frac{\sigma(t)}{\varepsilon_0}.$$

Both are very much similar like you might feel that this

$$C_{crp} = \frac{1}{E_{rel}}.$$

They might be dimensionally same but effect wise they are not the same and the rate at which the relaxation happens compared to the stabilization of creep is actually faster in case of stress relaxation so there is difference between both, one is not the reciprocal of the other.

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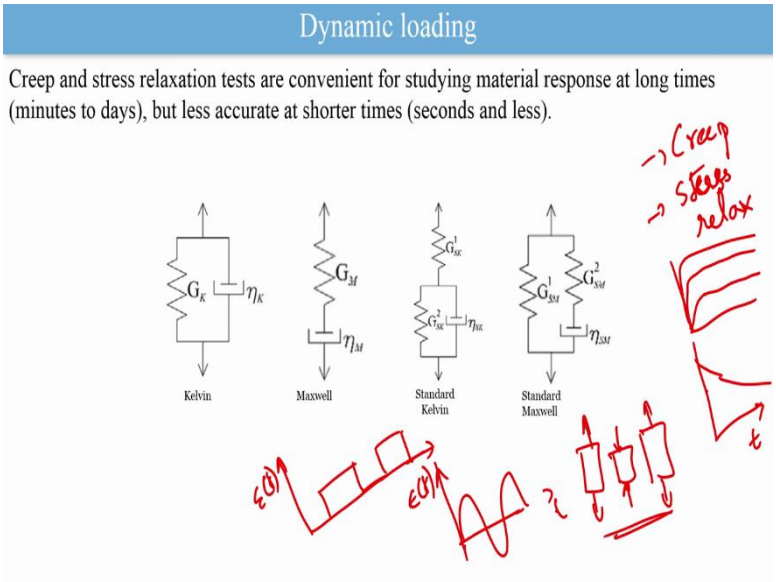

Now, both these characterization methods that we have discussed so far like the creep as well as stress relaxation. As you can easily see, you can see that they need time so either you apply some stress and weight for it to get the behavior or you apply some strain already and try – wait for it to give some results. So you need some time to characterize the materials.

So basically this time depending on the material it actually changes maybe from minutes to hours or to days. But in our daily application to understand the behavior we do not have that much time that is why we go towards dynamic loading of the material. In dynamic loading what we do is, we very quickly apply force like at an instant we stretch the material and another instant we compress the material and we will repeat this for very long time or so this is one form of dynamic test.

So there are various ways you can do it maybe you can apply a sinusoidal so this is the strain with as a function of time. You can apply a sinusoidal foam, a foam kind of strain or what you can do is, you can just give a step kind of response or the strain. So all these things we can try it out and these methods can also give us similar results or similar information as we can get through creep as well as stress relaxation test.

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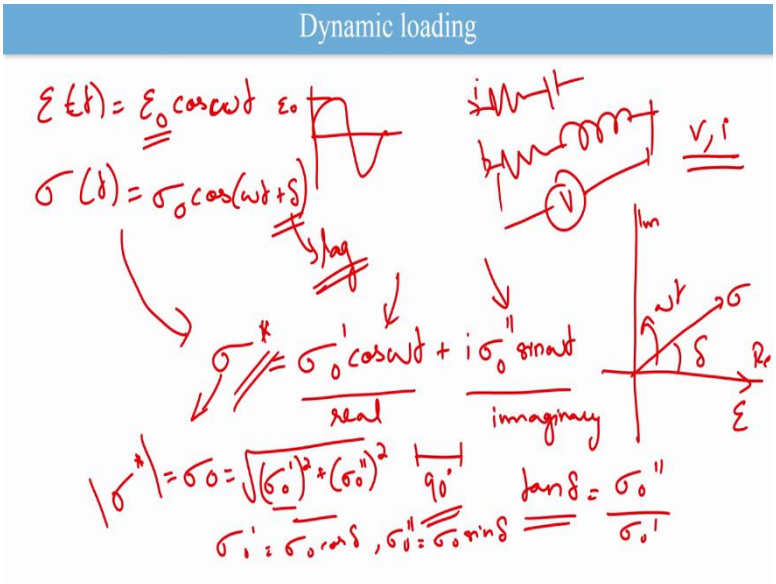

So now let us see if we can try an example with this. So if I apply a strain as a function of time, which is sinusoidal in nature. So this is a harmonic kind of strain that we are applying, so basically like this we are applying. Now this is the peak strain that we apply. Now it is observed that the stress that is developed in the material, $\sigma(t) = \sigma_0 \cos(\omega t+\delta)$ where δ is a lag.

It is very much similar to the electrically when we have a resistor and a capacitor or a resistor and a inductor. So we know that when we apply the current or when we apply a potential between these 2 there can be a lag or lead of current and potential. Similar to that in case of mechanical dynamic in loading the stress actually lax the strain that we apply by a amount of δ.

So this expression can also be written in a complex form that is $\sigma^* = \sigma_0'$. So if I represent – this generated stress as a function, as a complex function and which is having a real as well as imaginary part, which have a face of 90 degree apart. We can represent the same stress generated in this foam.

So if we have this thing, if we apply ε so there can be a stress with a lag of δ. This is the imaginary part and this is a real part. We move towards with a angular vacancy of ω and the $\tan(\delta)$ can be represented or the face in forming of $\tan(\delta)$ can be represented as

$$\tan(\delta) = \frac{\sigma_0''}{\sigma_0'}.$$

And also you can say the magnitude of this complex stress function as

$\sigma_0 = \sqrt{(\sigma_0')^2 + (\sigma_0'')^2}$. So basically what it is, it means is the $\sigma_0' = \sigma_0 cos\delta$ and $\sigma_0'' = \sigma_0 sin\delta$.

So I guess you might have understood this part. Now so basically the stress function can be grouped as a real part as well as imaginary part or a – so the stress is basically categorized into 2 different parts and so converting this stress into the modulus or elastic modulus that we discussed, modulus that we discussed earlier.

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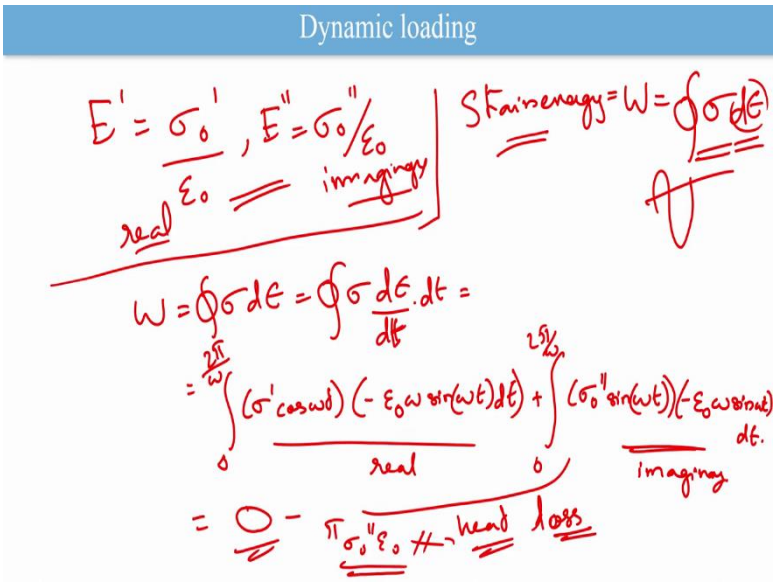

We can write it as, the real part of modulus is given by $\frac{\sigma_0'}{\varepsilon_0}$ and the imaginary part can be said as ε_0. So this is the rationale behind bringing these two apart like the complex part as well as or the imaginary part as well as the real part of the stress function will be understood after some – after a little bit later. So now you understood there will be a real part as well as imaginary part for the modulus also.

Now why this kind of a representation is good or why do we do this. This we can tell based on a quantity known as strain energy. So this is very much similar to the work done that we discussed earlier in kind of thermodynamic. So it is basically nothing but when you apply a strain so some sort of energy stored within the material.

And this strain energy say W can be represented by

$$W = \oint \sigma(d\epsilon)$$

so this stress generated in the material with respect to the strain that we apply. So when we take the $\oint \sigma(d\epsilon)$ for the cycle of dynamic loading, we will know the strain energy stored in the material during the cycle. So we will do the same for this dynamic loading that we discussed so far. W is equal to, we can also represent this as, I am just adding dt on numerator and denominator.

Now this σ we can represent it as 2 terms right. So I am just grouping them apart for a single cycle $2\pi\omega$. So this is the real part. When you integrate this, you understand that this entire real part this is the imaginary part. This entire real part becomes 0. You know this mathematically and this term comes out to be π and $\sigma_0'' \varepsilon_0$.

So you will see that so during an entire cycle the energy stored by the real part of the stress is actually 0. That means whatever force that we applied or whatever stress that we applied, which caused an energy change or storing of energy, the real part of stress actually became 0 over a complete cycle. That is whatever energy that was given in was given out once the cycle was complete.

But at the same time the imaginary part there he is something that remains that means there is an energy loss associated with the dynamic loading that is whenever you apply a load or a loading cycle there will be some amount of energy lost and this energy can be lost as heat. So there is the behaviour of tissue or material where some part of the energy that was transferred during this dynamic loading cycle that is being lost.

That is usually represented by the imaginary part or the loss part that is why we actually represent the stress as well as strain relationship in case of dynamic loading and the lag between them as a complex function, so that we will actually understand what is its actual significance. Now let us talk about – so now you might have understood the basic understanding of dynamic loading.

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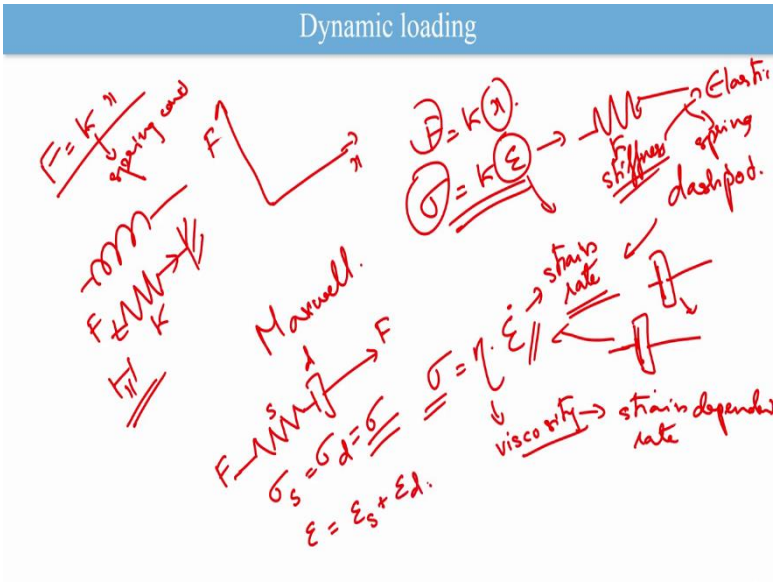

Now we move towards the different models that we use for this kind of applications. So basically you know Hooke's law where force is F = kx where k is the spring constant. So when you apply a force on a spring, the spring can be better represented like this because we are going to use that kind of a terminology later on of a stiffness k and when you apply a force, it displaces by a distance x.

So this is basically what our expression for Hooke's law is. Now the same expression analogous to F = kx. We can say $\sigma = k\varepsilon$. These are very analogous. Instead of force, we have stress and instead of displacement we have strain and this can be represented using a usual spring itself.

So any material can be represented as a combination of springs as well as something that we called as a – this is spring as well as something called as a dash pot. So, dash pot or a like damper. This is very – this kind of a mechanism is always used in electric models also. So you might be familiar with that during the previous lectures. So the dash pot what actually does is it does not consider anything relevant to the strain but it is not concerned towards strain rate.

So it is basically more like a damper so if you apply the force slowly, if you apply force at a slower rate the resistance that it will give is also slow. If you apply a very fast rate then the damper will give you a higher resistance. So in this case the stress can be represented as

$$\sigma = \eta\dot{\varepsilon}$$

where $\dot{\varepsilon}$ is nothing but the strain rate.

So a dash pot is an element where the stress is dependent upon the strain rate and instead of the stiffness K that we discussed, stiffness K that we discussed earlier in a dash pot, we define viscosity. So now we understand maybe like the material or the tissue or the rubber or any other polymer can have this stiffness or elastic properties as well as the viscosity or the strain dependent, strain rate dependent, strain rate dependent behaviour.

So combing this kind of elements Maxwell proposed the Maxwell model that he said that we can say that the materials can be a combination of a spring and a dash pot in series. So whatever force that you apply you apply the same force on both the elements. So if this is spring and this is the dash pot the stress in spring is must be stress in dash pot.

So we called it σ only. And the total stretch that happens when you apply a force, the total force that you apply and that when you apply that force is total strain that will be caused will be sum of these strains in spring and that in dash pot.

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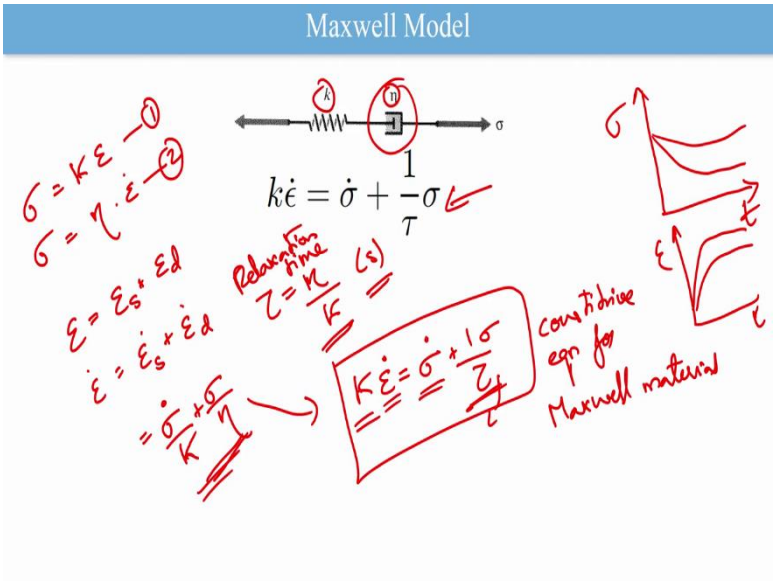

So that is what actually is the Maxwell model. So this is why – so as you have seen earlier, the material will have a plot of the varying say sigma versus time then why does it relax or when you plot the creep that is epsilon versus time why there is a change with respect to time. This is explained by the presence of the dash pot there.

So the dash pot actually is affected by the strain rate and that is why the behaviour of the material changes with time so that is why Maxwell proposed this model with a spring as well as dash pot in series with each other. Now from the previous expressions that we wrote already, that is equal to σ = kε that is 1 and $\sigma = \eta\dot{\varepsilon}$ that is 2.

We can say the – since we also discussed that $\varepsilon = \varepsilon_s + \varepsilon_d$ we can say the strain rate of the total system can be represented as correct and so this can also be written as,

$$= \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta}.$$

Now this is the stress rate in case of spring. Now to better give an idea about the k as well as or the stiffness as well as viscosity of this model, there is a thing called as relaxation time.

Unit or a parameter known as relaxation time is defined τ is given by $\tau = \frac{\eta}{k}$ which has same unit as time. So, maybe seconds or something or minute depending on the unit that you take for viscosity as well as stiffness. So this relaxation time will come everywhere. So if you just solve this, you will get

$$k\dot{\varepsilon} = \dot{\sigma} + \frac{1}{\tau}.\sigma.$$

So this expression that I have given here is known as the constitutive equation for Maxwell material. So basically the stiffness as well as the viscosity which is included in this relaxation time and the different strain rate as well as stress rate in the material will be related in this – as per constitutive equation for the material. Now, how is this significant, how does this be useful?

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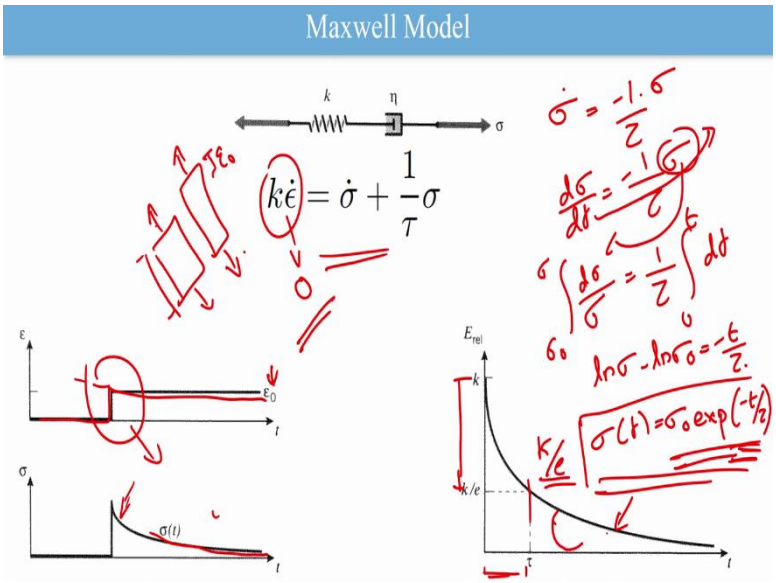

So for that I will just show an example. So there is a material and there was a material and you were loading it to a suddenly to a strain of ε_0, you strained it like this and you are keeping like

this. Now you have seen stress relaxation. So this stress will actually reduce with time and the stress is reducing with time.

So the σ(t) actually decreases with time, correct. So the same expression using this constitutive equation, we will try to solve how it works. Now in this case, and in this kind of strain that we applied you will know that after we apply the strain, there is no more strain rate or we are not applying strain anymore or there is no change in strain after we apply the constant strain ε_0.

So what happens is the $\dot{\varepsilon}$ term actually become 0, so whatever left out will be this will be $\dot{\sigma} = \frac{-1}{\tau}.\sigma$ or this can also be written as $\frac{d\sigma}{dt} = \frac{-1}{\tau}.\sigma$, correct. And we when we integrate it between the total change in stress with respect to total change in time what we can do is so basically I am just rearranging it separating the variables and integrating. It goes here.

And when you solve it, $\sigma - \sigma_0 = \frac{-t}{\tau}$ or this stress, let us say function of time can be given as $\sigma_0\left(\frac{-t}{\tau}\right)$. So basically the relaxation that happens in case of a Maxwell model for the applied strain as shown in this case is exponentially decaying of the stress that we applied initially or the stress generated initially by a strain of ε_0 actually reduces with time exponentially.

And this is the plot and this relaxation time can be physically quantified as the time at which the stiffness of the material the k or the stiffness of the material is reduced to – reduced by $\frac{1}{e}$ times or $\frac{k}{e}$ times that is usually defined as the relaxation time as in most of the electrical analog cases.

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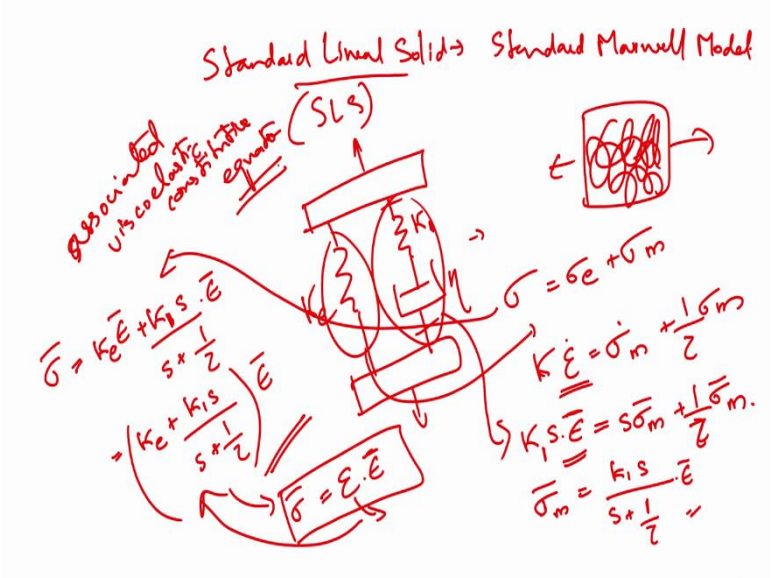

So this is one of the most basic mechanical model that people have proposed over the time. So there were modifications of – but this model or the Maxwell model cannot actually exactly represent the case of the stretch of polymers. The Maxwell model cannot actually show it properly. So that is why the standard linear solid model or standard Maxwell model of the expression was developed.

So in this model – this model was developed because most polymers do not exhibit the unrestricted flow as shown by the Maxwell model. So except for some materials like this Silly Putty or warm tar most of the other materials do not actually follow the simple Maxwell model as we discussed. So it is very limited to the materials that we can say of.

But when it comes to polymers that we discussed earlier there is a lot of entanglements between the polymers. So it cannot exactly be represented by the simple Maxwell model. So that is why this model standard linear solid or SLS model was proposed where we have a resistor in parallel with the Maxwell model that we already discussed.

So if this spring in series is having stiffness of k_1 and the viscosity of the damper is η and we have the stiffness of the other spring, the parallel spring is to be say K_e that I am showing as here and we can model the same model or the can do the same way as we have shown in the previous case. Now in the previous case what happened was whatever stress or the force that applied across the series of spring and damper created the same stress in both.

But the elongation was the sum of both. But in this case the stress will be distributed among the two arms of the model, correct, right. So the total stress in this case will be the stress across this individual spring σ_e plus that of the previous model that we already discussed in the Maxwell model that is σ_m. Now trying to solve this model using the previous approach will be actually little bit difficult.

So we will approach the same problem or solving this model using the Laplace transform model or taking the Laplace of the same thing. Now so how do we do that? So we start from the constitutification and we take the Laplace of the same thing. So here the – this was a constitutification and now in this specific case of this one or the right side branch with the spring and dash pot and series.

We can say the stiffness is K_1 and when we take the Laplace of the differential or the change in strain, what we get is s into the change in strain or the dash over the strain rate shows the Laplace transform of that same function. And if we solve for σ_m dash as we did in previous case, it will get so it is the – we are approaching the same problem that we did previously but using Laplace transform. So that we can do the combinations later on easily

So now if I try to find the stress here and here also if we take the Laplace the total strain in Laplace domain can be given by – and so basically this result can be represented as $\bar{\sigma} = \varepsilon\bar{\epsilon}$. So it is more like the previous expression that we wrote. So and where this e or the ε can be is equal to k_e plus or basically it is nothing but this constant within the bracket.

So you can actually this is basically model parameter that relates to stress as well as strain in the Laplace domain. So this expression totally is known as the associated viscoelastic constitutive equation. So similar to this SLS model or the Maxwell model various other models are there for the tissues that or the polymers that we know of.

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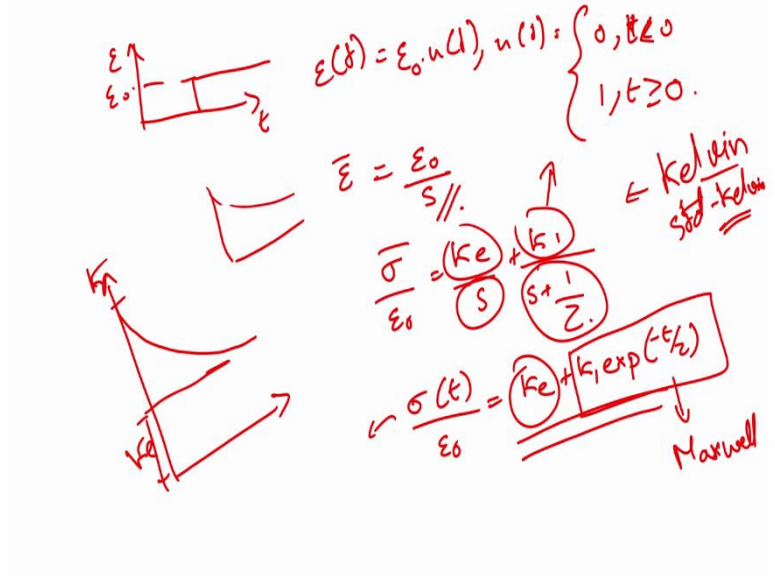

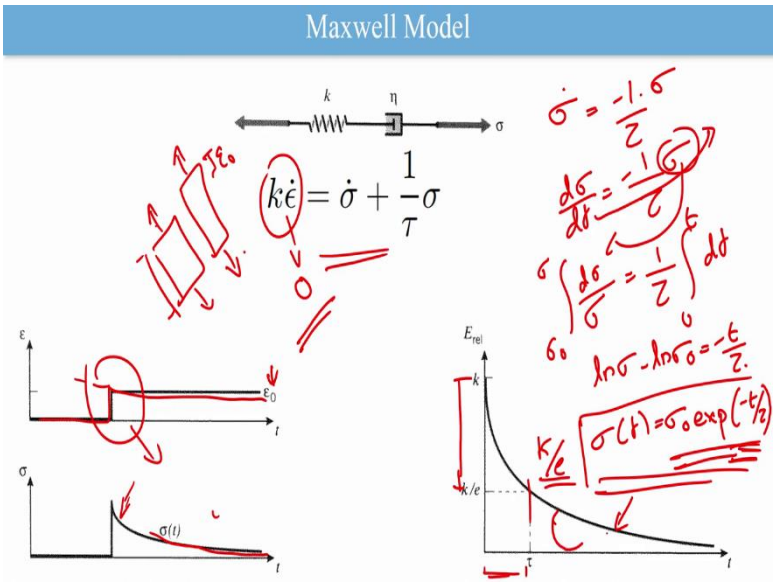

And so before winding up we will just do one example with this case. So as we discussed earlier, we applied a strain with respect to time like this, like we applied a step of ε_0. So instead of that this time we will say apply say unit function so I can write the function as $\varepsilon(t) = \varepsilon_0 u(t)$ where $u(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases}$.

So if we apply a unit step function and the Laplace of this function will be $\frac{\varepsilon_0}{S}$, correct. So the previous equation that we derived will become

$$\frac{\bar{\sigma}}{\varepsilon_0} = \frac{k_e}{S} + \frac{k_1}{S + \frac{1}{\tau}}.$$

So now we can take the inverse Laplace of these terms to get what we will get

$$\frac{\sigma(t)}{\varepsilon_0} = k_e + k_1 \exp\left(-t/\tau\right)$$

so basically this expression you are very similar familiar.

This expression is nothing but the expression for stress that you got for Maxwell expression. You might be remembering them. So when we derived it for Maxwell model, we got the expression the stress is a function of time, exponentially. So it is the same plot but what we have got now by it offset by so that it is a previous one but we have an offset of k.

So this is how the modified function or modified Maxwell model or the standard Maxwell model actually changes from the previous Maxwell model that we have discussed. So we have discussed lot of things so far so starting from how we categorize different materials what is the importance of characterizing materials mechanically, what are the differences between the metal and a polymer and how this polymers are similar to the tissues and how the polymer based models can be used for the applications for the tissue.

So we have discussed a lot of models. Similar to this Maxwell models we have Kelvin model and the standard Kelvin model that is little bit more complex than the Kelvin model or the Voigt model. So we have a lot of models like this which are used to depict the mechanical properties of tissue using dynamic testing.

So depending on the tissue behaviour one model might fit better than the other model. So different materials will have the different type of tissues even will have different correlation with different, different models. A deceased tissue might be a better fit using maybe a Maxwell model. I am just telling, I mean not really but maybe a normal tissue will be more towards SLS model.

So depending on the application, depending on the experimented data we can use different types of models like this and we can actually find the parameters K. So once you do the experiment then you get a result like this, what you can do is you can fit this curve and obtain different values for K_e or K_1 so that these models can be used further for various simulation applications as well as designing various tools. So, I hope you found this lecture or this session useful. Thank you.