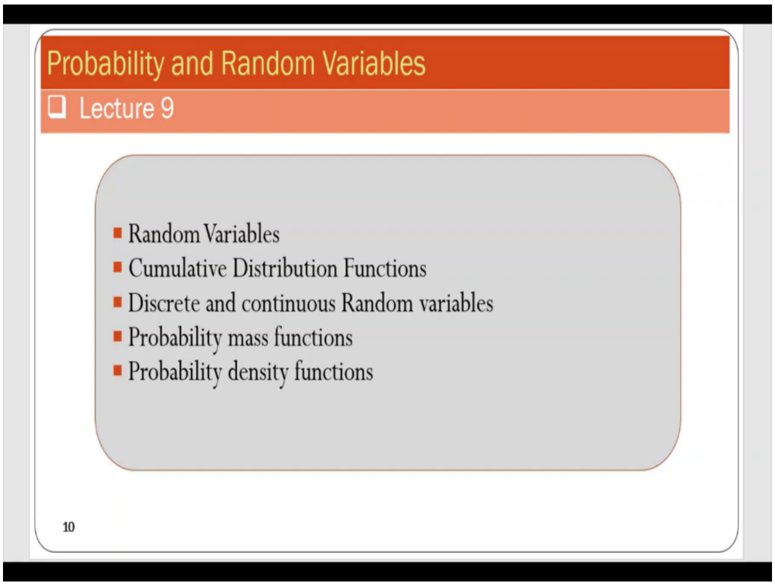
Mathematical Aspect of Biomedical Electronic System Design
Professor Chandramani Singh
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

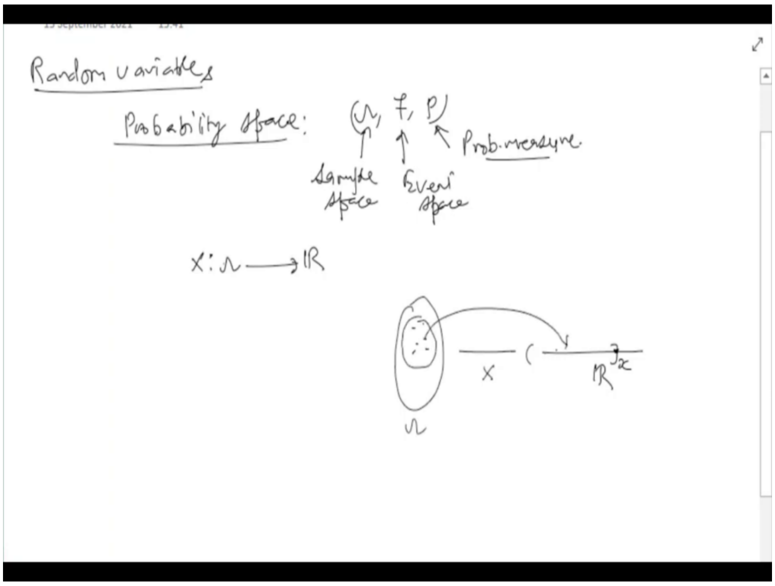
Hello everyone, welcome to another lecture of the course Mathematical Aspects of Biomedical Electronic System Designing.

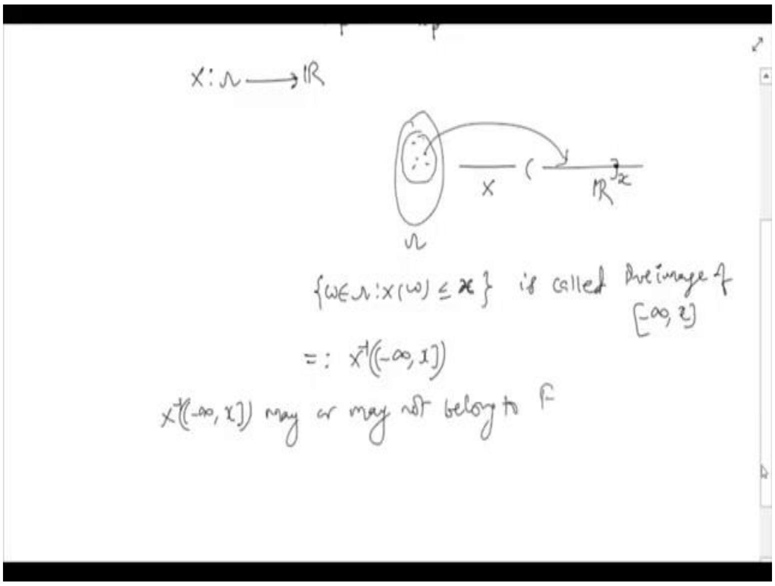
(Refer Slide Time: 00:36)



We are now going through the module on probability and random variables. In today's lecture, we will cover random variables, cumulative distribution functions, discrete and continuous random variables, probability mass functions, probability density functions etcetera. So, let us start today's lecture.

(Refer Slide Time: 00:56)



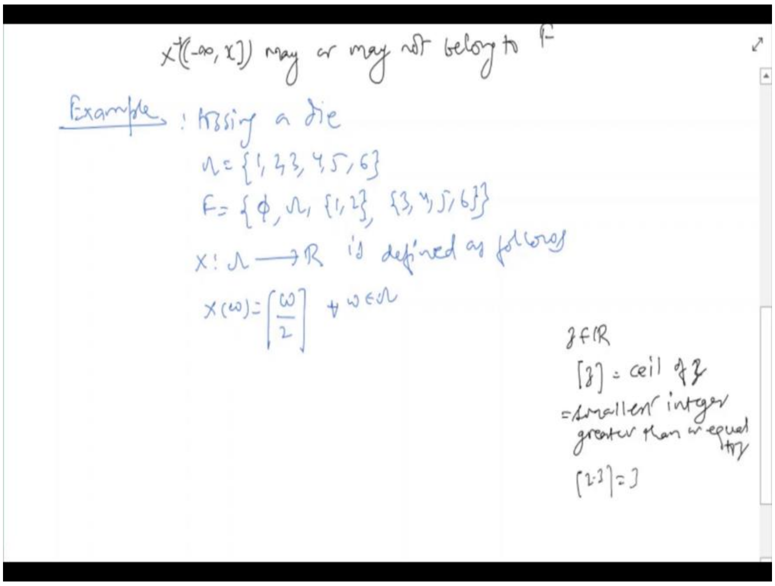


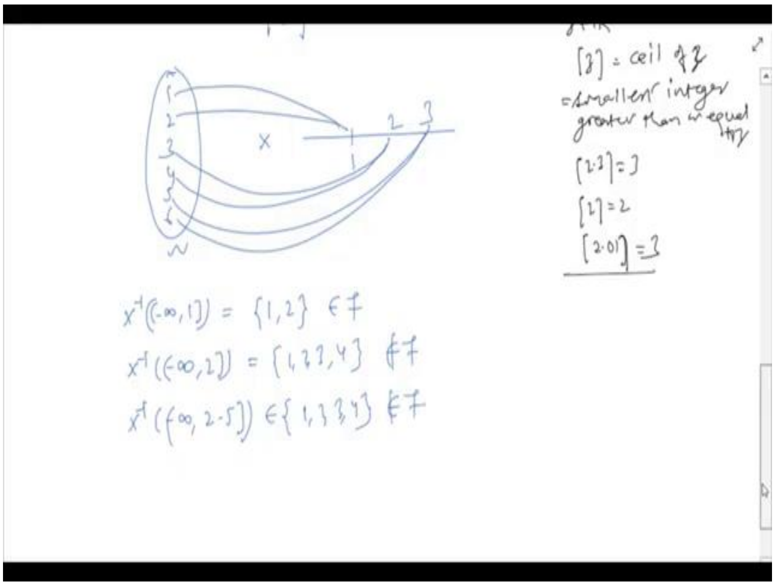
We will start with the notion of random variables. To understand random variables, let us recall the notion of probability space, recall that probability space is a Ω, script F and P where Ω is sample space, F is event space and P is a probability measure. Let us consider functions from sample space to real numbers.

So, on one side we have space of sample points, on other side we have the space of real numbers and X is a mapping from Ω to R. Let us fix a point small x on R and look at all the points, all the sample points in Ω that are mapped to the segment -∞ to x. So, basically, we want to see all the points here that are mapped to this.

Collection of these points is called preimage of [-∞, x]. $\{\omega \in \Omega : x(\omega) \leq x\}$, this is called preimage of [-∞, x]. It is denoted by x^{-1}([-∞, x]). Now, this preimage may or may not be an element of F that is event space. x^{-1}([-∞, x]) may or may not belong to F.

(Refer Slide Time: 03:20)





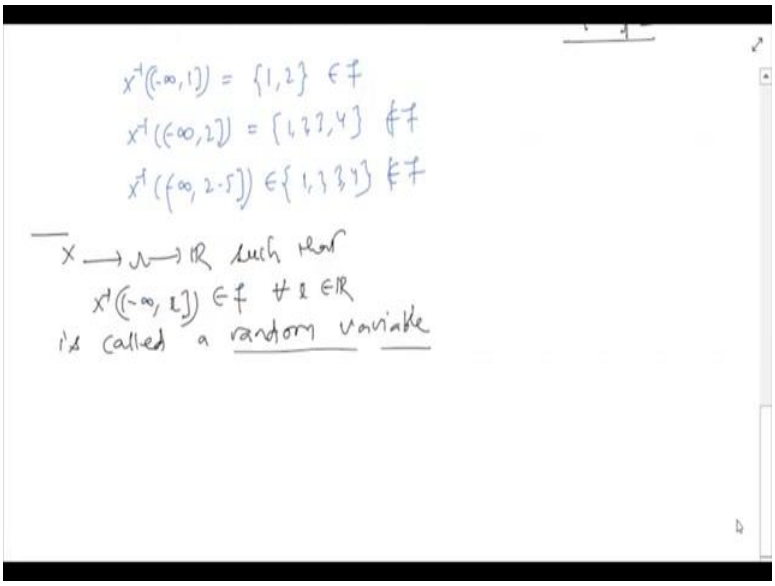
Let us understand it with an example. Consider the random experiment of tossing a die. In this case the sample space is {1, 2, 3, 4, 5, 6}. Let us consider that event space F = {Φ, Ω, {1, 2}, {3, 4}, {5, 6}}to be empty set. Now, we consider a function x : Ω → R defined as follows. We define $x(\omega) = \frac{\omega}{2}$, so that for a number z, ceil is the smallest integer that is ≥ z.

Ceil of z it is smallest integer that is ≥ z. For instance, ceil of 2.3 is 3. Ceil of 2 is 2. Ceil of 2.01 is also a 3. We consider function $x(\omega)$ which is ceil of $\frac{\omega}{2}$. So, clearly it is such a function will map the sample space to real numbers as follows 1, 2, 3, 4, 5, 6. This is my sample space 1 and 2 will be mapped to number 1; 3 and 4 will be mapped to number 2 and 5 and 6 will be mapped for number 3.

In this case, if we consider preimage of [-∞, 1], it is clearly the set {1, 2}, and as we can see it is in the event space. On the other hand if we consider preimage of [-∞, 2] it is 1, 2, 3, 4, 5 because all these numbers 1, 2, 3, 4 are mapped to either 1 or 2. Now, 1, 2, 3, 4 is not in script F.

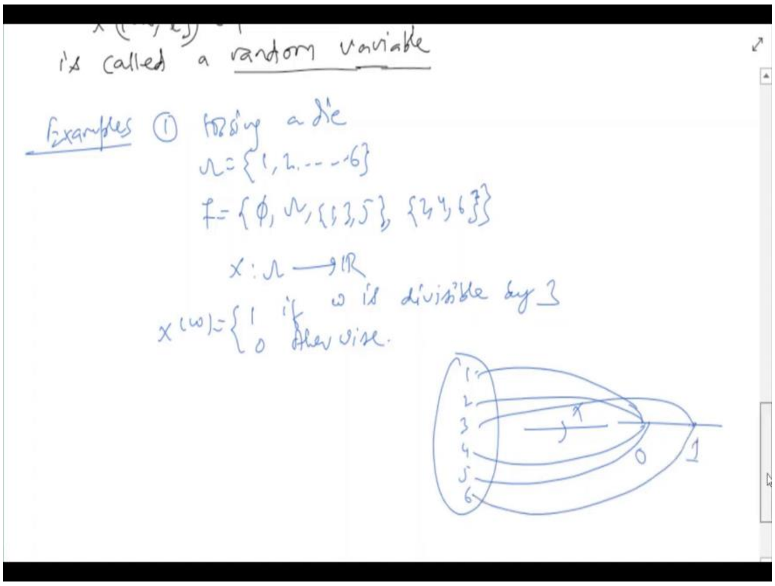
Similarly, if we consider the preimage of [-∞, 2.5] this will also be 1, 2, 3, 4 this is also not in F. So, we see that preimage of x^{-1}([-∞, x]) may or may not belong to the event space. The special functions for which preimage of [-∞, x] is always part of the event space are called random variables.

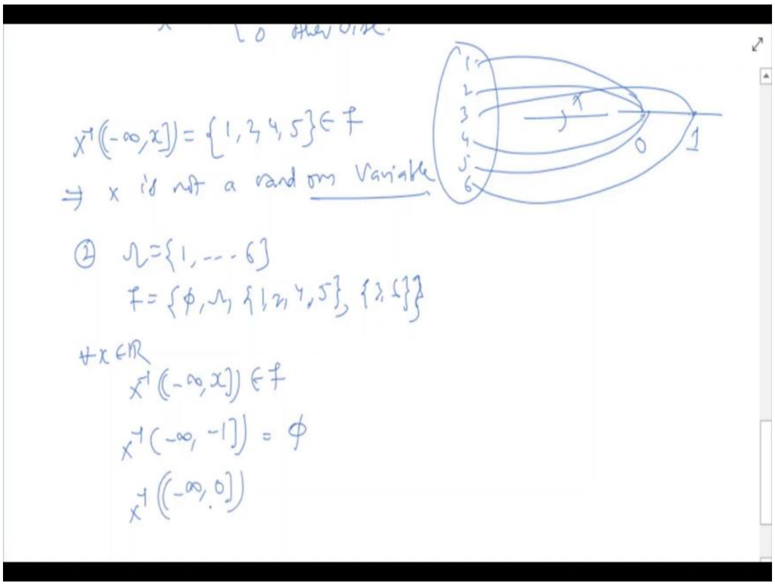
(Refer Slide Time: 6:41)

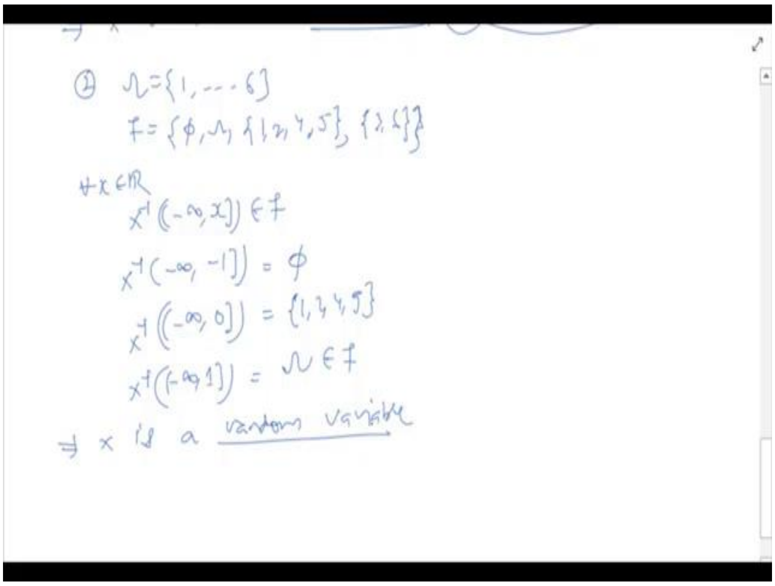


So, let us write this definition X → Ω → ℝ such that preimage of [-∞, x] is part of the event space for all X and R is called a random variable. Let us illustrate this definition of random variables with a few more examples.

(Refer Slide Time: 07:19)







Let us again consider the experiment of tossing your die. As before the sample space is 1 to 6 let us consider the event space to be empty set, the whole set {1, 3, 5} and {2, 4, 6}. It can easily verified that, such a script F is a sigma algebra that is it in an event space. Now, let us define a function x: $\Omega \rightarrow \mathbb{R}$ as follows.

$$x(\omega) \begin{cases} 1 \; if \; \omega \text{ is divisible by 3} \\ 0 \quad\quad\quad\quad \text{otherwise} \end{cases}.$$

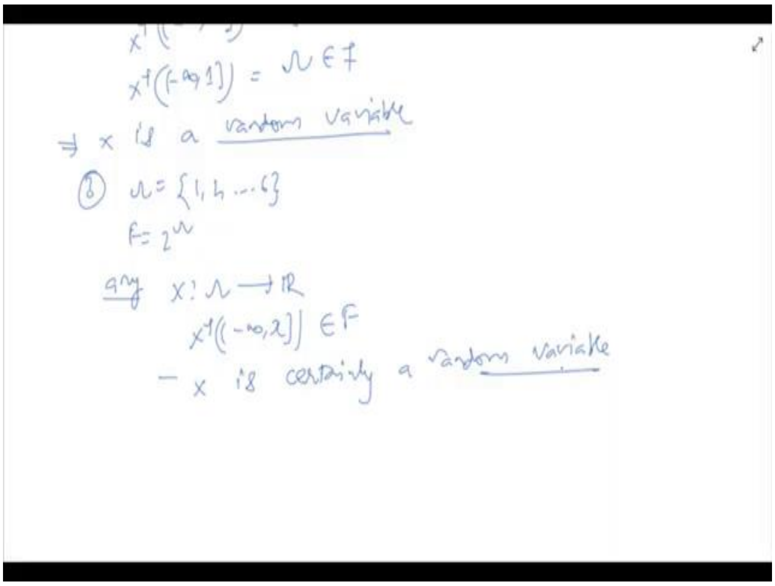
Clearly, under this function 1, 2, 4 and 5 are mapped to numbers 0 whereas 3 and 6 are mapped to number 1. Now, if we consider preimage of the set [-∞, x] then it is a collection of all points that are mapped to points under real and data, either 0 or left to 0, so, this preimage is {1, 2, 4, 5}.

And it is not in F. So, X is not a random variable. However, let us consider the following example. Again, as before Ω = {1,…..,6}, but, the event space now is empty set was set 1, 2, 4, 5, 3, 6. Now, it can be checked that for any x on real line preimage of the set [-∞, x] will be part of script F. For instance, $x^{-1}[-\infty, -1]$, this would be set of all points whose image is ≤ -1.

So, this is clearly empty set. No pointer remains ≤ -1. $x^{-1}[-\infty, 0]$. Preimage of [-∞, 0], this would be {1, 2, 4, 5}. And this is now clearly an element of the event space. Similarly, $x^{-1}[-\infty, 1]$ this will be the whole space Ω. So, clearly this is also script F.

In the same way, we can check the preimage of any set of the form [-∞, x] will be an element of the sample space. So, x is a random variable.

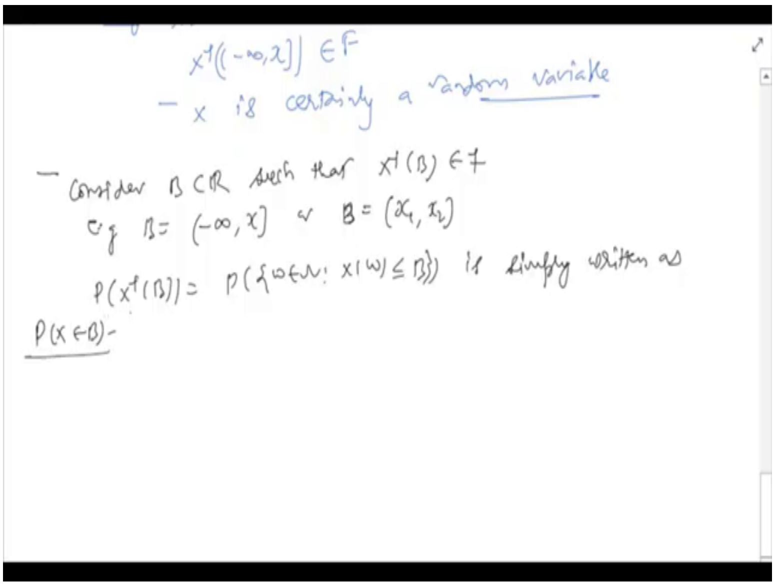
(Refer Slide Time: 10:59)



Let us consider yet another example. I keep the sample space same as before, but my event space is the power set of Ω. But my event space is the power set of Ω that it is collection of all subsets of Ω. In this case, if we consider any x : Ω → ℝ.

Since $x^{-1}[-\infty, x]$ that is preimage of [-∞, x] will certainly be a subset of Ω. It is certainly an element of the event space. In this case, x is certainly a random variable. So, we see that the same function from Ω → ℝ may or may not be a random variable depending on the event space.

(Refer Slide Time: 12:07)

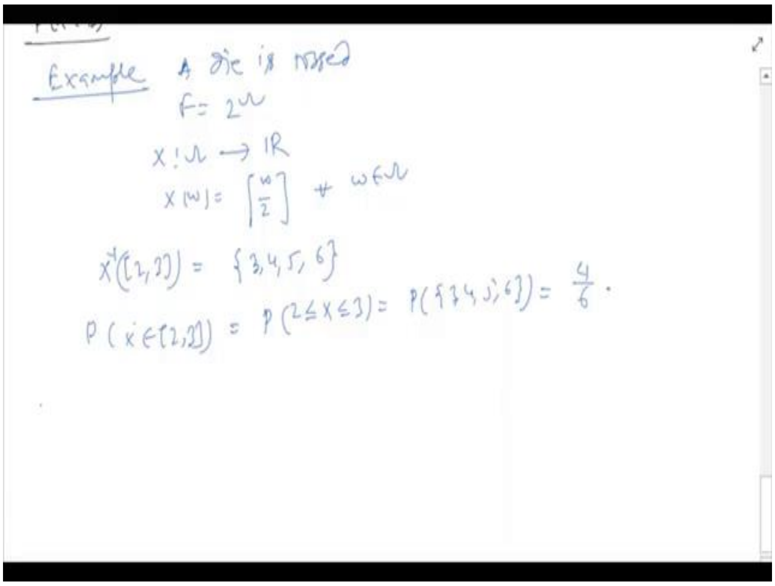


Now, consider $B \subset C$ such that $x^{-1}(B) \in F$. For example, B = [-∞, x] or B = (x_1,x_2). It can be seen that if X is a random variable, then for all subsets there preimage will be elements of event space.

In this case probability of the preimage that is

$P(x^{-1}(B)) = P(\{\omega \in \Omega: x(\omega) \leq B\})$ is often simply return as $P(x \in B)$. So, whenever we see a notation, $P(x \in B)$, we should understand that it means the probability of the preimage of B.

(Refer Slide Time: 13:25)

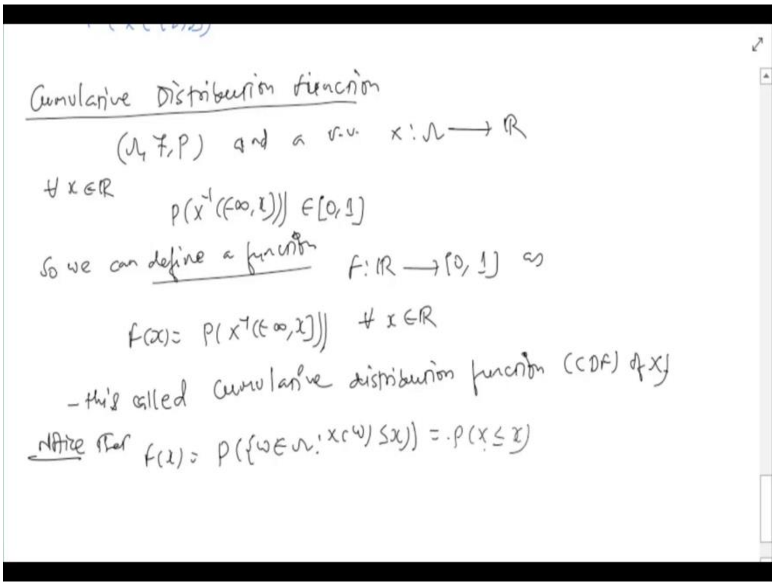


Let us see an example. It is again consider the example of a die being tossed. And event space to be the power set of the sample space. As in the first example, we consider a function x : Ω → ℝ defined as $x(\omega) = \frac{\omega}{2} \;\; \forall \;\; \omega \in \Omega$.

Then preimage of this [2, 3] will be collection of all sample points that are mapped to numbers between 2 and 3 that is that are mapped to 2 or 3 and it can be easily observed that this preimage is {3, 4, 5, 6}. So, by

$$P(x \in [2,3]) = P(2 \leq x \leq 3) = P(\{3,4,5,6\}) = \frac{4}{6}.$$

(Refer Slide Time: 14:51)



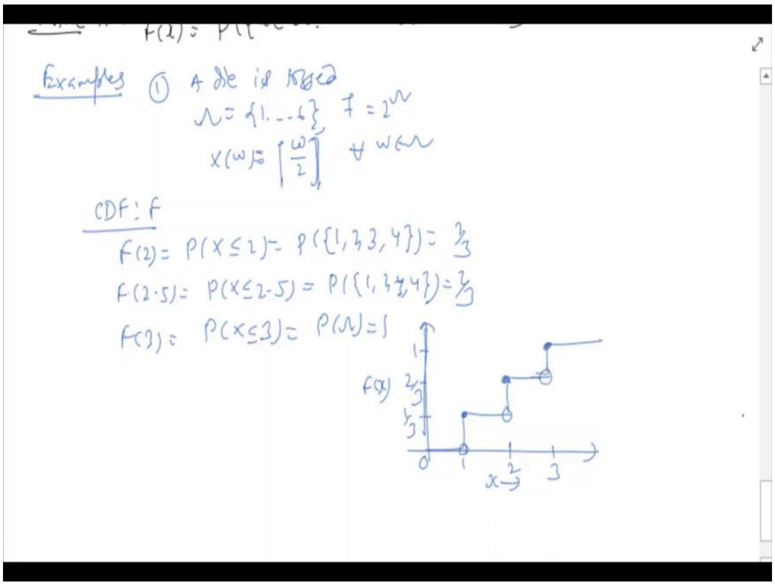
Next we consider the notion of cumulative distribution functions of random variables. Cumulative distribution function: It is considered a probability space and x : $\Omega \to \mathbb{R}$ $\forall x \in R$. From the definition of random variable $x^{-1}[-\infty, x]$ that is preimage of $[-\infty, x]$ is an element of script F.

And so it is probability is defined. Moreover, the probability is the number between 0 and 1. So, we can define a function, function capital F : $\mathbb{R} \to [0,1]$ as follows $F(x) = P(x^{-1}[-\infty, x])$ $\forall x \in R$. This function is called cumulative distribution function of x.

Cumulative distribution function in short, CDF of X. Notice that

$$F(x) = P(\{\omega \in \Omega: x(\omega) \leq x\}) = P(X \leq x).$$

(Refer Slide Time: 17:06)



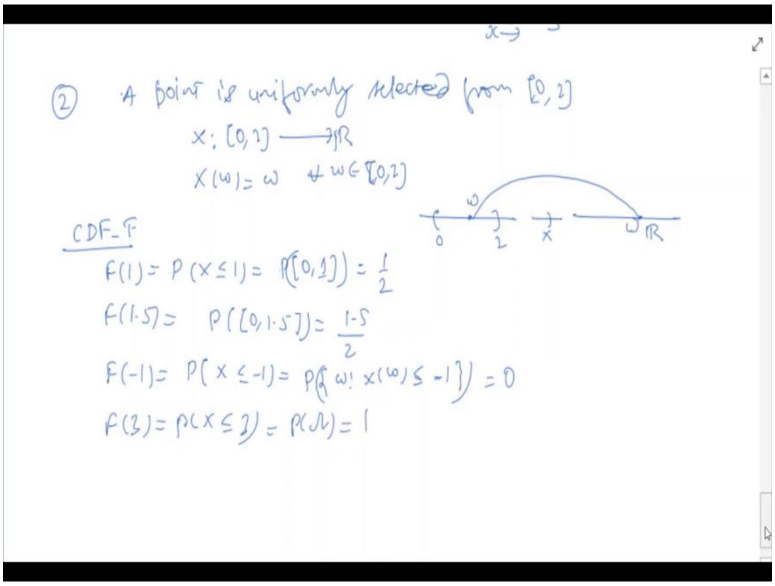
Let us see a few examples, the first example is the die tossing example, seen before. Here, Ω = {1,…..,6} we take script F to be the sigma algebra. And as the first example, we consider the function $x(\omega) = \frac{\omega}{2}\ \ \forall\ \ \omega \in \Omega$. We have already seen that such an x qualifies for a random variable.

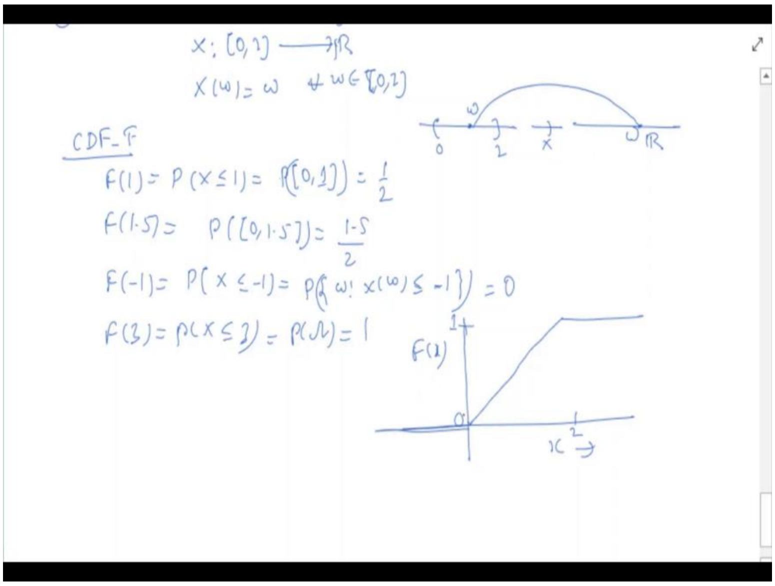
We can define the cumulative distribution function F of this random variable as follows. So, for instance, $\mathrm{F}(2) = P(X \leq 2) = P\{1,2,3,4\} = \frac{2}{3}$; $\mathrm{F}(2.5) = P(X \leq 2.5) = P\{1,2,3,4\} = \frac{2}{3}$.

$\mathrm{F}(3) = P(X \leq 3) = P\{\Omega\} = 1$. In fact, it can be seen that the function F looks as follows. Here we have points 0, 1, 2, 3. Here we have $\frac{1}{3}, \frac{2}{3}$, 1 and F is of the form of this step function.

Value of this function at x = 1 is $\frac{1}{3}$. The value of this function at x = 2 is $\frac{2}{3}$, and the value of the function at x = 3 is 1. The value of the function at all the points below 1 is 0. Value of the function at all the points above 3 is 1. So, this is a cumulative distribution function F of the random variable that we defined.

(Refer Slide Time: 19:35)

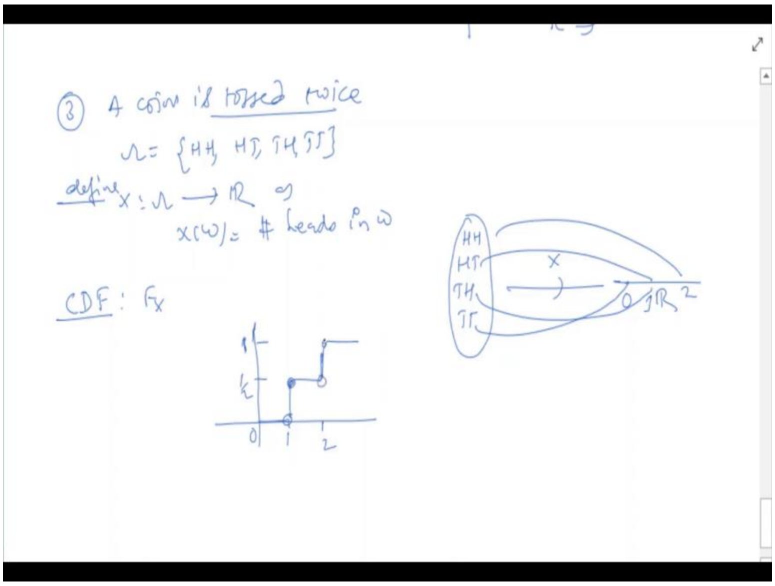




Let us see another example. Now we considered the experiment of a point being selected uniformly from interval [0, 2]. We consider x : [0,2] → ℝ defined as follows. $x(\omega) = \omega\ \forall\ \omega \in [0,2]$. In other words, this F takes elements from an interval 0 to, and it maps them to elements in R and any point ω in 0 to map the 2, ω in R.

In this case, we can define the cumulative distribution function, F of the random variable X as follows. $\mathrm{F}(1) = P(X \leq 1) = P([0,1])$ and since the point is uniformly selected from 0, 2 this probability would be similarly, $\mathrm{F}(1.5) = \mathrm{P}([0,1.5]) = \frac{1.5}{2}$. $\mathrm{F}(-1) = P(X \leq -1) = P(\{\omega: x(\omega) \leq -1\})$. Clearly, for no value of ω, x(ω) ≤ -1. So, this probability will be 0, $\mathrm{F}(3) = P(X \leq 3) = P(\Omega) = 1$. In fact, it can be seen that the cumulative distribution function F in this case looks as follows. It is a RAM function between 0 and 2. Starts at 0, it ends up at 1. It is a flatline outside this interval.

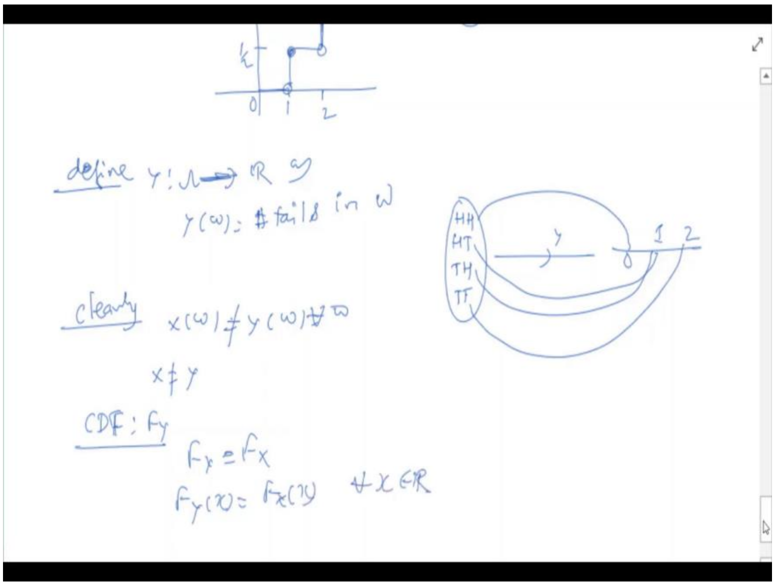
(Refer Slide Time: 22:00)

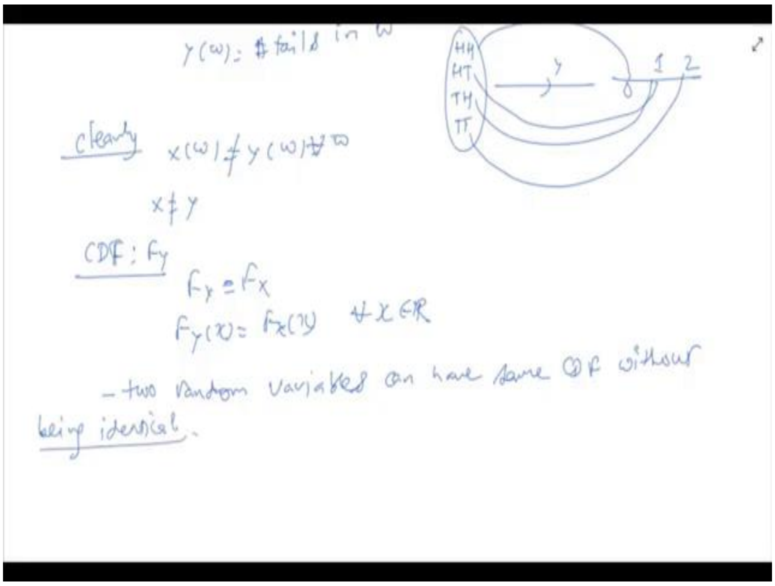


Let us see yet another example. Now, we consider the experiment of a coin being tossed twice, is tossed last twice. Clearly, in this case, the sample space {H H, H T, T H, T T}. We define x : $\Omega \rightarrow \mathbb{R}$. x($\omega$) is number of threads in ω.

In other words, we have a set {H H, H T, T H, T T} and X is a mapping from this set to set of real numbers, where H H is mapped to 2 H T and T H are mapped to 1. And the T T is mapped to 0. In this case, it can be seen that CDF of X F_x will have the form as shown in this graph. So, we will have 0, 1, 2 and F_x will be form of a step function with steps edges being half. So, this is half, half. F(0) = 0, F(1) = $\frac{1}{2}$, F(2) = 1.

(Refer Slide Time: 23:41)

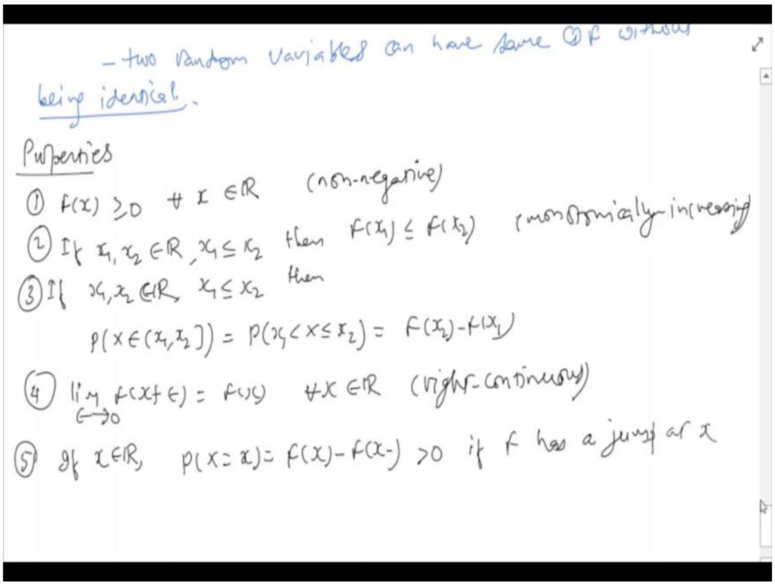


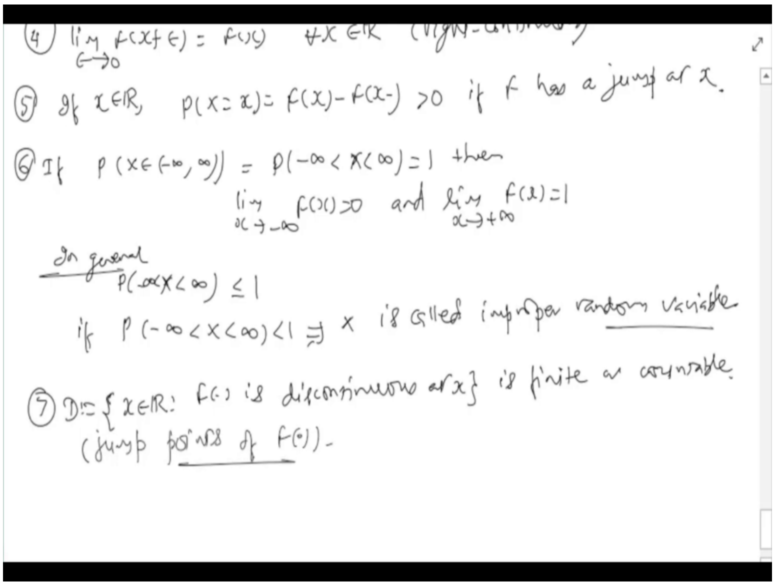


Similarly, we can define another random variable y to be number of tails in ω. So, y(ω) is number of tails in ω. Now, the function y looks as follows. Its domain is same as that of x. But it maps H H to 0, H T and T H to 1 and T T to 2. Clearly, $x(\omega) \neq y(\omega)\ \forall \omega$. This means $x \neq y$.

The random variables x and y are different. However, if we consider the CDF cumulative distribution function of y, say F_y then $F_y = F_x$. In other words, $F_y(x) = F_x(x)\ \forall x \in \mathbb{R}$. So, we see that if 2 random variables have same CDF, that does not mean that they are identical. So, 2 random variables and have same CDF without being identical.

(Refer Slide Time: 25:25)





Let us now see a few properties of cumulative distribution functions. Properties: First property which one can see immediately is $F(x) \geq 0\ \forall x \in \mathbb{R}$, that is CDF are more negative. The second property is if $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$. Then $F(x_1) \leq F(x_2)$.

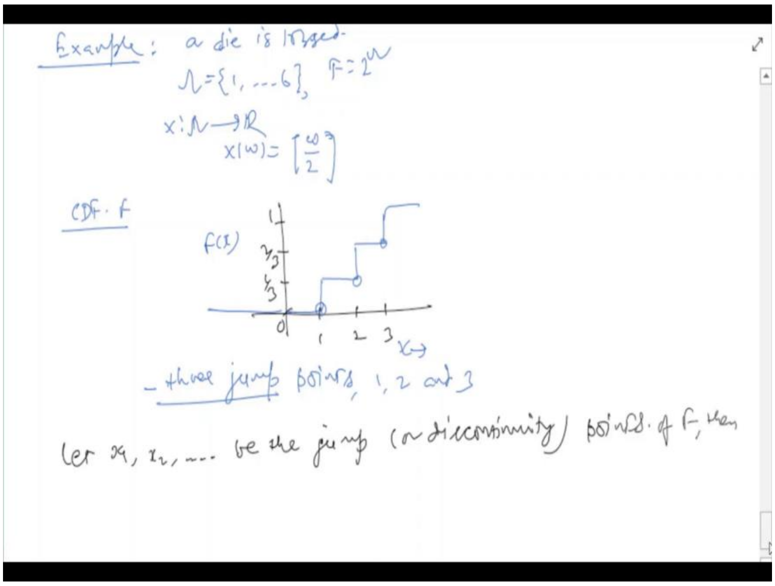
That is CDFs are monotonically increasing. The third property is if as in the second property $x_1, x_2 \in \mathbb{R}$ and $x_1 \leq x_2$. Then

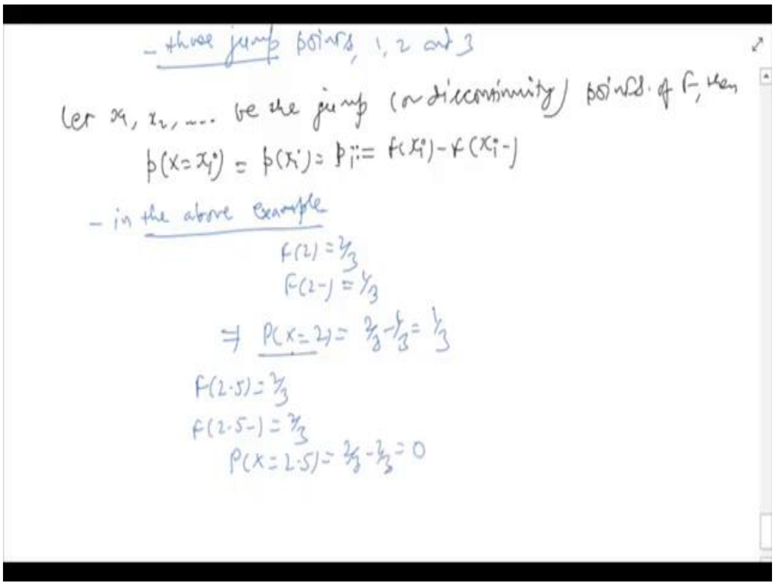
$$P(x \in [x_1, x_2]) = P(x_1 < x \leq x_2) = F(x_2) - F(x_1).$$

Fourth property is that $\lim_{\epsilon \to 0} F(x + \epsilon) = F(x)\ \forall x \in \mathbb{R}$. That is CDFs are right continuous. Property number 5 says that if $x \in \mathbb{R}$, then $P(X = x) = F(x) - F(x-) > 0$, if F has a jump or a discontinuity at x, jump at x. Property number 6 is that if $P\big(x \in (-\infty, \infty)\big) = P(-\infty < x < \infty) = 1$ then $\lim_{x \to -\infty} F(x) = 0$ *and* $\lim_{x \to +\infty} F(x) = 1$. Note that $P(-\infty < x < \infty) < 1$. That is in general $P(-\infty < x < \infty) \leq 1$. If $P(-\infty < x < \infty) < 1$ then x is called improper random variable, improper random variable.

However, we will restrict ourselves to proper random variables. So, for us probability of x being finite will be 1. Let us now see one more property which says that $D! = \{x \in \mathbb{R}: F(.)$is discontinuous at x$\}$. Finite or countable, that is there can be at most countably many jump points of F.

(Refer Slide Time: 29:53)





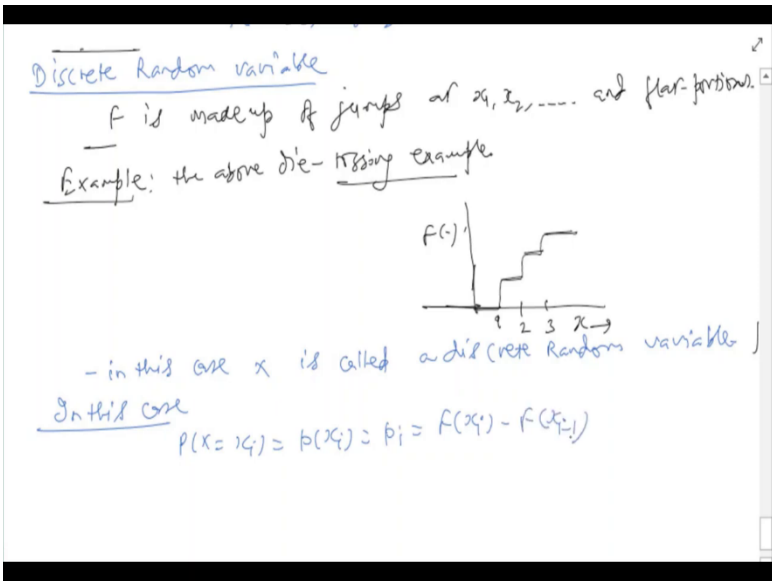
For example, let us recall the random experiment of a die being tossed. Your sample space was $\Omega = \{1,\ldots..,6\}$. Let us assume the event space to be $F=2^{\Omega}$ and $x : \Omega \to \mathbb{R}$. You find as $x(\omega) = \frac{\omega}{2}$. All that in this case the CDF F(x) at the following form 0, 1, 2, 3, 1 by 3, 2 by 3, 1. So, this was F(x).

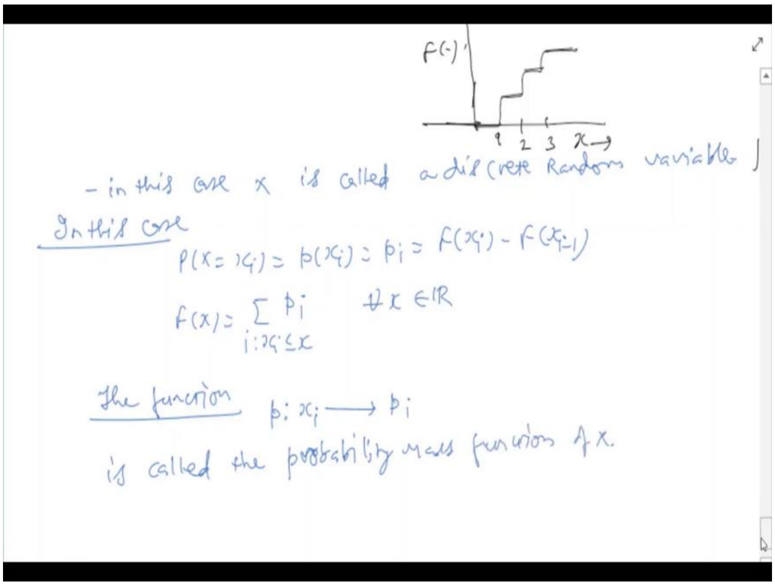
Here there are 3 jump points, 1, 2 and the 3. The other property says that in general there can be either finite or countably many jump points. Further, let x_1, x_2 ….. be the jump points or discontinuity points, points of F, then $P(X = x_i) = P(x_i) = P_i \coloneqq F(x_i) - F(x_i -)$.

For instance, in the above example, F(2) = 2/3, F(2-) = 1/3. Hence, $P(X = 2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$. Similarly, F(2.5) = 2/3 and F(2.5-) = 2/3. So, $P(X = 2.5) = \frac{2}{3} - \frac{2}{3} = 0$.

So, we see that the P(X) taking certain values will be non-zero only if those values are jump points. In the following we will be interested in 2 special types of cumulative distribution functions.

(Refer Slide Time: 33:15)





The first special type is when F is made up of jumps say at x_1, x_2, … and flat portions. An example is the above die tossing example, example where F was of the form as desired here. So, F here is made up of jumps at points 1, 2 and 3, and it is flat everywhere else. So, F is made up of jumps and flat portions.

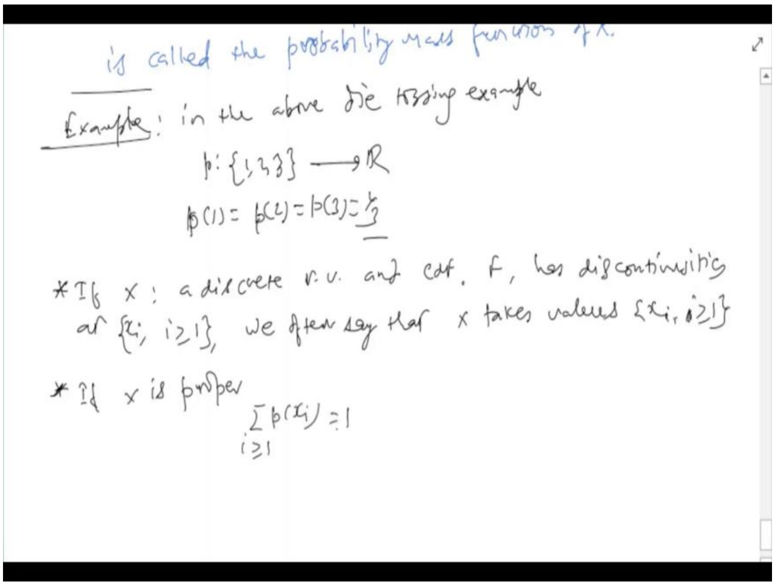
In such a case, the random variable X is called discrete random variable. In other words, X is called a discrete random variable. If its CDF is made up of jumps and flat portions. In this case, X is called a discrete random variable. In this case, $P(X = x_i) = P(x_i) = P_i \coloneqq F(x_i) - F(x_i -)$.

And, F(x) can be expressed in terms of these probabilities as follows.

$$F(x) = \sum_{i:x_i \le x} P_i \ \ \forall x \in \mathbb{R}.$$

The function $P: x_i \rightarrow P_i$ is called the probability mass function of X, mass function of.

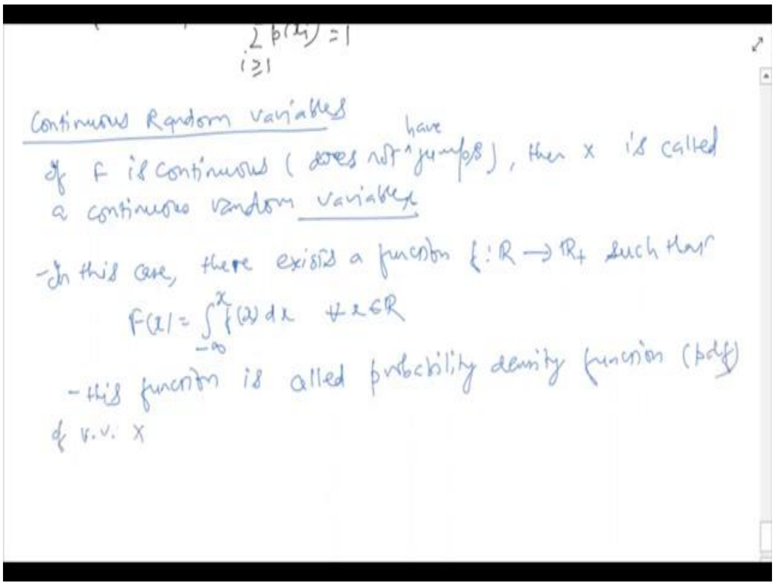
(Refer Slide Time: 36:11)

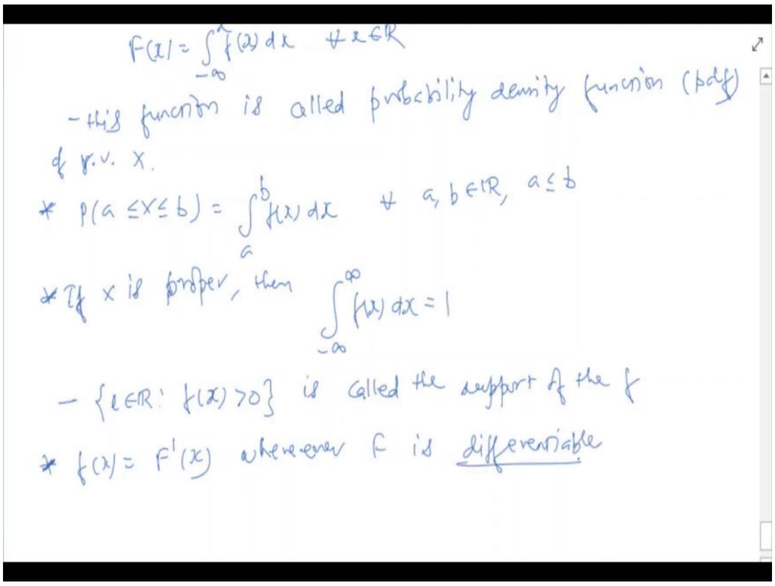


For instance, in the above die tossing example P : {1, 2, 3} → R and P(1) = P(2) = P(3) = 1/3. If X is a discrete random variable and its CDF say F has discontinuities at $\{x_i, i \geq 1\}$, we often say that X takes values $\{x_i, i \geq 1\}$. X takes values $\{x_i, i \geq 1\}$, if the discrete random variable X is proper then clearly

$$\sum_{i \geq 1} P(x_i) = 1.$$

(Refer Slide Time: 37:47)





The second type of CDFs that we will be interested in are those that are continuous. So, F is continuous, if F continuous that is does not have jump points, does not have jumps then the random variable X is called a continuous random variable, continuous random variable. We are now seeing continuous random variables.

In the case of continuous random variables, there exists a function f : R →R+ such that

$$F(x) = \int_{-\infty}^{x} f(x)dx \quad \forall x \in \mathbb{R}.$$

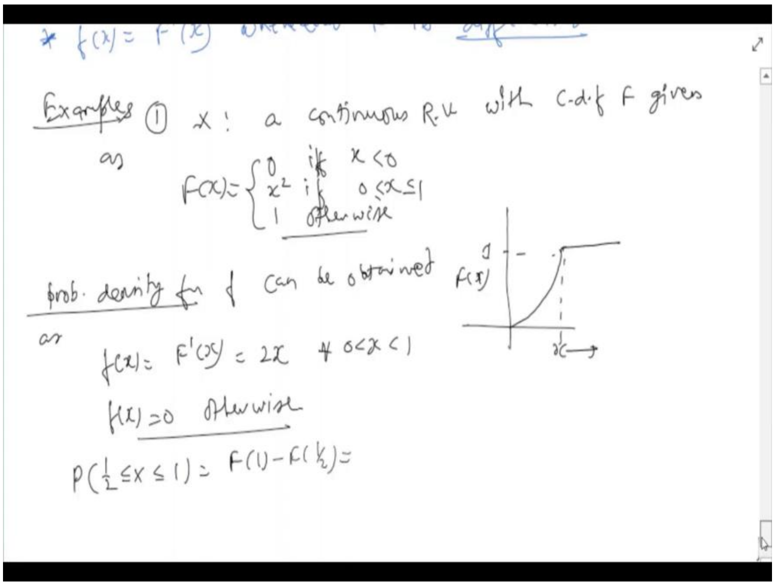
This function f is called probability density function of the random variable X, this function is called probability density function, function in short PDF of random variable X.

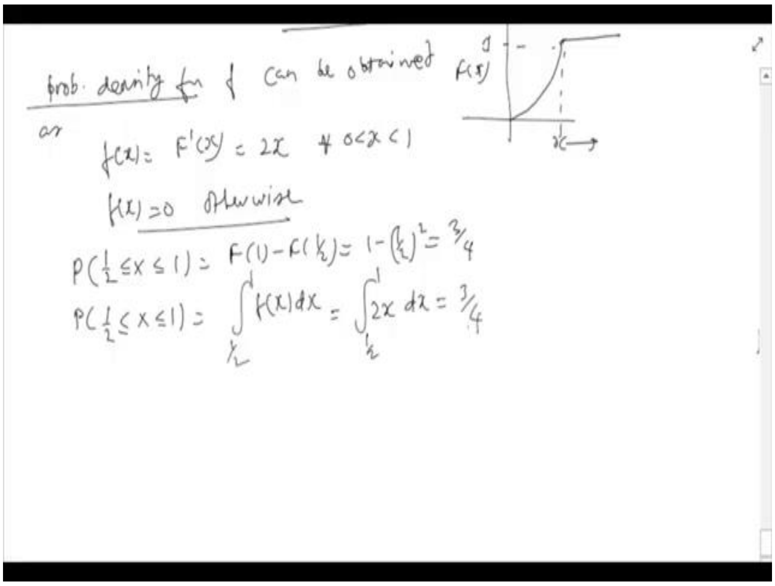
Observe that $P(a \leq x \leq b) = \int_a^b f(x)dx \quad \forall a, b \in \mathbb{R}, a \leq b$. If X is proper if the continuous random variable X is proper then

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The collection of values at which F(x) is strictly positive is called the support of the PDF F or often called support of the random variable X. Further the PDF F(x) is related to the CDF F(x) as follows f(x) =F´(x) wherever CDF is differentiable.

(Refer Slide Time: 41:17)





Let us see a couple of examples. Let us first consider a random variable X that is a continuous random variable with CDF F given as

$$F(x) = \begin{cases} 0, & if\ x < 0 \\ x^2, & if\ 0 \le x \le 1. \\ 1\ , & otherwise \end{cases}$$

If we plot the CDF is looks like this this point is 1 so, is this. As we can see, the CDF does not have any jump it is continuous.

So, X is continuous random variable in this case probability density function f can be obtained f(x) =F´(x) at all the points x at which F is differentiable. So, this is $= 2x\ \ \forall\ 0 < x < 1$ and f(x) = 0 outside this window.

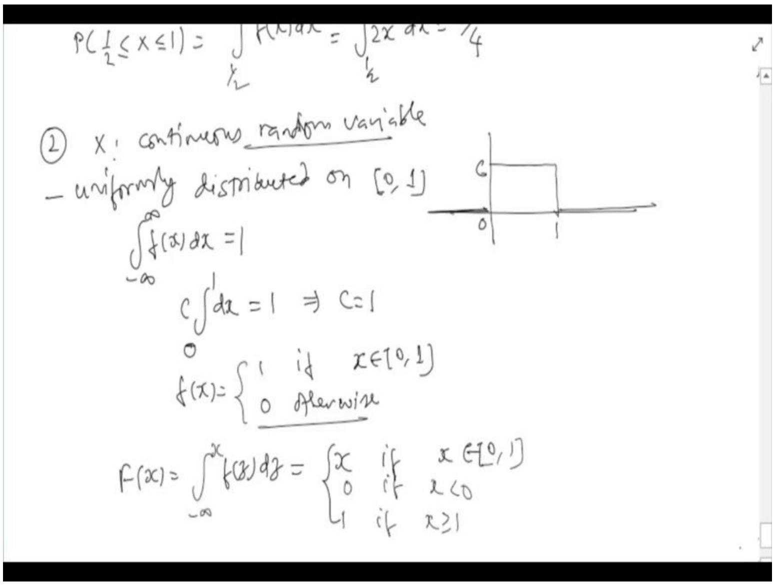
$$P\left(\frac{1}{2} \le X \le 1\right) = F(1) - F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

The same can also be obtained by

$$P\left(\frac{1}{2} \le X \le 1\right) = \int_{\frac{1}{2}}^{1} f(x)dx = \int_{\frac{1}{2}}^{1} 2xdx = \frac{3}{4}.$$

Let us see one more example.

(Refer Slide Time: 43:49)



Let us now consider a random variable X again continuous random variable, whose PDF is a constant between 0 and 1. So, it is a constant between 0 and 1. So, let us say it takes value C between 0 and 1 and it is 0 outside this interval [0, 1]. Such a random variable X is called uniformly distributed on [0, 1]. Recall that the PDF must satisfy this relation

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

So, in this case

$$C\int_{0}^{1} dx = 1 \quad \Rightarrow C = 1.$$

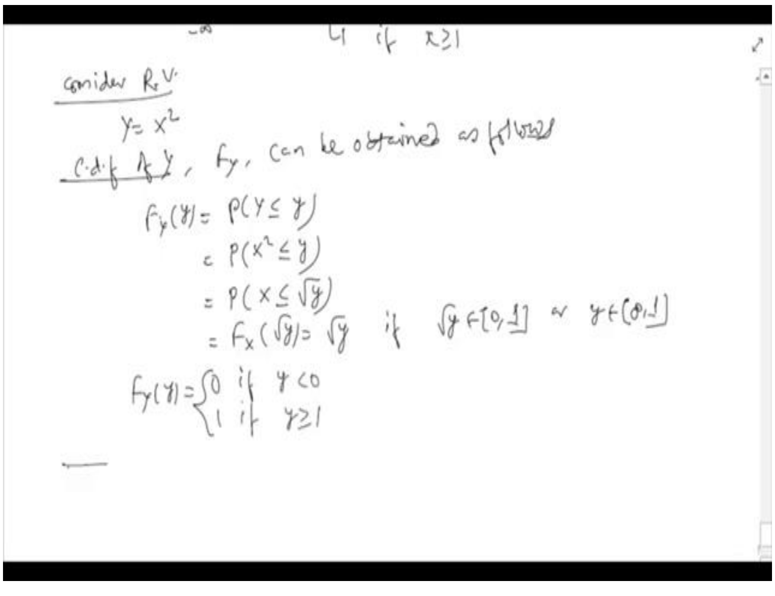
So, the PDF now is

$$f(x) = \begin{cases} 1, & if\ x \in [0,1] \\ 0, & otherwise \end{cases} .$$

In this case, we can obtain the CDF of X from PDF as follows

$$F(x) = \int_{-\infty}^{x} f(z)dz = \begin{cases} x, & if\ x \in [0,1] \\ 0, & if\ x < 0 \\ 1, & if\ x \geq 1 \end{cases} .$$

(Refer Slide Time: 45:31)



Now suppose Y = X^2. Consider another random variable Y = X^2. Then CDF of Y can be obtained from CDF of X as follows. CDF of Y say F_y can be obtained as follows.

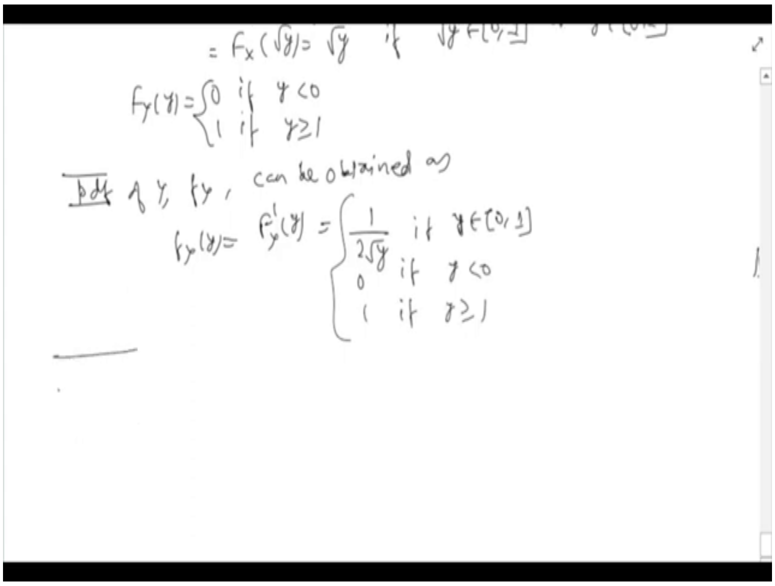
$F_y(y) = P(Y \leq y)$.

So, this is $= P(X^2 \leq y)$, which is $= P\left(X \leq \sqrt{y}\right)$, which is $= F_x\left(\sqrt{y}\right) = \sqrt{y}$ from the graph above, if $\sqrt{y} \in [0,1]$ or $y \in [0,1]$. It can also easily be seen that

$$F_y(y) = \begin{cases} 0 \; if \; y < 0 \\ 1 \; if \; y \geq 1 \end{cases}.$$

From the CDF of y, we can obtain its PDF by differentiate.

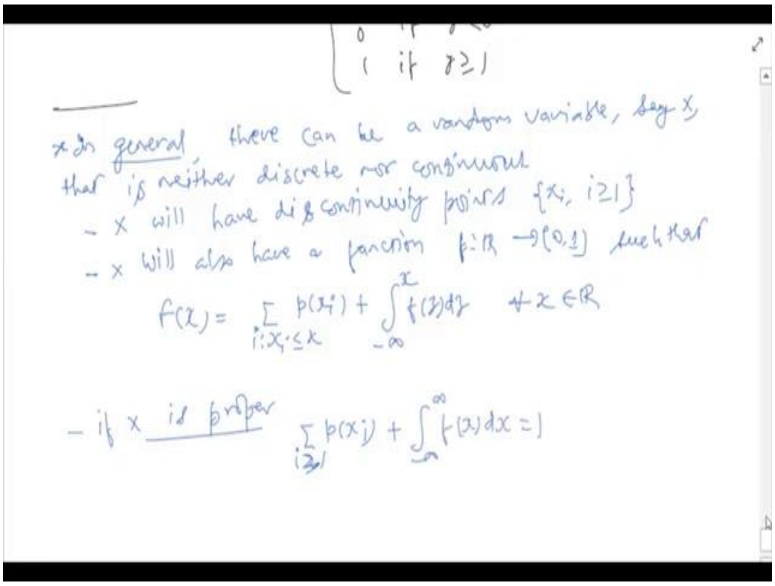
(Refer Slide Time: 47:01)



So, PDF of y is less, let us say f_y, can be obtained as follows.

$$f_y(y) = F'_y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} \; if \; y \in [0,1] \\ 0 \; if \; y < 0 \\ 1 \; if \; y \geq 1 \end{cases}.$$

(Refer Slide Time: 47:43)



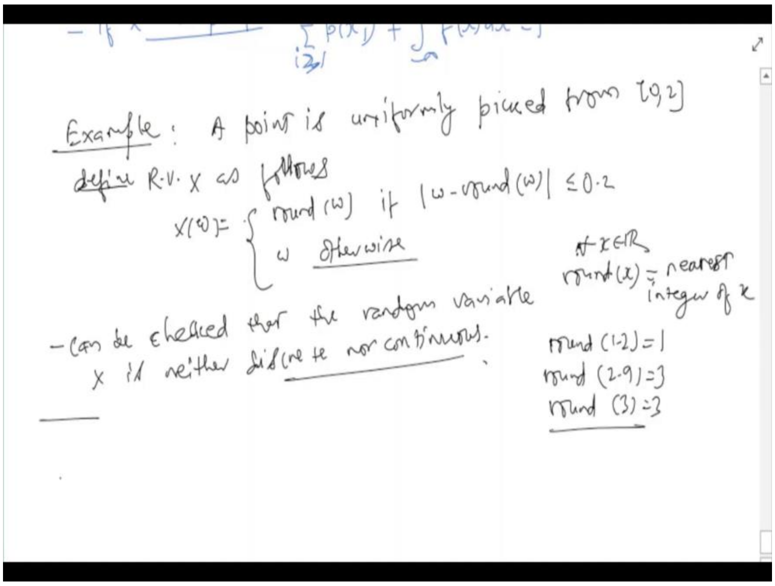
In general, we can have a random variable that is neither continuous nor discrete. In general, there can be a random variable X say X, there can be a random variable say X that is neither discrete nor continuous. In other words, such a random variable will have discontinuity points $\{x_i, i \geq 1\}$ and also a function $f : \mathbb{R} \rightarrow [0,1]$ such that it is a CDF that

$$F(x) = \sum_{i:x_i \leq x} P(x_i) + \int_{-\infty}^{x} f(z)dz \quad \forall x \in \mathbb{R}.$$

In this case, if X is proper, then

$$\sum_{i \geq 1} P(x_i) + \int_{-\infty}^{\infty} f(x)dx = 1.$$

(Refer Slide Time: 49:41)



Let us consider an example. Suppose a point is uniformly picked from interval [0,2], and I define a random variable X as follows, define a random variable X as follows.

$$X(\omega) = \begin{cases} round(\omega) & if\ |\omega - round(\omega)| \leq 0.2 \\ \omega & otherwise \end{cases}.$$

Let me tell what we mean by round of a number.

So, $\forall x \in \mathbb{R}$ round(x), is defined to be the nearest integer, nearest integer of x. For instance, round(1.2) = 1, round(2.9) = 3, round(3) = 3. So, we define a random variable

$$X(\omega) = \begin{cases} round(\omega) & if\ |\omega - round(\omega)| \leq 0.2 \\ \omega & otherwise \end{cases}.$$

Otherwise, it can be checked that X is neither continuous nor discrete, checked that the random variable X is neither discrete nor continuous. We will not continue this discussion as we will only be interested in random variables that are either continuous or discrete. In other words, we will not consider such mixed random variables. This brings us to the end of this lecture. Thank you.