Mathematical Aspects of Biomedical Electronic System Design
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Lecture 19
System of Linear Equations

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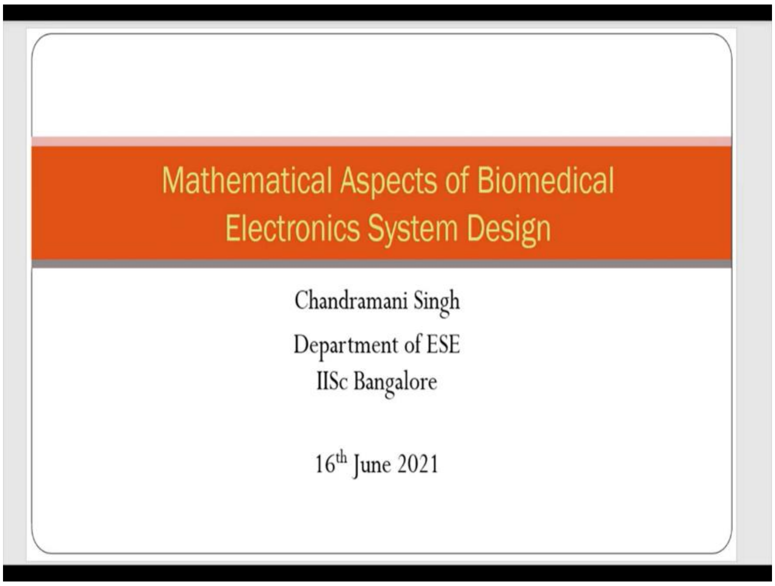

Hello everyone. Welcome to another lecture of the course mathematical aspects of biomedical electronic system design.

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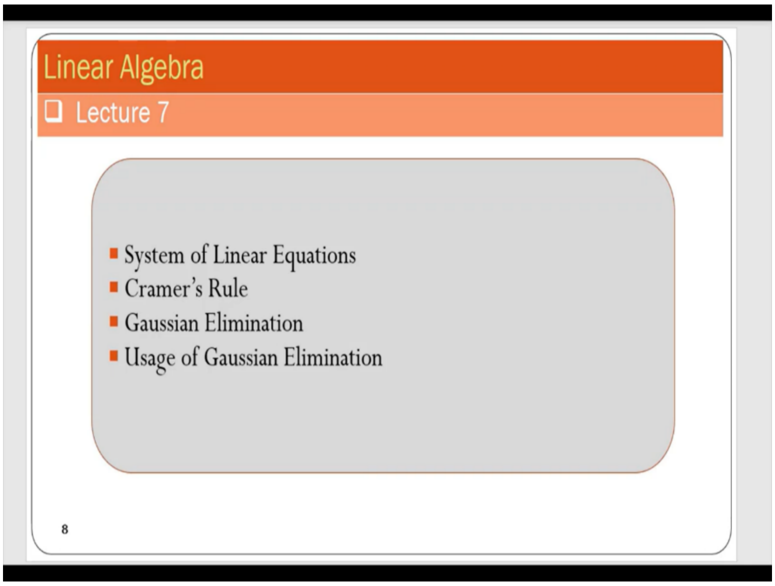

Today's lecture will be the last one on linear algebra. In this lecture, we will look at system of linear equations, Cramer's rule, Gaussian Elimination something called row echelon form of matrices, we will also see various uses of Gaussian Elimination. So, let us begin today's lecture.

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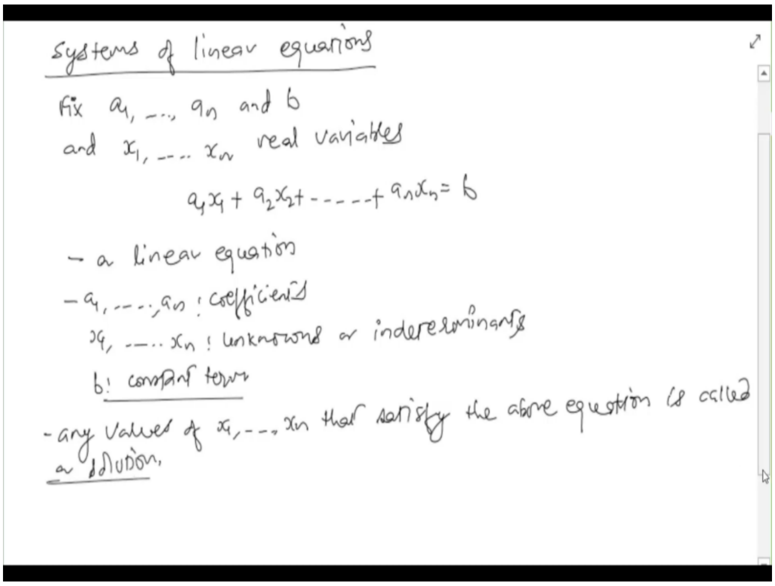

We will start with the system of linear equations. So, to begin with let us fix numbers, a_1 to a_n. These are real numbers and another one b and also consider variables x_1 to x_n, these are real variables and consider the following equation

$$a_1x_1 + a_2x_2 + \cdots \dots \dots a_nx_n = b.$$

This equation is called a linear equation in variables x_1 to x_n.

a_1 to a_n are called coefficients, x_1 to x_n are called unknowns or indeterminants. b is called constant term it is again a real number it is called constant term. Now, any values of x_1 to x_n that satisfy this equation are called solution to this equation. Any values of x_1,....., x_n that satisfy the above equation is called a solution.

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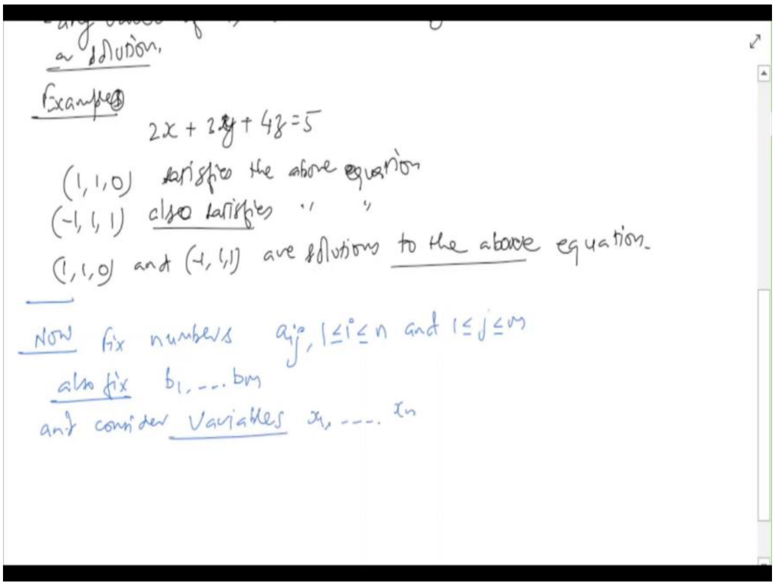

Let us consider an example. Consider following equation $2x + 3y + 4z = 5$. It can be easily seen that (1, 1, 0) that is x = 1, y = 1 and z = 0 satisfies the above equation. Moreover, (-1, 1, 1) also satisfies the above equation. -2+3+4 = 5. So, this also satisfies the above equation. So, as per the statement above (1, 1, 0) and (-1, 1, 1) are solutions to the above equation.

So, we see that an equation can have more than one solution. Now, let us extend this these observations to more than one equations. So, now we will fix numbers. Numbers a_{ij} where i each ranging from 1 to n and the j is ranging from 1 to m, also fix b_1 … bm and consider variables as before n variables x_1 to x_n.

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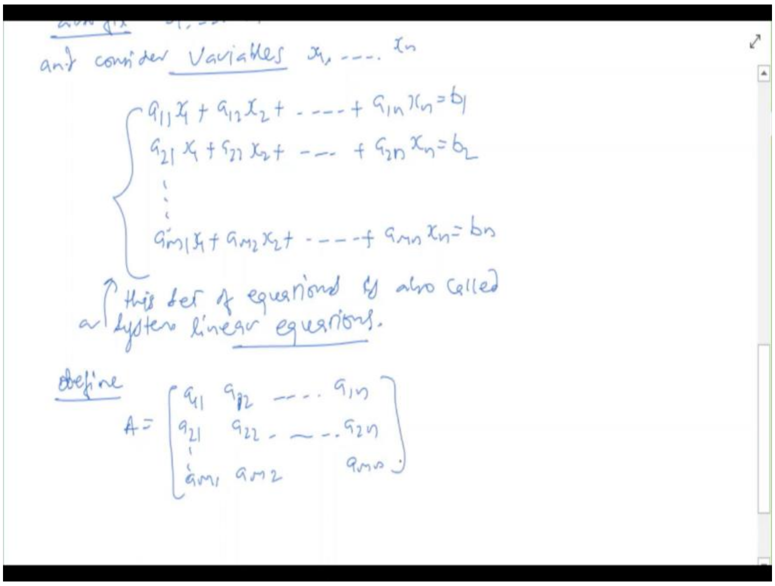

With these numbers and variables let us write the following equations. $a_{11}x_1 + a_{12}x_2 \dots a_{1n}x_n = b_1$. $a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$ and so on. We have m equations like this the last one being $a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_n$. Now, we rather than one equation we have m equations. This system is called the set is called system of linear equations.

This set of equations is called, also called a system of linear equations. Let us now define a matrix A and vectors x and b as follows, define A to be the following matrix,

$$A = \begin{bmatrix} a_{11}\, a_{12} \dots \dots \dots\ a_{1n} \\ a_{21}\, a_{22} \dots \dots \dots\ a_{2n} \\ a_{m1}\, a_{m2} \dots \dots \dots\ a_{mn} \end{bmatrix}.$$

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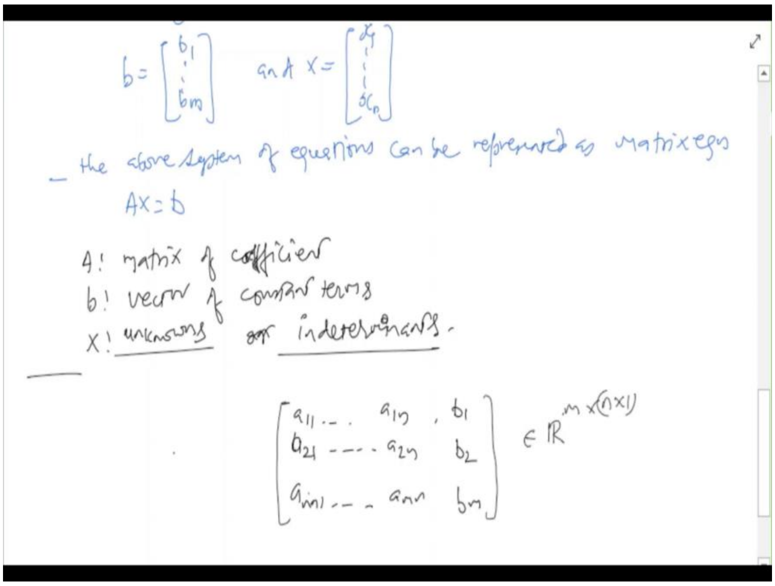

And vectors b to be

$$b = \begin{bmatrix} b_1 \\ . \\ . \\ . \\ . \\ b_m \end{bmatrix} \quad and\ \ x = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}.$$

Then the above system of equations can be compactly represented as the matrix equation Ax = b. The above system of equations can be compactly represented as matrix equation Ax = b. Following the terminology for single linear equation, here A is called matrix of coefficients, b is called vector of constant terms and x is called vector of unknowns or indeterminants.

When analyzing the above system of linear equations in the following, we will come across another matrix which is obtained as follows. Here we write a which is

$$\begin{bmatrix} a_{11}\, a_{12} \dots \dots \dots\ a_{1n}\ .\ b_1 \\ a_{21}\, a_{22} \dots \dots \dots\ a_{2n}\ .\ b_2 \\ a_{m1}\, a_{m2} \dots \dots \dots\ a_{mn}\ .\ b_m \end{bmatrix} \in \mathbb{R}^{m\times(n+1)}.$$

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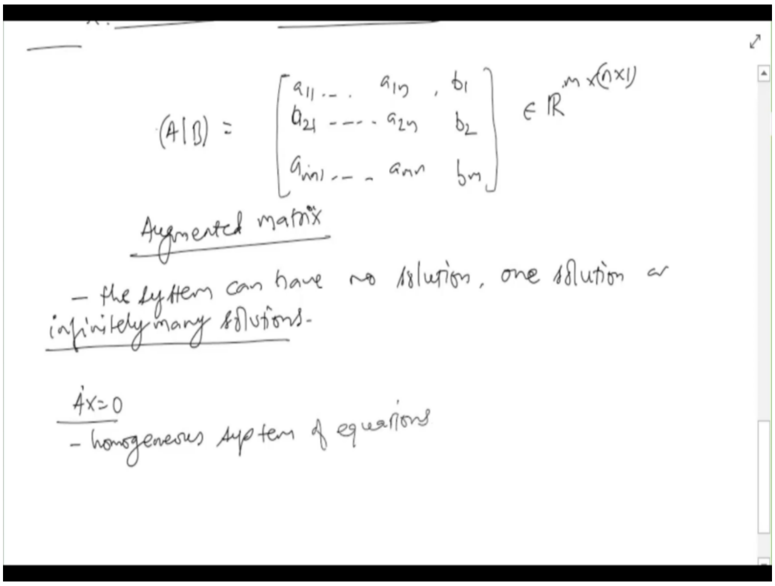

It is compactly represented as follows and it is referred to as augmented matrix. In case of a system of linear equations the system can have no solution, one solution or infinitely many solutions. Let me elaborate. The system of equations Ax = b will always have a solution if b = 0, in that case the system of equations is called a homogeneous system of equations.

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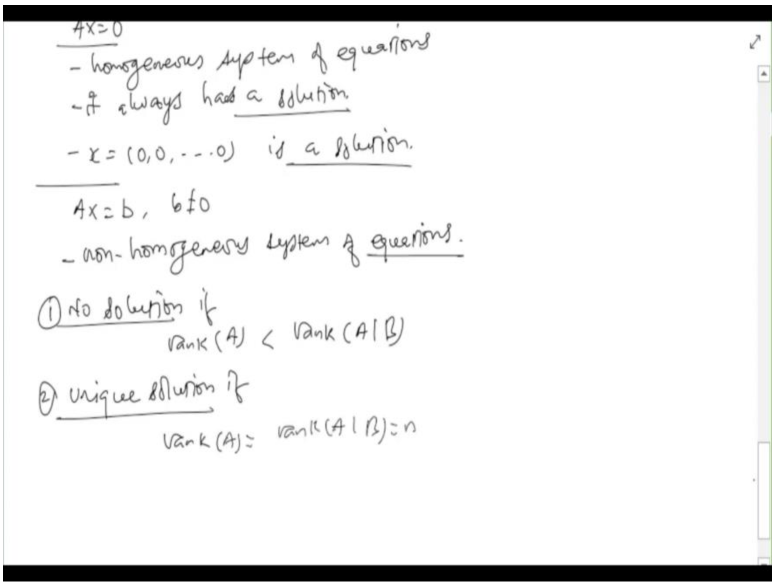

It always has a solution, in particular x = 0 vector itself is a solution. If b ≠ 0, then the system of equations is called non-homogeneous set of equations. In this case, there will be no solution, if the rank(A) < rank of augmented matrix AB. There will be unique solution, that is, one solution if the rank(A) = rank(AB) = n.

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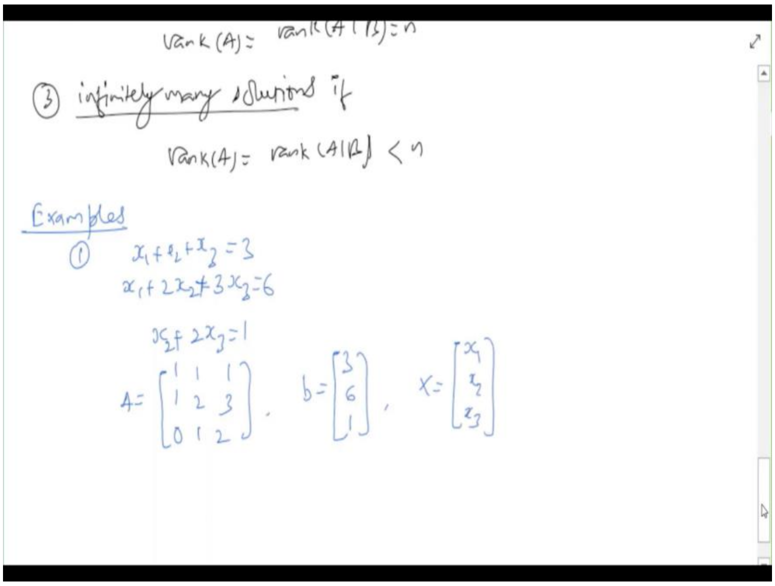

And there will be infinitely many solutions if a rank(A) = rank(AB) < n. Let us see examples. First example is the set of equations $x_1 + x_2 + x_3 = 3$. $x_1 + 2x_2 + 3x_3 = 6$. And $x_2 + 2x_3 = 1$. In this case, the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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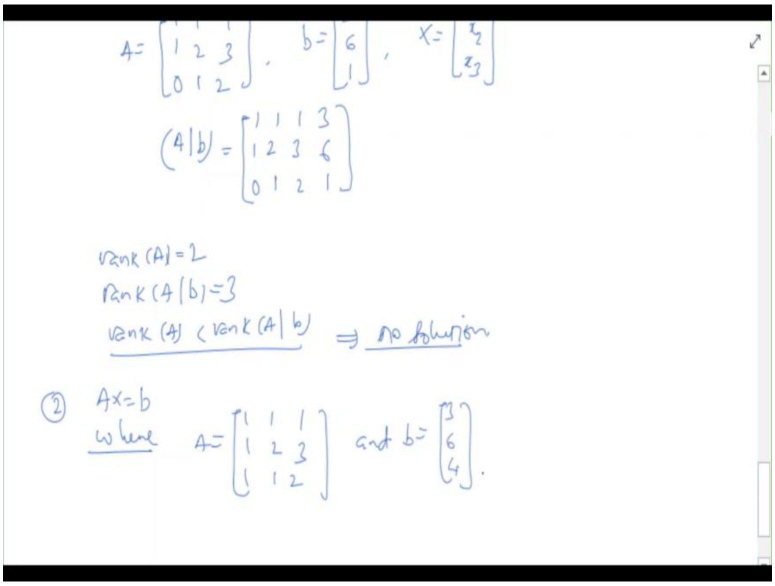

The augmented matrix

$$(A|B) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Notice that in A, all three rows are not linearly independent. In fact, if I add the first and third rows, I get the second row. So, rank(A) = 2, on the other hand, it can be checked that rank(A|B) = 3. So, rank(A) < rank(A|B) and so, the system has no solution. Let us see another example. Now we have system of equations Ax = b, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}.$$

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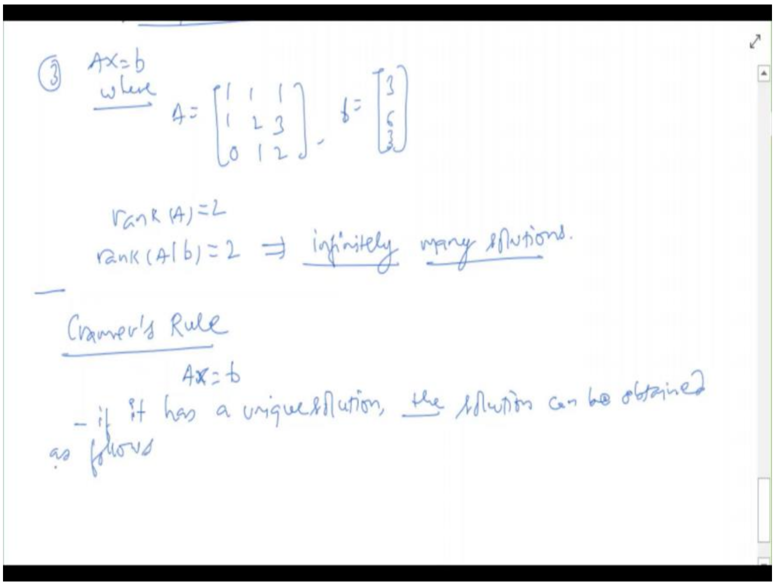

In this case, it can be check that rank(A) and the rank(A|B) = 3. So, the system has a unique solution. Finally, let us consider Ax = b, where A is same as in the first example, that is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}.$$

Rank(A) = 2 and now it can be seen that rank(A|B) = 2. So, A and the augmented matrix have same rank but both is less than the number of variables 3. So, in this case we will have infinitely many solutions.

Now, the question is given a system of linear equations, we want to see if the system will have no solution, unique solution or infinitely many solutions and if it has a solution, we want to find a solution. We will first see an algorithm that works in the case of unique solution that is, it gives the unique solution if the system of linear equations has unique solution and this method is called Cramer's rule. Let us see what Cramer's rule is. Cramer’s rule says that for a system of equations Ax = b, it is if it has a unique solution, the solution can be obtained as follows.

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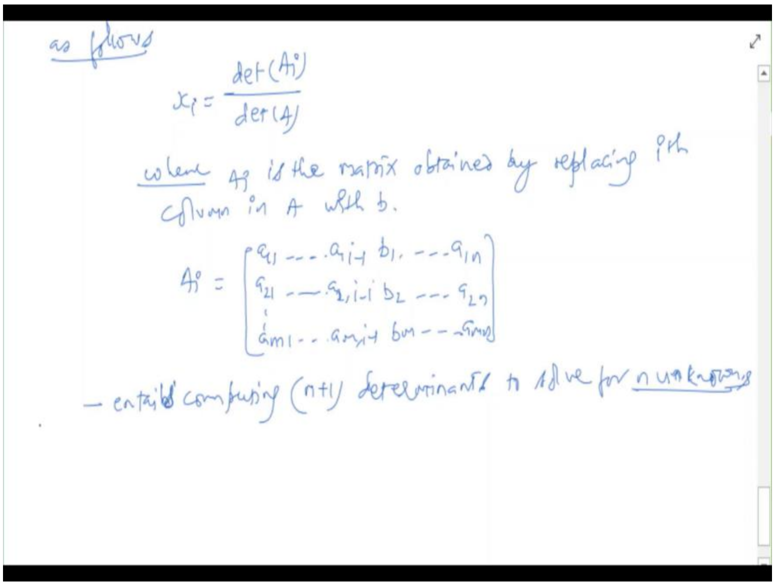

It says that ith unknown x_i can be obtained by

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix that is obtained by replacing the i^{th} column of A with b. That is A_i is the following matrix

$$A_i = \begin{bmatrix} a_{11} \dots \dots\ a_{1i-1} b_1 \dots \dots\ a_{1n} \\ a_{21} \dots \dots\ a_{2i-1} b_2 \dots \dots\ a_{2n} \\ . \\ . \\ . \\ a_{m1} \dots \dots\ a_{mi-1} b_m \dots \dots\ a_{mn} \end{bmatrix}$$

So, the Cramer’s involves computing n + 1 determinants to obtain n unknowns, to solve for n unknowns. As I stated Cramer’s rule works only if the system of equations has a solution and has a unique solution. Next, we will see about a method that tells us whether a system of equations has no solution, has unique solution or had infinitely many solutions. And in the latter cases, it also provides us all the solutions to the equations.

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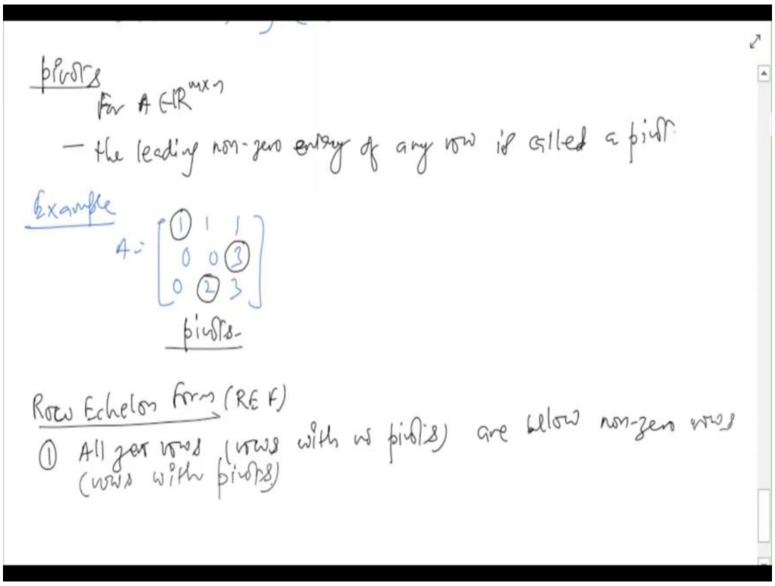

We will now see the notion of matrices in row echelon form or simply the row echelon matrices. However, I will start with defining what it means by pivot entry of the matrix. So, for a matrix say m × n matrix, the leading non-zero entry of any row is called pivot. The leading I mean the first non-zero entry of any row is called a pivot.

For instance, if I consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

then this 1 is the first non-zero entry in the first row, this is first non-zero entry in the second row and this is first non-zero entry in the third row. These are the pivots. Next, a matrix is called a row echelon form, in short, REF, or it is simply called RE matrix, row echelon matrix if the following two conditions are satisfied. First condition is all zero rows, that is, rows with no pivots are below non-zero rows.

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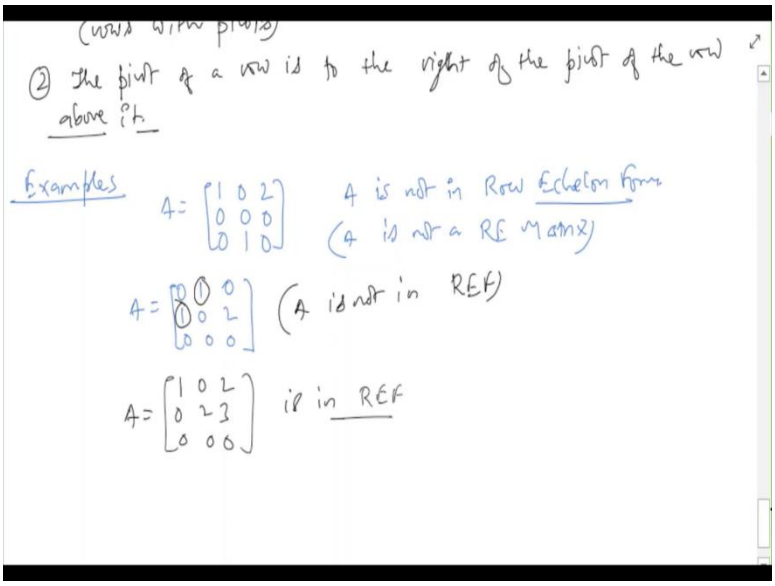

And the second condition is the pivot of a row is to the right of the pivot of the row above it. Let us see the examples. Consider

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Here we see that the zero row, second row is zero row whereas third row is not. So, first condition is violated. So, A is not in the row echelon form or we can simply say that A is not a row echelon matrix.

Similarly, if I consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

again I see that zero rows are below all non-zero rows, but the pivot entry of the second row is not on the right of the pivot entry of the row above it. So, this A is also not in row echelon form. On the other hand, if you look at the matrix A which is

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is in row echelon form.

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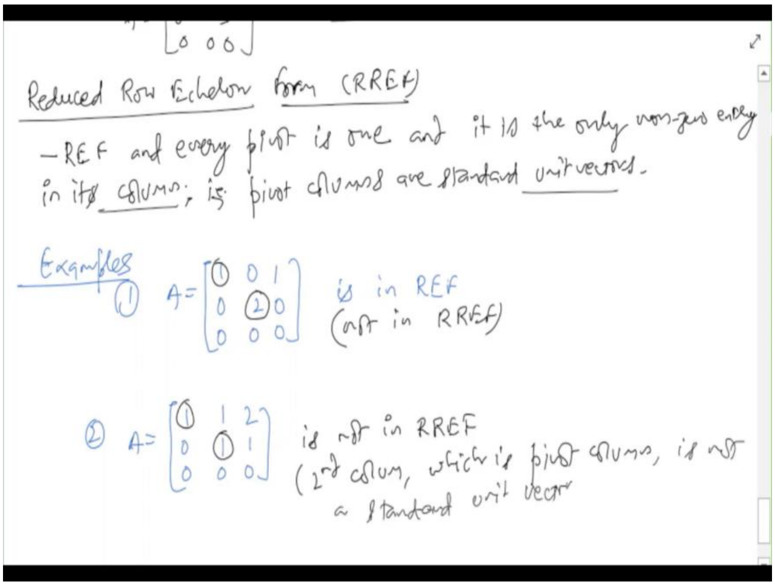

Next, we will see the motion of reduced row echelon matrices which are special cases of row echelon matrices. Reduced row echelon form in short RREF. The matrix A is called to be an RREF if it is in row echelon form and also satisfies the extra conditions that every pivot is 1 and it is the only non-zero entry in its column. In other words, the pivot columns are standard unit vectors. The columns of the pivot are called pivot columns, for instance, in this matrix A all three columns are pivot columns. Let us consider again a few examples.

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this matrix is in row echelon form but it is not in row echelon form because the pivot entry in the second row is not 1 not in reduced row echelon form. Let us see another example. Now

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we see that the pivot entries are all 1 but from the second column which is a pivot column is not the standard unit vector.

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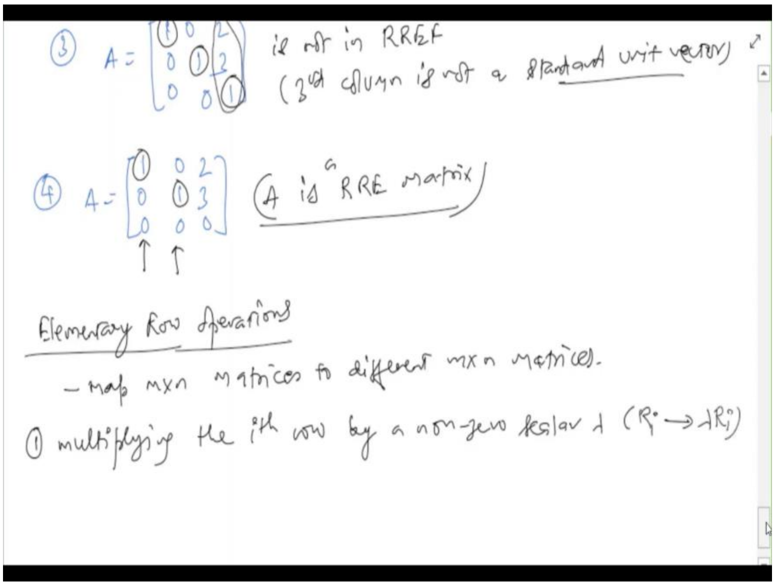

It is not in reduced row echelon form, second column which is a pivot column is not a standard unit vector. Let us see one more example. Now,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again, we see that the pivot entries are all 1 but this is also not in reduced row echelon form because the last column which is a pivot column has other non-zero entries also, other than pivot it has other non-zero entries also. It is not in RREF because third column is not a standard unit vector. Finally, let us see one more example.

Where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, we see that all the conditions are satisfied. Namely, the zero row is below the two non-zero rows. The pivot entries are all 1s, pivot entry in a row is on the right of the pivot entry in the row above and all the pivot columns there are two pivot columns here and both are standard unit vectors, that is, in both these columns other than pivots all other entries are zero.

So, A is RRE matrix or A is in reduced row echelon form. Now, we introduce elementary row operations. Elementary row operations are functions that map $m \times n$ matrices to different $m \times m$ matrices. There are three elementary row operations which are as follows. The first one is multiplying the i^{th} row by non-zero scalar λ. This operation is denoted as R_i to λR_i.

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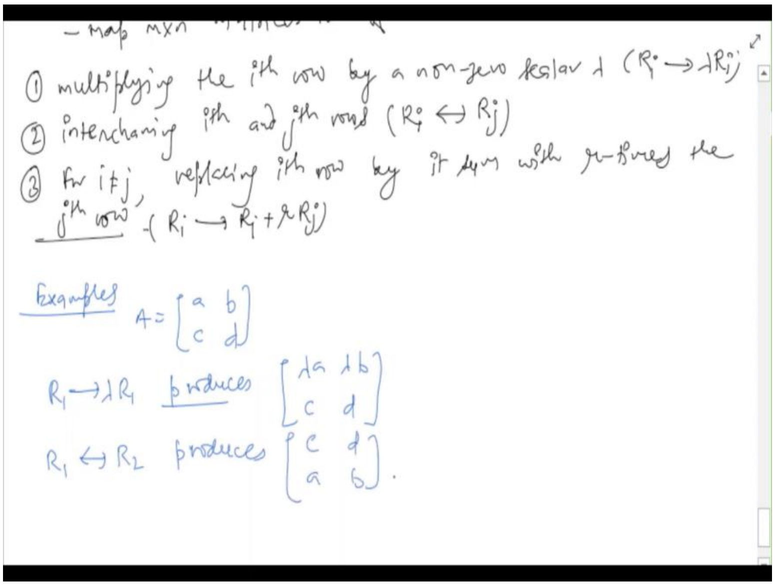

The second elementary row operation is interchanging i[th] row and j[th] rows. This one is denoted as this and the third elementary row operation is for i ≠ j, replace i[th] row by its sum with μ times the j[th] row. This one is denoted as R_i to $R_i + \mu R_j$. Let me illustrate these operations via simple examples.

So, let us consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in this case the elementary row operation R_1 to λR_1 produces $\begin{bmatrix} \lambda a & \lambda b \\ c & d \end{bmatrix}$. Similarly, the interchange operation produces $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

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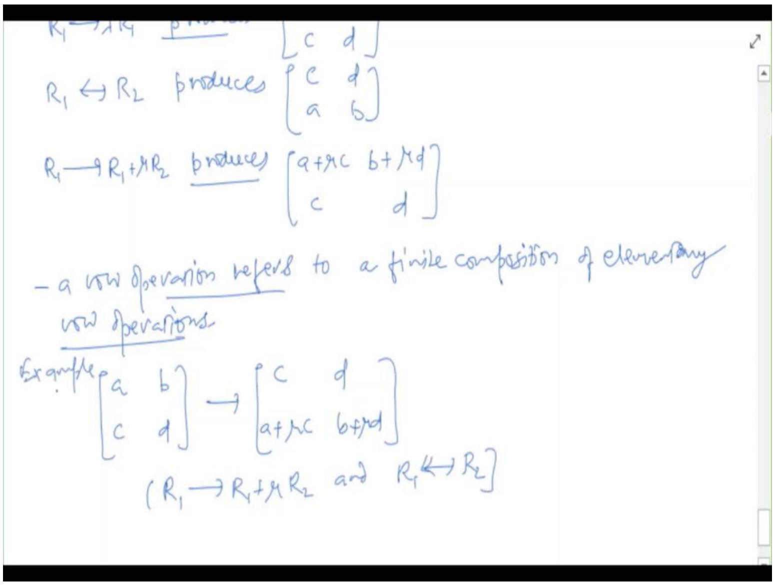

And the operation R_1 to $R_1 + \mu R_2$ produces

$$\begin{bmatrix} a + \mu c & b + \mu d \\ c & d \end{bmatrix}.$$

Any finite composition of elementary row operations is called a row operation. A row operation refers to finite composition of elementary row operations.

For example, an operation that produces that map's matrix to

$$\begin{bmatrix} c & d \\ a + \mu c & b + \mu d \end{bmatrix}$$

is row operation because its conversion is obtained by iteratively applying two elementary row operations namely R_1 to $R_1 + \mu R_2$ and then exchange of R_1 and R_2. This is an example of a row operation. We will see that any matrix can be mapped to a reduced row echelon matrix using a row operation.

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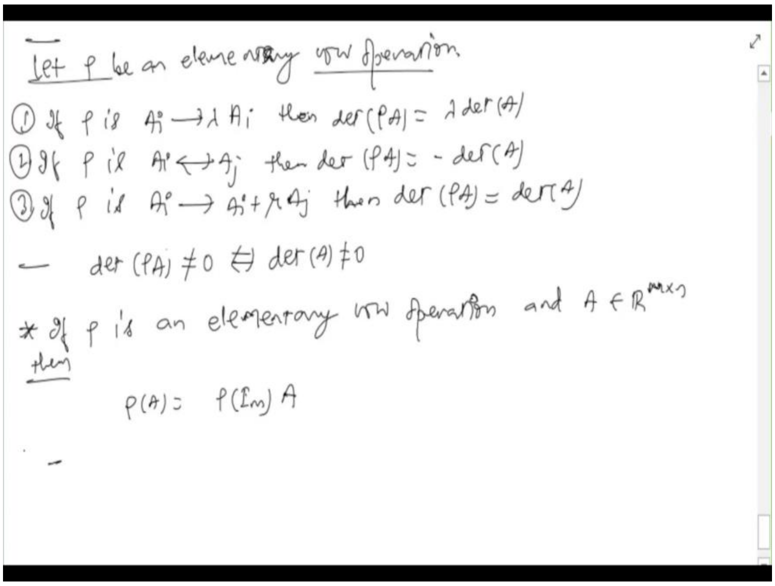

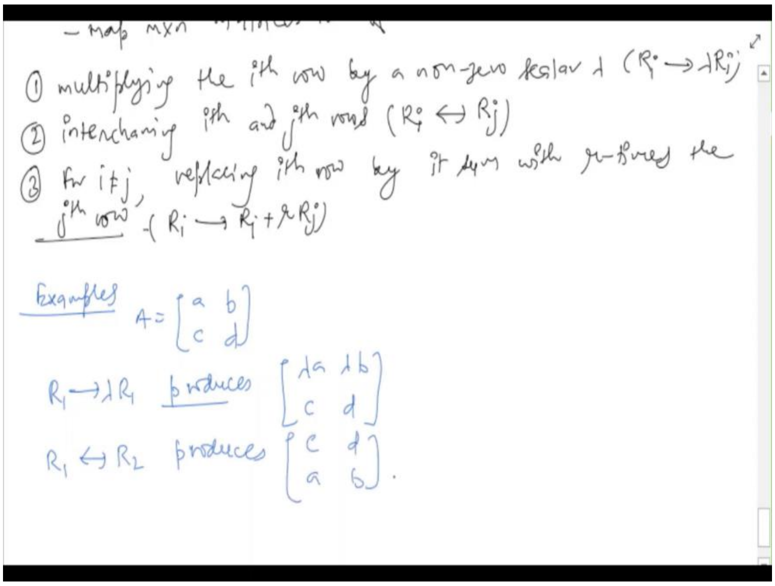

Let us see how these elementary row operations affect determinants of the mattress. So, let ρ be an elementary row operation. So, ρ could be one of these three operations. If ρ is A_i going to λA_i, then $\det(\rho A) = \lambda \det(A)$. If ρ is interchange of two rows, then $\det(\rho A) = -\det(A)$.

On the other hand, if ρ is A_i going to $A_i + \mu A_j$ then $\det(\rho A) = \det(A)$. So, from these we can infer that $\det(\rho A) \neq 0$ if and only if $\det(A) \neq 0$.

We further see that if ρ is an elementary row operation and A is an $m \times n$ matrix then

$$\rho(A) = \rho(I_m)A.$$

So, in this way we see that if we know the effect of ρ on the identity matrix, then we also know its effect on the matrix A.

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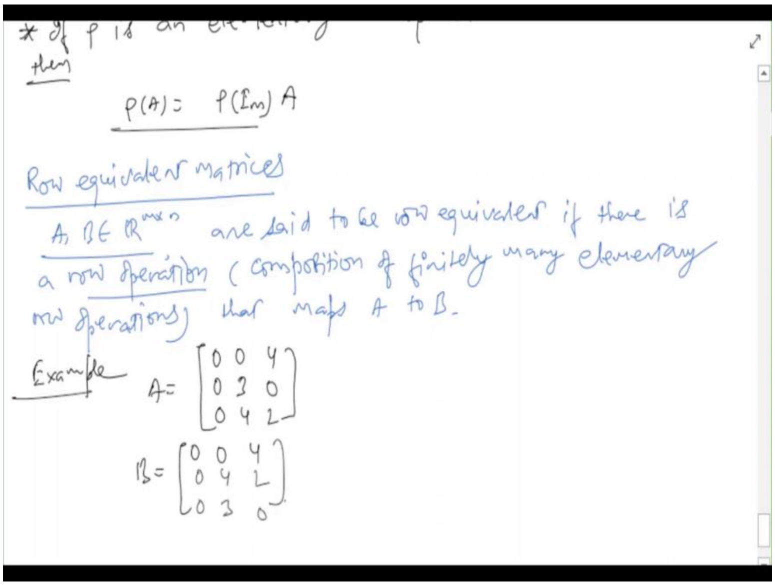

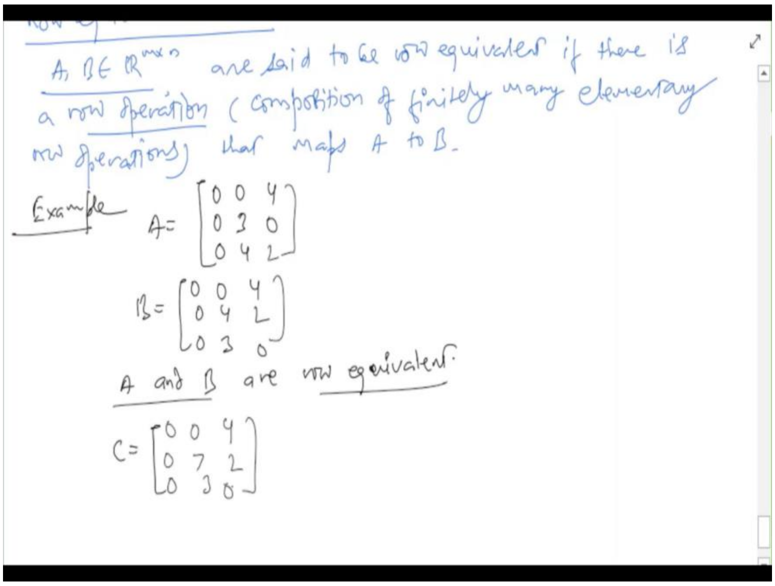

Next, we will see the notion of a row equivalent matrices. So, $A, B \in \mathbb{R}^{m\times n}$ are matrices of say order are said to be row equivalent if there is row operation that is a composition of finitely many elementary row operations that maps A to B. Let us see an example. Let us consider

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 4 & 2 \end{bmatrix} \; and \; B = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 2 \\ 0 & 3 & 0 \end{bmatrix}$$

then clearly A and B are row equivalent as B is obtained from A by interchanging second and third rows. A and B equivalent.

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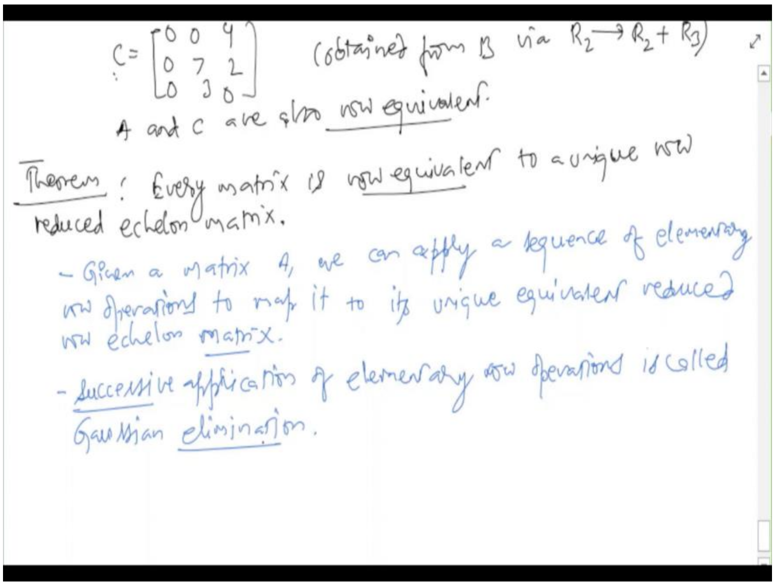

Similarly, if I consider

$$C = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 7 & 2 \\ 0 & 3 & 0 \end{bmatrix}$$

notice that C is obtained by obtained from B via going $R_2 + R_3$. So, C has also been obtained from A via a composition of elementary row operations. So, A and C are also row equivalent. Next, we see a very important measure which will prove to be quite useful in the remaining of the lecture that says that and I will state it as a theorem.

Every matrix is row equivalent to a unique row reduced echelon matrix. In fact, given a matrix A we can apply a sequence of elementary row operations to map it or to convert it to its unique equivalent reduced row echelon matrix. This successive application of elementary row operations is what is referred to as Gaussian Elimination. Successive application of elementary row operations is called Gaussian elimination.

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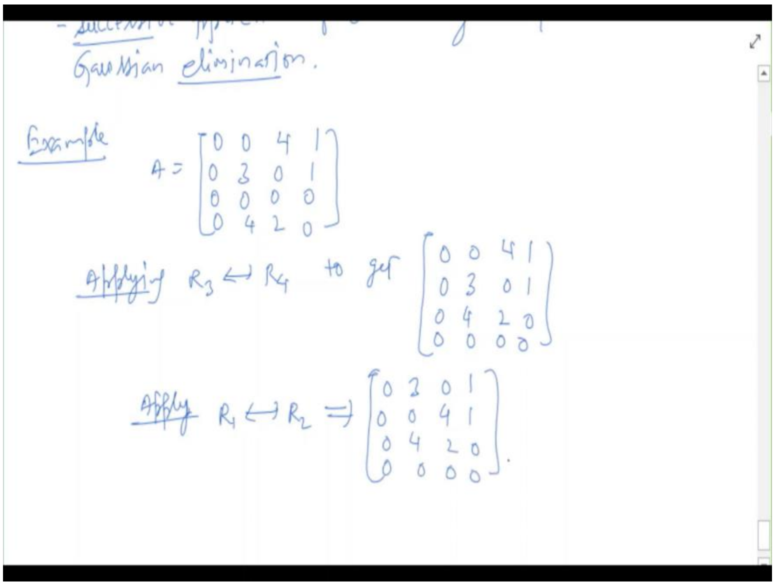

We now illustrate Gaussian elimination via an example. Consider

$$A = \begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{bmatrix}.$$

Now, we will do a sequence of elementary row operations to convert it to a row reduced row echelon matrix. Apply interchange of R_3 and R_4 and we will get

$$\begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next, apply interchange of R_1 and R_2 and this way we will get

$$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the pivot element in the first row is 3. So, we apply R_1 to $\frac{1}{3}R_1$ to get

$$\begin{bmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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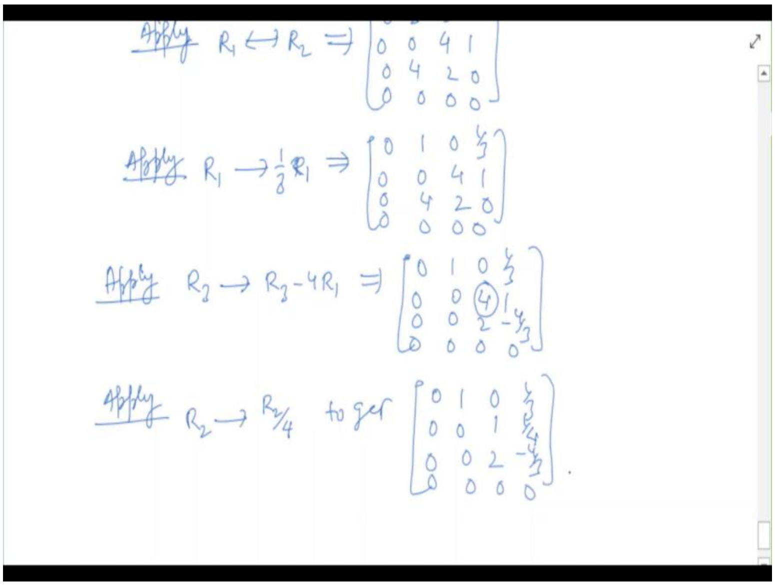

Next, we see that the second column which is a pivot column, it is not a standard unit vector. To fix this, we apply R_3 to R_3 - $4R_1$ and we get

$$\begin{bmatrix} 0\ 1\ 0\ \frac{1}{3} \\ 0\ 0\ 4\ 1 \\ 0\ 0\ 2 - \frac{4}{3} \\ 0\ 0\ 0\ 0 \end{bmatrix}.$$

Now we will focus on second row and we see that the pivot element here is 4. So, we apply R_2 to $\frac{R_2}{4}$ to get

$$\begin{bmatrix} 0\ 1\ 0\ \frac{1}{3} \\ 0\ 0\ 1\ \frac{1}{4} \\ 0\ 0\ 2 - \frac{4}{3} \\ 0\ 0\ 0\ 0 \end{bmatrix}.$$

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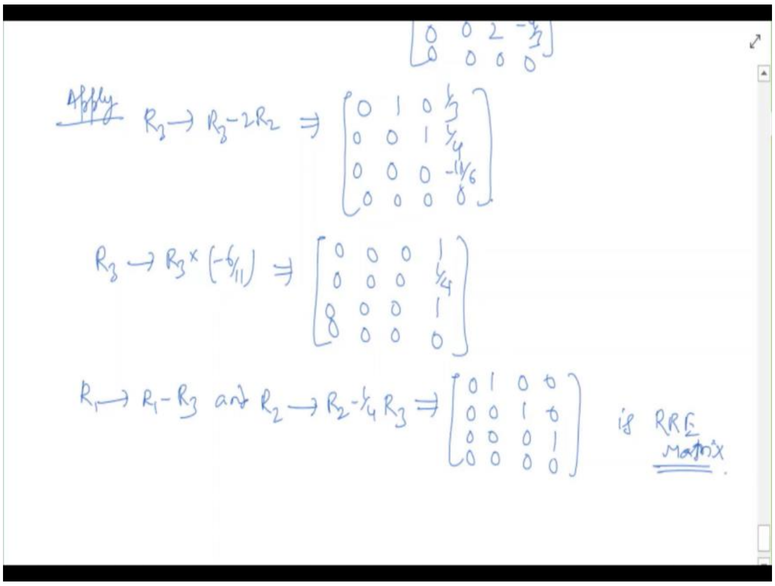

Now, we see that again the third column which is a pivot column is not a standard unit vector to fix this we apply R_3 - $2R_2$ to get

$$\begin{bmatrix} 0\ 1\ 0\ \frac{1}{3} \\ 0\ 0\ 1\ \frac{1}{4} \\ 0\ 0\ 0 - \frac{11}{6} \\ 0\ 0\ 0\ 0 \end{bmatrix}.$$

Now, we will look at the third row and notice that the pivot entry is not 1. To fix that we apply R_3 to $-\frac{6}{11}$ and this way we get

$$\begin{bmatrix} 0\ 0\ 0\ \frac{1}{3} \\ 0\ 0\ 1\ \frac{1}{4} \\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix}.$$

Finally, we see that the fourth column which is also now a pivot column is not a standard unit vector. To fix this we apply R_1 to $R_1 - \frac{1}{3} R_3$ and R_2 to $R_2 - \frac{1}{4} R_3$ to get

$$\begin{bmatrix} 0\ 1\ 0\ \ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix}$$

and this is the reduced row echelon matrix. So, we saw how we could apply a sequence of elementary row operations to convert A to its equivalent reduced row echelon matrix.

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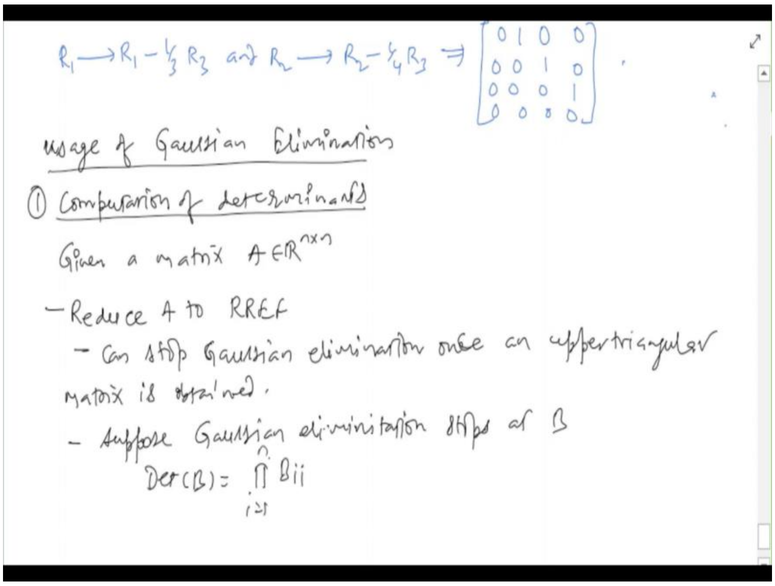

We will now see several uses of Gaussian elimination or uses of reduction to RRE form. The first application is compute computation of determinants. We have seen how elementary row operations affect the determinant of matrices, we can now adopt the following procedure to compute determinant of any matrix. Given a matrix A say of order $n \times n$, we can reduce A to reduce row echelon form.

In fact, we can stop the Gaussian elimination procedure as soon as we will get a upper triangular matrix. Stop Gaussian elimination once an upper triangular matrix is obtained. So, we need not go all the way to reduced row echelon matrix. Now, we can easily write the determinant of the upper triangular matrix swapped in. So suppose Gaussian elimination stops at B, then we know the determinant of B is just the product of B_{ii}, i = 1 to n.

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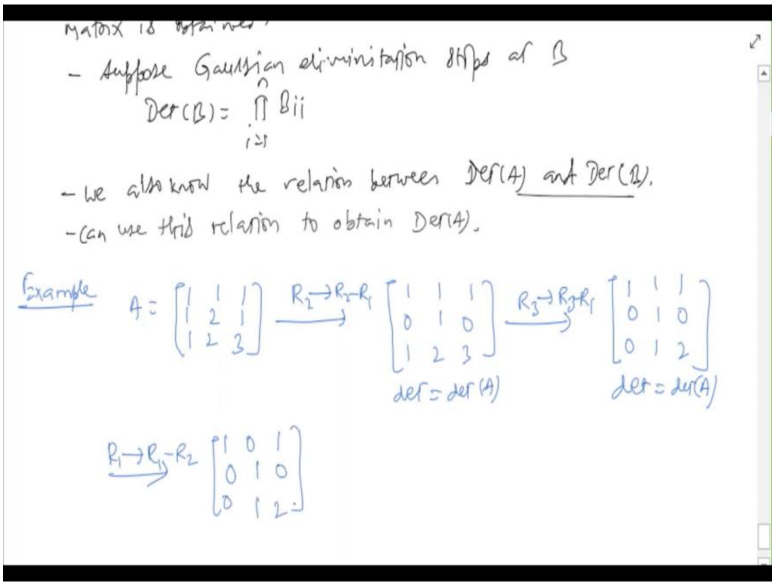

Now, we also know how the determinant of A is related to determinant of B. We also know the relation between determinant of A and determinant of B. We can use this relation to recover determinant of A from determinant of B. So, we can use this relation to obtain determinant of A. Let us see an example. Consider matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Let us start Gaussian elimination in an attempt to convert A to row reduced echelon matrix.

So, we will first do R_2 to R_2 - R_1 to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

and we know that determinant of this new matrix is same as det(A). Next, we will do R_3 to R_3 - R_1 to obtain

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and the determinant of this new matrix also remains unchanged. So, it is same as det(A). Then we will do R_1 to R_1 - R_2 and this way we will get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

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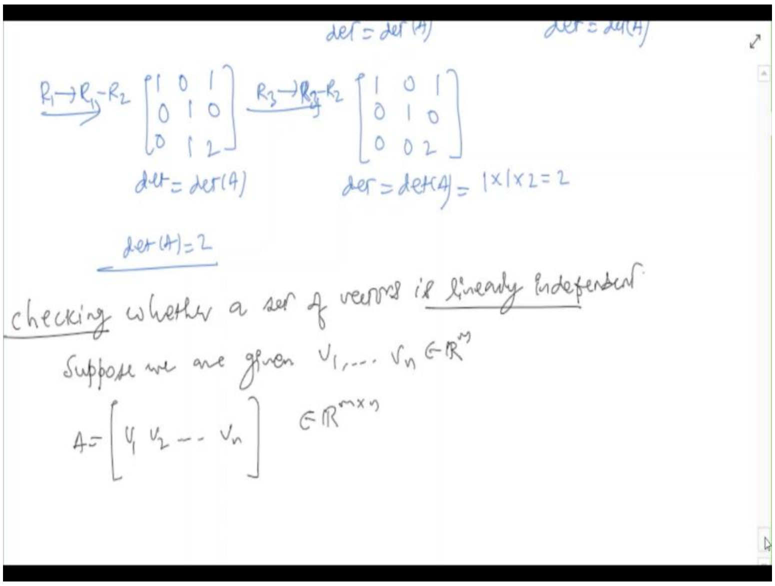

Again, this operation does not alter the determinant. So, determinant remain same as determinant of A. Next, we do R_3 to $R_3 - R_2$ to get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

this operation also does not alter the determinant. So, determinant remain same as determinant of A and now we have an upper triangular matrix. So, we can stop the Gaussian elimination process. And can also readily compute the determinant of this terminal matrix as $1 \times 1 \times 2 = 2$. So, this we will as det(A) = 2.

The next application is checking whether a set of vectors is linearly independent. Checking whether a set of vectors is linearly independent. So, suppose we are given n vectors, v_1 to v_n. $v_1, \dots \dots, v_n \in \mathbb{R}^m$ and we have to determine these vectors are linearly independent or not. What we do is we write a matrix $A = [v_1, \dots \dots, v_n] \in \mathbb{R}^{m \times n}$.

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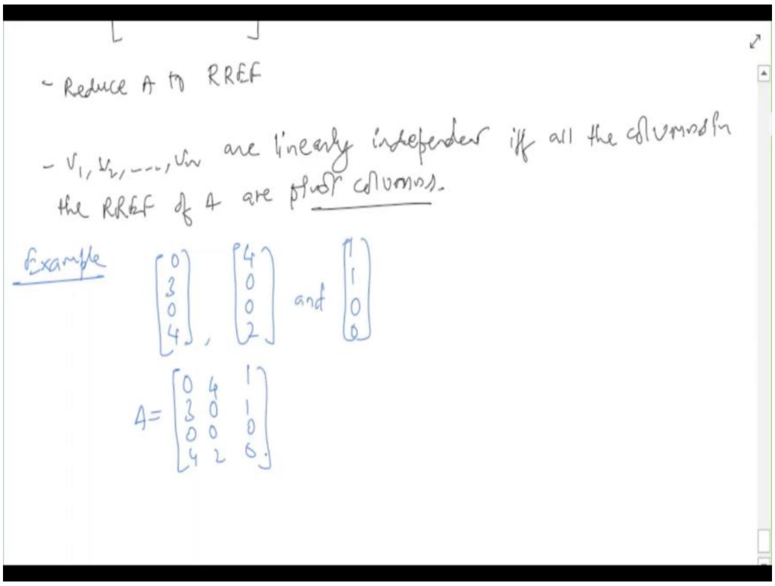

Now, we reduced A to reduced row echelon form. It turns out that v_1 to v_n are linearly independent, if and only if all the columns in the reduced row echelon form of A are pivot columns. All the columns in RREF of A are pivot columns. Let us consider an example. Suppose, we are given three vectors,

$$\begin{bmatrix}0\\3\\0\\4\end{bmatrix}, \begin{bmatrix}4\\0\\0\\2\end{bmatrix} \ and \ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}.$$

We will form a matrix A with these three vectors as columns,

$$A = \begin{bmatrix}0 & 4 & 1\\3 & 0 & 1\\0 & 0 & 0\\4 & 2 & 0\end{bmatrix}.$$

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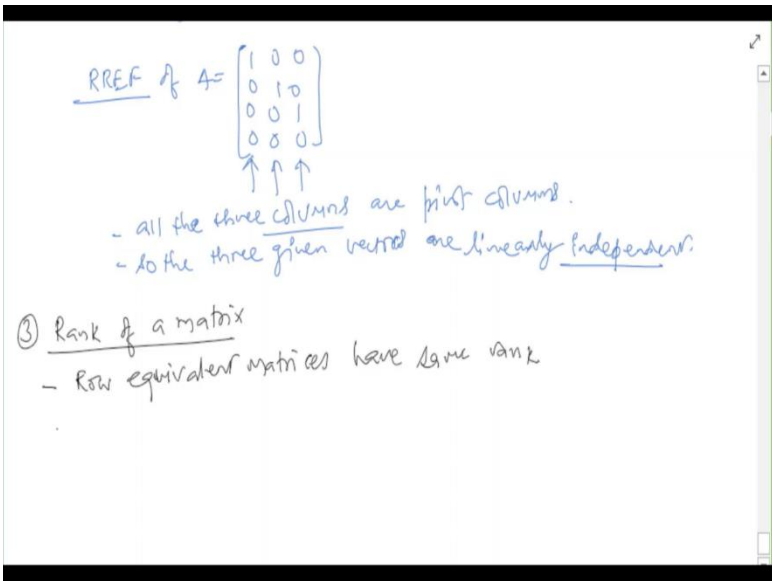

Observe that this A is same as the second third and fourth columns of matrix A that we took an example a while ago. So, reduced row echelon form of A as we saw then will be

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We clearly see that all the three columns of A are pivot columns dependent. All the three columns are pivot columns. So, the three vectors are linearly independent. So, the three given vectors are linearly independent.

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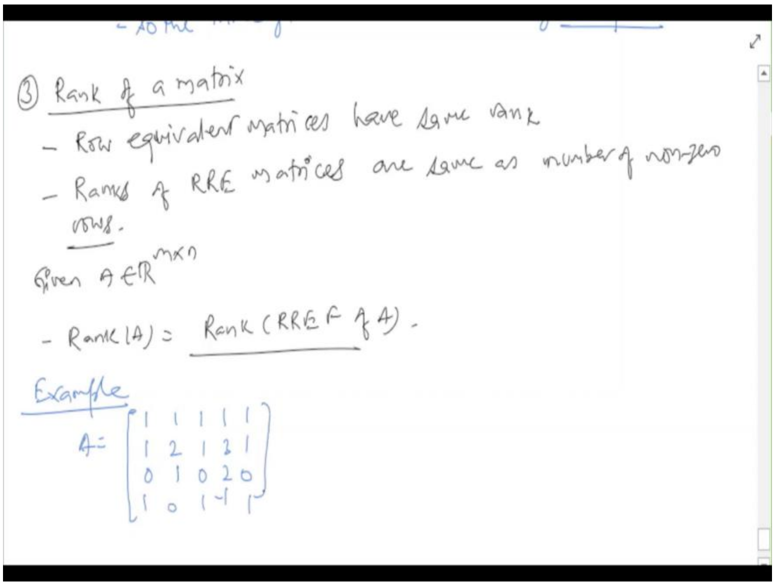

The third application that we will consider determine the rank of a matrix. It turns out that if two matrices are row equivalent, then their ranks are same. Row equivalent matrices have same rank. Also, the ranks of reduced row echelon matrices are same as number of non-zero rows.

Finally, if we are given an m × n matrix, given A that is an m × n matrix, the rank of A is same as rank of reduced row echelon form of A. This tells that we can use Gaussian elimination to reduce A to a reduced row echelon form and thereby to get the rank of A. Let us consider an example. Suppose A is a 4 × 5 matrix which is as follows,

$$A = \begin{bmatrix} 1\,1\,1\,1\,1 \\ 1\,2\,1\,3\,1 \\ 0\,1\,0\,2\,0 \\ 1\,0\,1\,-1\,1 \end{bmatrix}.$$

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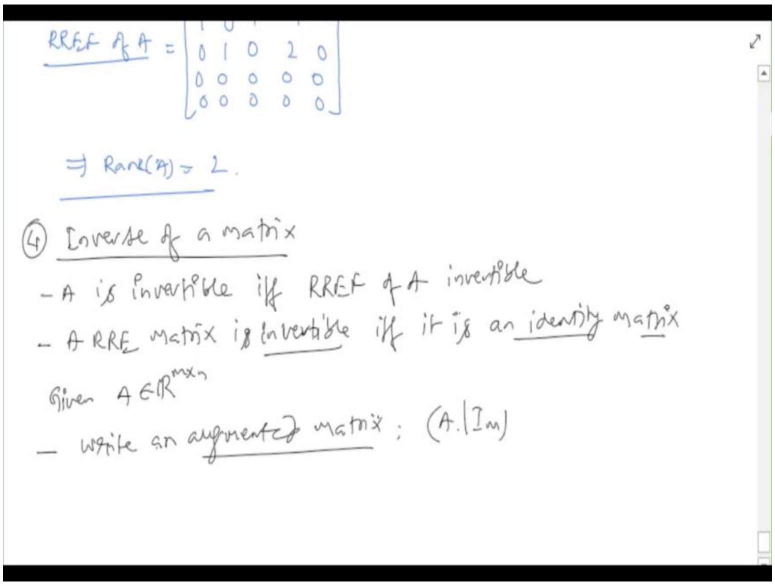

It can be seen that the reduced row echelon form of A happens to be

$$= \begin{bmatrix} 1\,0\,1\,-1\,1 \\ 0\,1\,0\,2\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}.$$

So, we see that reduced row echelon form of A has to non-zero rows, so its rank is 2. So, rank of A is also 2. The next application that we see is computing the inverse of a matrix.

We have already seen that determinant of a matrix is non-zero if and only if determinant of its row reduced row echelon form is non-zero. This tells that A is invertible if and only if reduced row echelon form of A is invertible.

A is invertible if and only if reduced row echelon form of A is invertible. A reduced row echelon matrix is invertible if and only if it is an identity matrix. Moreover, we can adopt the following procedure to determine if a matrix A say m × n matrix is invertible or not and if it is invertible what is its inverse. So, given A which is m × n, here is the procedure.

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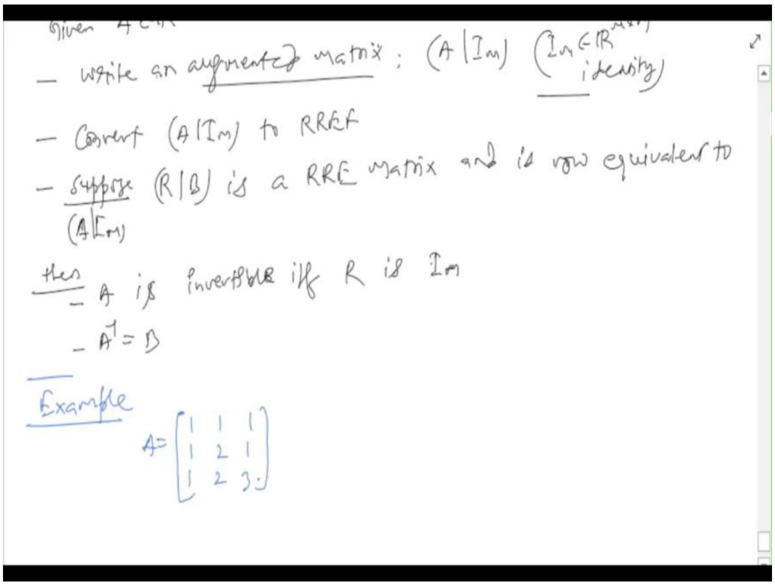

We first write an augmented matrix, matrix which is $(A|I_m)$. I_m is m × n identity matrix. Then we convert $(A|I_m)$ to reduced row echelon form. Suppose, (R|B) is a reduced row echelon matrix and is row equivalent to $(A|I_m)$. That is suppose the Gaussian elimination of $(A|I_m)$ terminates at (R|B), then A is invertible if and only if R is identity matrix that is, R is I_m. Moreover, A^{-1} is let us understand it via an example. Let us again consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

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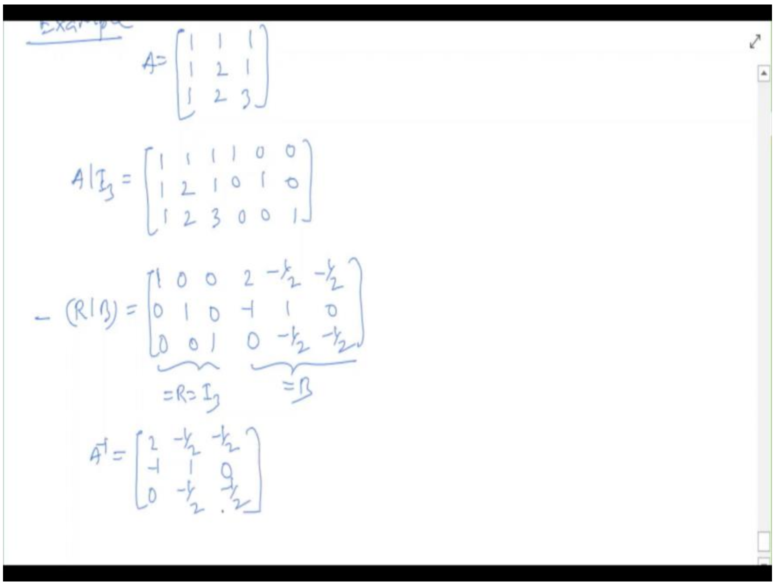

We will first write the augmented matrix (A|I_3) which clearly is

$$A = \begin{bmatrix} 1\ 1\ 1\ 1\ 0\ 0 \\ 1\ 2\ 1\ 0\ 1\ 0 \\ 1\ 2\ 3\ 0\ 0\ 1 \end{bmatrix}.$$

If we apply a sequence of elementary row operations it can be seen that (A|I_3) is row equivalent to matrix (R|B) that is

$$(\mathrm{R}|\mathrm{B}) = \begin{bmatrix} 1\ 0\ 0\ 2\ \frac{-1}{2}\ \frac{-1}{2} \\ 0\ 1\ 0 - 1\ 1\ 0 \\ 0\ 0\ 1\ 0\ \frac{-1}{2}\ \frac{-1}{2} \end{bmatrix}.$$

Clearly R here is an identity matrix. So, A is invertible and moreover, this sub-matrix B is inverse of A.

$$A^{-1} = \begin{bmatrix} 2\ \frac{-1}{2}\ \frac{-1}{2} \\ -1\ 1\ 0 \\ 0\ \frac{-1}{2}\ \frac{-1}{2} \end{bmatrix}.$$

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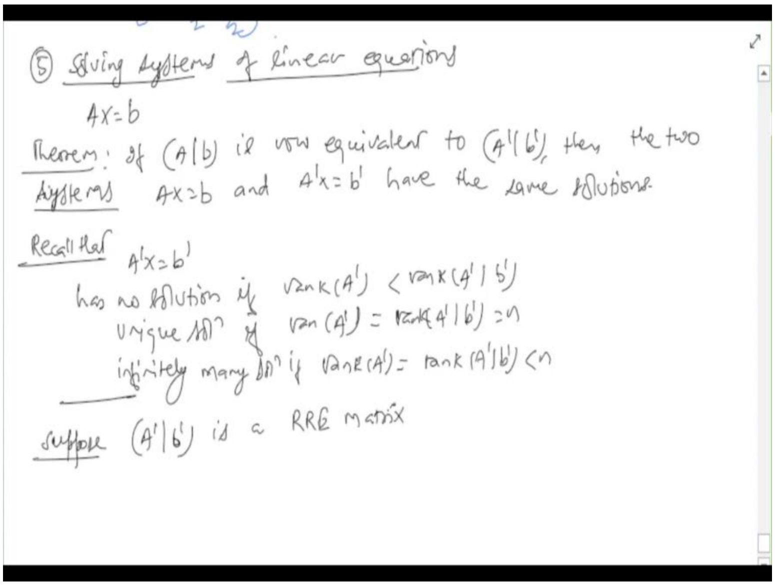

We now discuss another application of Gaussian elimination, namely, solving systems of equations. Solving systems of linear equations. Consider a system of linear equations Ax = b, we start with a very important result that we stated a theorem. It says that if (A|B) is row equivalent to (A´|b´), then the two systems it is Ax = b and A´x = b´ have identical solutions. Recall that the number of solutions of A´x = b´ is governed by the rank of (A´) and (A´|b´).

In particular, it has no solution if rank(A´) < rank(A´|b´). Unique solution if the rank(A´) = rank(A´|b´) = n and it has infinitely many solutions if rank(A´) = rank(A´|b´) < n. Now, suppose (A´|b´) in row reduced echelon form. Suppose (A´|b´) is a row reduced echelon matrix.

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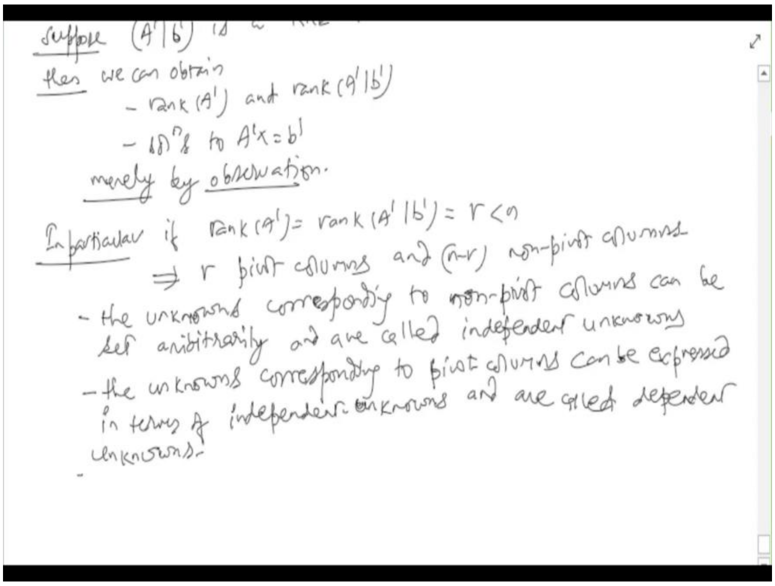

Then, we can obtain the rank(A´) and the rank(A´|b´) and also the solutions to A´x = b´ merely by observation. In particular, if the rank(A´) = rank(A´|b´) = r < n, then we have r pivot columns and (n – r) non-pivot columns.

In this case, the unknowns corresponding to non-pivot columns can be set arbitrarily and are called independent unknowns and the unknowns corresponding to the pivot columns can be expressed in terms of independent unknowns and are called dependent unknowns.

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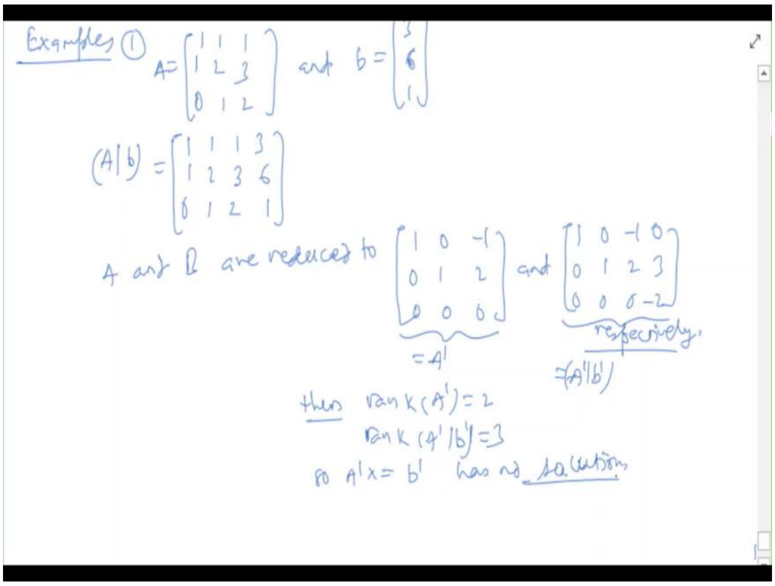

We will illustrate these notions through a few examples. In fact, we will revisit the examples seen at the beginning of the lecture. So, let us start with the first example. Here we take

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \ and\ b = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}.$$

In this case, augmented matrix

$$(A|b) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

If we apply elementary row operations A and A|b are reduced to reduced row echelon matrices.

A and b are reduced to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ and\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Let us call these matrices A´ and A´b´ respectively. Then, since A´ has two non-zero rows it can be directly seen that rank(b´) = 2 and following a similar argument, the rank(A´b´) = 3. So, A´x = b´ has no solution and so, Ax = b also does not have any solution.

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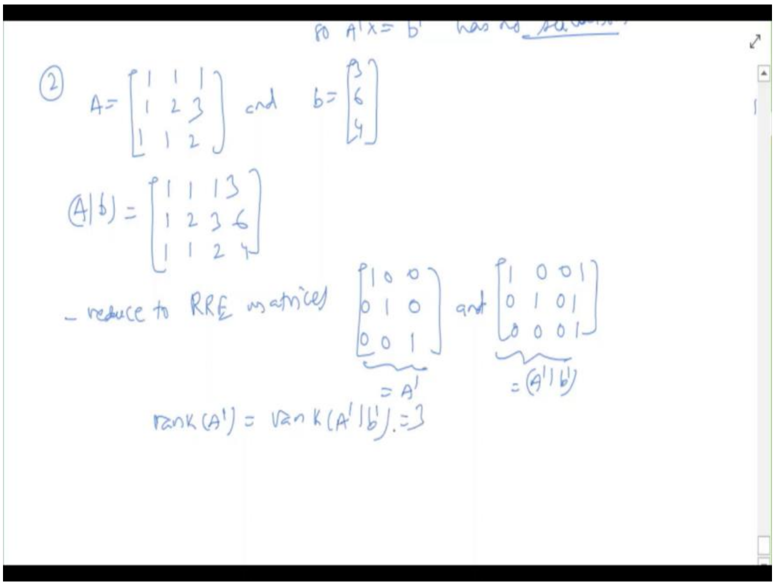

Let us look at another example. Now,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \ and\ b = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}.$$

The augmented matrix now is

$$(A|b) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 1 & 2 & 4 \end{bmatrix}.$$

Again, if we apply a series of elementary row operations A and A|b reduced to reduced row echelon matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ and \ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If I define the first one to be A´ and second one to be A´b´ now, I see that a rank(A´) = rank(A´b´). Both have same number of non-zero rows that is 3.

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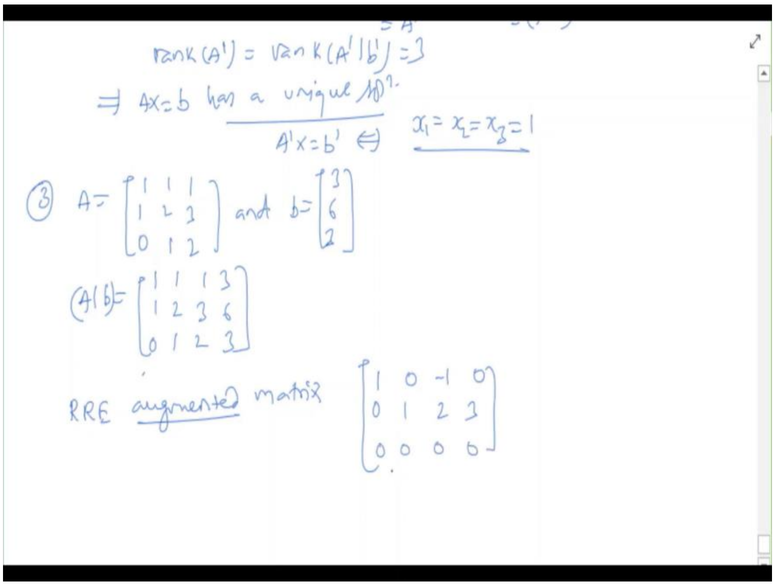

And so, Ax = b as a solution has a unique solution and this unique solution can be directly read from A´x = b´ which is nothing but $x_1 = x_2 = x_3 = 1$ in this case. Let us now see one more example. Now

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \ and \ b = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}.$$

The augmented matrix now is

$$(A|b) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Now, if we apply a series of elementary row transformation, we get reduced row echelon augmented matrix that is RRE, augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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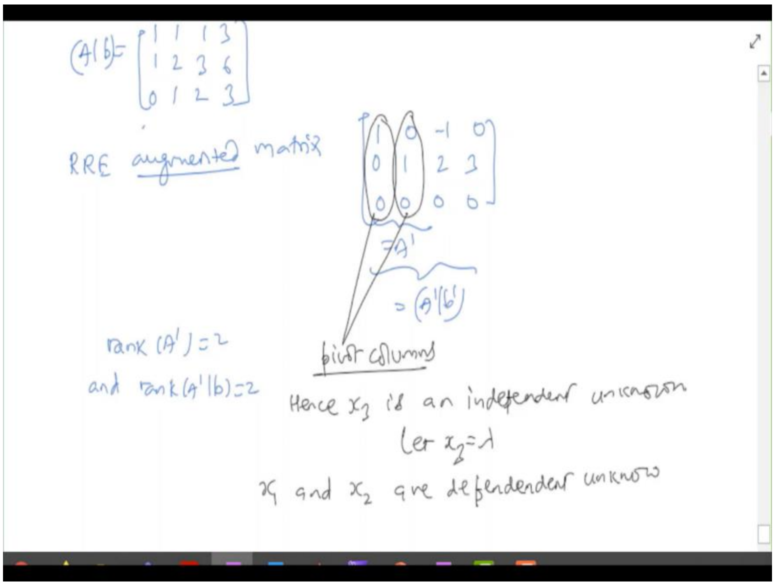

As before, if I use A´ to denote this submatrix and A´b´ to denote the whole matrix, then I see that the rank(A´) = 2 and rank(A´b´) = 2. These can be seen merely by observing the matrices A´ and A´b´. Moreover, notice that the first and second columns of this matrix are pivot columns. Hence, x_3 is an independent unknown and it can be set arbitrarily. Let $x_3 = \lambda$, on the other hand, x_1 and x_2 are dependent unknowns.

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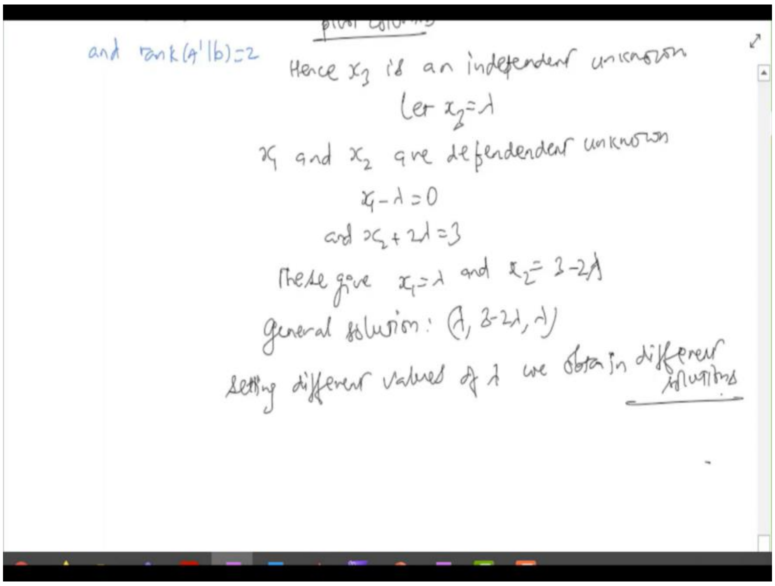

The value of x_1 and x_2 depend on x_3 as follows, $x_1 - \lambda = 0$ and $x_2 + 2\lambda = 3$. These two equations together give $x_1 = \lambda$ and $x_2 = 3 - 2\lambda$. Clearly in this case, we have infinitely many solutions. A general solution will be of the form $(\lambda, 3 - 2\lambda, \lambda)$, setting different values of λ we obtain different solutions. This brings us to the end of this lecture. Thank you.