Photonic Integrated Circuit
Professor. Shankar Kumar Selvaraja
Centre for Nano Science and Engineering
Indian Institute of Science and Engineering, Bengaluru
Lecture No. 04
Electromagnetic Theory Review 2

Hello all. So, let us look at wave propagation again using electromagnetic theory now. So, we understood that any solution should satisfy the wave equation. So, let us look at this wave equation itself, so how we came about this wave equation by using Maxwell, so let us look at that.

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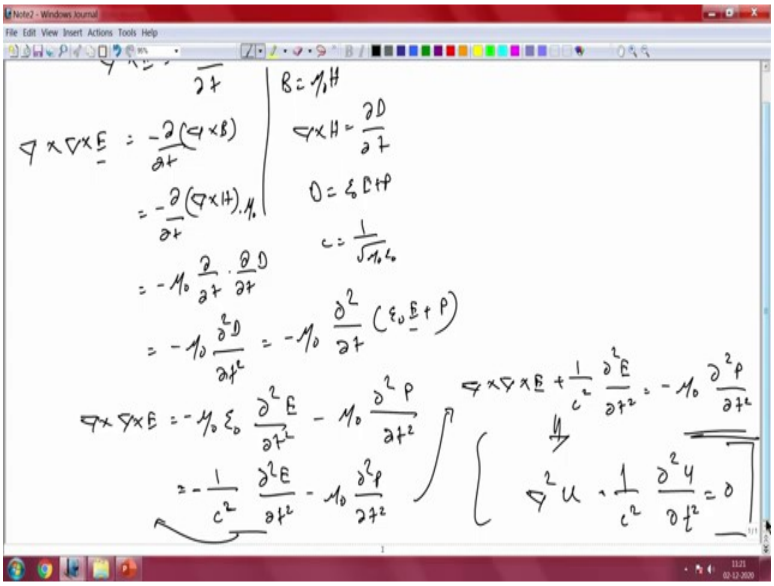

$$\nabla \times \nabla \times \underline{E} = -\frac{\partial}{\partial t}(\nabla \times B)$$
$$= -\frac{\partial}{\partial t}(\nabla \times H).\mu_0$$
$$= -\mu_0 \frac{\partial}{\partial t}\cdot\frac{\partial D}{\partial t}$$
$$= -\mu_0 \frac{\partial^2 D}{\partial t^2} = -\mu_0 \frac{\partial^2}{\partial t}(\varepsilon_0 \underline{E} + P)$$
$$\nabla \times \nabla \times E = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$
$$= -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$B = \mu_0 H$$
$$\nabla \times H = \frac{\partial D}{\partial t}$$
$$D = \varepsilon_0 E + P$$
$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\nabla \times \nabla \times \underline{E} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2}$$
$$\Downarrow$$
$$\left[\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0\right]$$

So, we start with the electric field representation and let us do a cross operator here, $\nabla x \nabla x E$, so that is what we would like to do. This would become $\frac{-\partial}{\partial t}(\nabla x B)$, so where B is nothing but $\mu_0 H$. So, we can also write that we know that $\nabla \times H = \frac{\partial D}{\partial t}$ and we also know $D = \varepsilon_0 E + P$, so we are going to use this. So, $\frac{-\partial(\nabla \times H)}{\partial t}\mu_0$, so we are doing this.

And now we can even make it using $-\mu_0 \frac{\partial}{\partial t}\frac{\partial}{\partial t}(D)$ t and let us expand this further or $-\mu_0 \frac{\partial^2}{\partial t^2} D$, so this is basically what we have. In other words, $-\mu_0 \frac{\partial^2}{\partial t^2}(\varepsilon_0 E + P)$. So, now we have the electric field as a function of electric field and the polarization. So, we now have a form, so that is what we would like to do from these detections. So, if some of you might be, why are we doing this, the whole idea of doing this is to connecting the effect of electric field and how the material reacts to this.

And $\nabla x \nabla x E = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} E - \mu_0 \frac{\partial^2}{\partial t^2} P$, I am just expanding this thing, so $\nabla x \nabla x E = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} E - \mu_0 \frac{\partial^2}{\partial t^2} P$. So, this is our equation. Now, so we all know $u_0 \varepsilon_0 = \frac{1}{c^2}$, so basically we knew that $\frac{-1}{c^2} \frac{\partial^2}{\partial t^2} E - \mu_0 \frac{\partial^2}{\partial t^2} P$. So, this is what we all knew. And then the rest of the things remain the same.

So, when you rearrange these factors, we can take this factor to move to left, to right, so then you get del squared. Let me first write it, $\nabla x \nabla x E + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E = -\mu_0 \frac{\partial^2}{\partial t^2} P$, so I am just writing it here, $\nabla x \nabla x E + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E = -\mu_0 \frac{\partial^2}{\partial t^2} P$. So, now this is a wavy, a form of wave equation now. So, what it represents is a source, a term, the source that is driving the optical wave and the left hand side is your electric field itself, how you can represent this.

So, basically this deduce to del squared as a generalized form, $\nabla^2 U = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} U$. So, when you do not have any source inside the medium, then you can make this as 0, when there is no polarization inside the medium, let us say in a free space. If there is no polarization involved and μ_0 is basically, not going to play a role, then you basically get this wave equation. So, one need to satisfy this equation in order to get a propagating wave or an electromagnetic wave.

So, once you have this electromagnetic wave, you want to understand how fast or how slow this electromagnetic wave is going to move, that is something that we briefly discussed in the earlier lectures on scalar wave optics. So, we talked about phase velocity, how a wave propagates through a medium and how the speed is going to be changed as a function of refractive index of the medium. So, let us look at the wave propagation using whatever we have understood so far.

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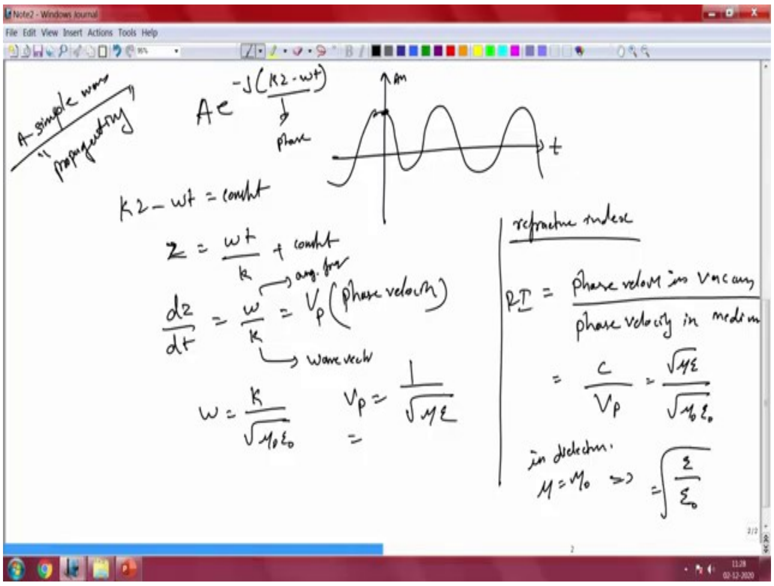

A simple wave " propagating

$Ae^{-j(kz-\omega t)}$ → phase

$kz - \omega t = \text{constant}$

$z = \frac{\omega t}{k} + \text{constant}$

$\frac{dz}{dt} = \frac{\omega}{k} = V_p$ (phase velocity)

ω → ang. freq

k → wave vector

$\omega = \frac{k}{\sqrt{\mu_0 \varepsilon_0}}$

$V_p = \frac{1}{\sqrt{\mu \varepsilon}}$

=

refractive index

$RI = \frac{\text{phase velocity in vacuum}}{\text{phase velocity in medium}}$

$= \frac{c}{V_p} = \frac{\sqrt{\mu \varepsilon}}{\sqrt{\mu_0 \varepsilon_0}}$

in dielectric.

$\mu = \mu_0 \Rightarrow = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$

So, a simple wave could be represented by this simple way, let us say a simple wave, let us say a simple wave very importantly propagating, so let us say it needs to be propagating. So, you could represent it as $Ae^{-j(kz-\omega t)}$ is what you say a propagating wave. And let us say this is as a function of time, there is an amplitude and this amplitude is changing as a function of time at a certain location and so on.

But then if you, the speed at which a particular point moves, so how fast this point is moving is very important and that means this should force this to a constant in order to find that, so that means you see no change in the wave propagating through the medium, so if you put this as a constant, then you can find out over k + a constant, so how far it took. So, now if you want to know the velocity, how far it has moved, but if you want to know the velocity you just differentiate with respect to time.

So, this is basically trying to find the velocity and this is basically your phase, so this we said phase and how fast the phase is changing, so if you differentiate this you would get $\frac{\omega}{k}$, so this is a constant so this will go away and this is basically your phase velocity. And this relates your angular frequency and the magnitude of your wave vector, so this k is nothing but your wave vector and this is your angular frequency. So, now the phase velocity actually links your angular frequency and your wave vector. And there are different forms that you can use this for various applications.

And now if you are going by the material property then the phase velocity is basically your velocity of light in the medium. So, this is the speed of light in in free space $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, but now it is $\frac{1}{\sqrt{\mu\varepsilon}}$, so $\boldsymbol{\mu}$ and **ε** are relative permeability of that. So, the speed of light from the wave equation is the phase velocity, the speed of light and the phase velocity are identical when it comes to a wave that has a single frequency, so we are talking about w.

So, what is a refract, from here you could also find the refractive index, so what is refractive index? So, one of the postulates of the wave theory is that any homogeneous, isotropic and transparent material could be represented by a single constant which is called a refractive index. So, refractive index is nothing but your phase velocity in vacuum to phase velocity in medium.

So, basically what you are saying is phase velocity of light in vacuum is c, so that is speed of light and phase velocity in the medium, just now we found out that is V_p, that is the phase velocity. And that is nothing but $\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$. So, if you consider a dielectric medium, so when $\mu=\mu_0$ it is a non-magnetizing material let us say, then this would become $\sqrt{\varepsilon}$ rather, $\sqrt{\frac{\varepsilon}{\varepsilon_0}}$. So, this is basically you are refractive index.

So, if you know the permittivity of a material, then it is basically root of that, so this is something that you all know and it is a relative permeability, so if you square root of relative permeability is your refractive index. And let us look at how your wave vector is going to look like, so this is the wave vector, so we want to know how these two are related. So, now let us look at k now.

So, $k=\omega\sqrt{\mu_0\varepsilon}$, so this is in a medium, so that is why I am putting it as an **ε** while $\boldsymbol{\mu}$ = μ_0. So, if this is the case then you can represent mu omega times root of mu nought epsilon nought. No, I think I made a mistake here, so this is basically root of epsilon over epsilon nought. So, if you deduce this what you would get is $\omega\sqrt{\mu_0\varepsilon_0}$. I do not have space here, let me write it here. .$\omega\sqrt{\mu_0\varepsilon_0}$n,

So, this is how you could modify whatever we have for k and now you can look at this, so this is basically your free space wave vector. So, that means k = k_0n, I am not sure whether you are able to see this.

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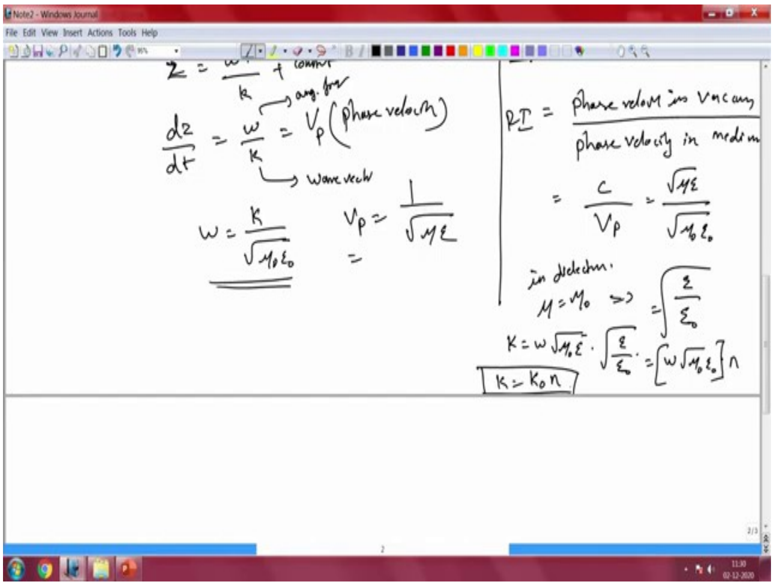

$z = \frac{\omega}{k} +$ constnt

ang. freq

$\frac{dz}{dt} = \frac{\omega}{k} = V_p$ (phase velocity)

wave vector

$\omega = \frac{k}{\sqrt{\mu_0 \varepsilon_0}}$

$V_p = \frac{1}{\sqrt{\mu \varepsilon}}$

$=$

RI $= \frac{\text{phase velocity in vacuum}}{\text{phase velocity in medium}}$

$= \frac{c}{V_p} = \frac{\sqrt{\mu \varepsilon}}{\sqrt{\mu_0 \varepsilon_0}}$

in dielectric.

$\mu = \mu_0 \Rightarrow = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$

$K = \omega \sqrt{\mu_0 \varepsilon} \cdot \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \left[\omega \sqrt{\mu_0 \varepsilon_0}\right] n$

$\boxed{K = K_0 n}$

So, this is nothing but k_0 n, so this is your wave vector inside the medium. So, wave vector inside the medium is nothing but free space wave vector with the refractive index here. So, this is the vacuum wave vector and n is the refractive index. So, as we discussed even in the earlier lectures, the refractive index is going to change the wavelength of light, the speed of light or in this case phase velocity of light is going to change and also propagation constant or the wave vector is also going to be changed because of your refractive index.

So, it is important to understand the effect of the material on the wave propagation. So, let us look at group velocity now. So, this is all phase velocity and there is a something called group velocity.

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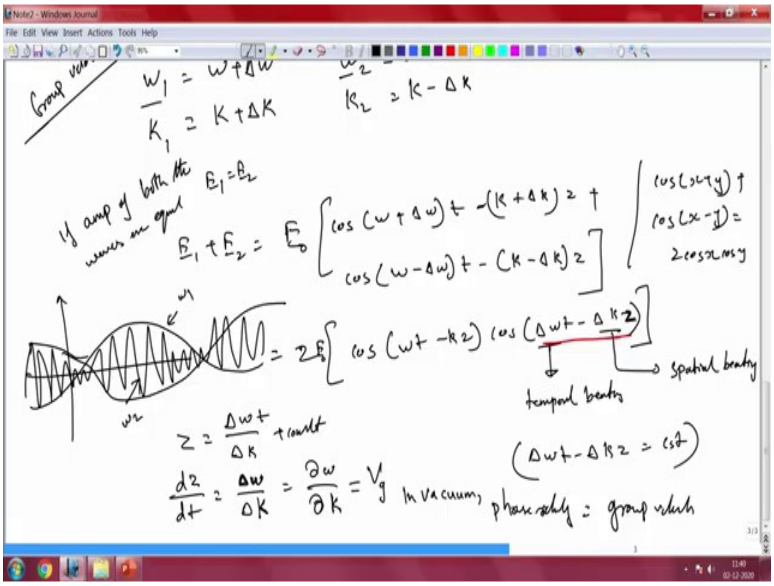

Group velo

$\omega_1 = \omega + \Delta\omega$ $\quad$ $\omega_2 =$

$K_1 = K + \Delta K$ $\quad$ $K_2 = K - \Delta K$

If amp of both the waves are equal $E_1 = E_2$

$$E_1 + E_2 = E_0 \left[\cos(\omega+\Delta\omega)t - (K+\Delta K)z + \cos(\omega-\Delta\omega)t - (K-\Delta K)z \right]$$

$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$

$$= 2E_0 \left[\cos(\omega t - kz) \cos(\underbrace{\Delta\omega t}_{\text{temporal beating}} - \underbrace{\Delta kz}_{\text{spatial beating}}) \right]$$

$$z = \frac{\Delta\omega t}{\Delta K} + \text{const}$$

$(\Delta\omega t - \Delta kz = \text{cst})$

$$\frac{dz}{dt} = \frac{\Delta\omega}{\Delta K} = \frac{\partial\omega}{\partial k} = V_g$$

In vacuum, phase velocity = group velocity

So, what is group velocity? So, phase velocity, the term is rather straightforward to understand because it is a single wave and also it has a phase and the phase is moving, and how fast the phase is moving is called phase velocity. But now we are talking about group velocity. Why should we talk about group velocity? So, group velocity is nothing but how a pulse propagates, so more often we use pulses of light.

Let us say if you have a pulse of light in time domain, you will have not a single frequency or multiple frequencies in the frequency domain. So, a pulse in the time domain will have a broader frequency representation in frequency domain. So, let us say there are waves that are propagating, so the frequency, the pulse let us say, you will have a certain bunch of frequencies there and the k or the propagation constant is also having a spread in that.

And you could have more than one, there are pulses, let us say there are two waves in this case. So, there is another frequency here and it will have its own wave vector here. So, let us say this is $k \pm \Delta k$. So, it has a certain spread, so there are two waves in this case but then it has a certain spread; that is fair enough. But what about the amplitude? So, you could have two different amplitudes but for simplicity sake let us assume that the amplitudes are equal. If amplitude of both the waves are equal, let us say E1 equals E2 let us say.

So, now I am going to put this two waves together, so if I am going to put these two waves together then I can write this as simple addition of cos and sin terms $\cos((\omega+\Delta\omega)t¿-(k+\Delta k)z)+\cos((\omega-\Delta\omega)t¿-(k-\Delta k)z)¿¿$. So, this is how your wave is going to propagate, a very simple a cosine representation.

We said the wave is moving together, they are moving together that means how it is going to look like, so you can rearrange this simply because the k terms are going to go away here and they are going to add up, so - k - k becomes 2k and the factor 2 you can take it out, so now you can write two waves as or simply like this. This is just an identity function, cos (x + y) + cos(x – y) will be equal to 2 cos x cos y, so this is just an identity that I applied here and when you apply this identity you get something of this kind.

So, what all this Δ k and $\Delta\omega$, so this is your beating of these two waves, it is called temporal beating and this is spatial beating. So, beating is nothing but interference of these two waves.

So, there are two components, they are beating with each other. So, if you look at beating its nothing but what is the beat length, so temporally how far or how these guys are beating. So, that is the two beating factor. So, now one can represent the wave that we just saw as a function of time, this is 2, $2E_o$, we miss that one. So, l et us look at the similar way how your distance can be done.

So, you know the frequency, you know the term here + constant, so the same trick that we applied earlier, so $\Delta\omega t+\Delta k z$ should be equal to constant. So, we took it from here so you do not have to worry too much, this is same from the phase discussion that we had, so if you want me to write I can do that, $\Delta\omega t+\Delta k z=c$ should be equal to a constant. So, now that is what we are using here in order to find the speed at which this wave is moving. Then we have to differentiate with respect to time.

If we do that, then this becomes $\Delta\omega/\Delta k$, so the constant is gone. Or in other words, $\frac{\partial\omega}{\partial k}$, which is nothing but the group velocity, so you see now when you have a group of waves or pulses traveling, then it is not any more single frequency that is dictating the speed at which the wave is moving.

Now which is this bunch of the spread, the bunch of frequencies are moving together, if that is the case then you will have a group velocity instead of phase velocity. So, the velocity at which this group of waves move together, they are moving together, they are not splitting, they are moving together but that speed of that movement, that bunch or that envelope is different, so if we normally represent it that beating.

So, you can see here there is an envelope that you can see as you move along, so you can see there are two waves here, so one is at lower frequency and one is at higher frequency, you can see this is $\omega 1$ and this is $\omega 2$, so when you have this together, it is almost like modulated signal. So, how is the modulated signal going to propagate through? There are two frequencies that you are mixing together, so how they are going to propagate?

So, that propagation is given by this group velocity. So, but you can just deduce this group velocity to phase velocity really quickly, so when $d\omega/dk$ is constant, so when you when you do not have this $\Delta\omega$, so if the $\Delta\omega$ goes to 0, then dω/dk becomes a phase velocity. Now so in vacuum your phase velocity equals to group velocity, in vacuum your phase velocity will be equal to group velocity.

So, they are all these waves, there is only one single wave and you do not have to worry about the group velocity in this case. But then in some complex medium this can be a little complicated. So, we saw in this case your k is represented quite easily through ωn/c.

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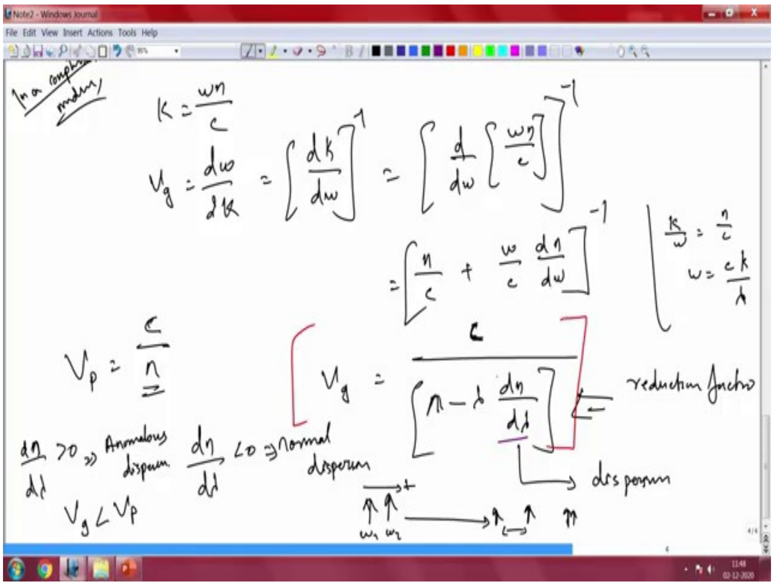

In a complex medium

$$k = \frac{\omega n}{c}$$

$$V_g = \frac{d\omega}{dk} = \left[\frac{dk}{d\omega}\right]^{-1} = \left[\frac{d}{d\omega}\left[\frac{\omega n}{c}\right]\right]^{-1}$$

$$= \left[\frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega}\right]^{-1}$$

$$\frac{k}{\omega} = \frac{n}{c} \qquad \omega = \frac{ck}{\lambda}$$

$$V_p = \frac{c}{n}$$

$$V_g = \frac{c}{\left[n - \lambda \frac{dn}{d\lambda}\right]}$$

reduction factor

dispersion

$\frac{dn}{d\lambda} > 0 \Rightarrow$ Anomalous dispersion

$\frac{dn}{d\lambda} < 0 \Rightarrow$ normal dispersion

$V_g < V_p$

ω_1 ω_2

So, this is your, let us say in a complex medium your k can be written as follows ωn/c. So, we know the group velocity here dω/dk, but we know the k, so let us invert this and we will use whatever we know about kωn/c, so if we use the, differentiate this equation n/c, so this is one term and the next term is (¿c)dn/dω ¿ $^{-1}$. So, this is the group velocity that we see here, so if we can rearrange this group velocity you can nicely re present it in this form.

So, we are just changing your omega to n by just moving around the terms. So, this is basically k/ω = n/c, so $\omega = ck/\lambda$. For some of you who might want to refer back. So, this is your group velocity when it is propagating through a medium. So, the medium is now interacting with the wave and it is slowing down, so the c is your free space wavelength, free space velocity and this is your reduction factor. So, earlier on we saw the phase velocity is nothing but c/n.

So, whenever material is propagating through a medium the speed of light is going to be reduced proportionally to the refractive index n, but then when you look at this, it also represent the same but then the denominator is little complex, so we have n, but then there is an additional factor dn/dλ and this is what we call dispersion or chromatic dispersion.

So, the refractive index is a function of wavelength, so because of this your group velocity is going to be a function of the wavelength that you have in the wave that is propagating or this group of waves that are propagating. So, if the wave, if the two wavelengths are going to be there, one will go faster compared to the other. And this is the property that we use in lot of pulse compression and pulse stretching and all those kind of interesting applications.

So, this dn/dλ can have two slopes, so dn/dλ could be less than 0 or you can have greater than 0. So, when it is less than 0 we call it as normal dispersion and when it is greater we call this as anomalous dispersion. So, when dn/dλ is 0 we call that as anomalous dispersion, in this case the group velocity will be less than the phase velocity. So, they will always move in that direction.

So, this is an important relation that you may want to pay attention to where your group velocity depends on the refractive index of the medium but an additional term because it is a group of waves, so and these waves will have different wavelengths, let us say and because of this difference in the frequency or difference in the wavelength that you have within this bunch of waves you are going to experience an additional factor or additional reduction factor in the velocity equation that we saw, group velocity equation we saw.

So, we have negative sign here, so based on the sign of dn/dλ you could make it faster or slower, so you can never get it faster than c, but the reduction factor can be managed by looking at whether this is positive o dn/dλ r it is negative, we will see that quickly in light matter interaction where you will have both anomalous and normal dispersion when you look at the material properties, but just to keep in mind that one could actually, if there are two waves, one wave could be traveling faster than the other wave.

And in that case you will, let us say if these two are λ1 or let us say ω1 and ω2 and if it travels after a certain distance you can either make them far apart, this is as a function of time, these waves can move apart or you can bring them closer, so these are all two possibilities and this bringing them closer or putting them far apart depends on whether the slope is positive or negative as we just discussed here, whether you are in normal dispersion or you are in anomalous dispersion.

So, with that we conclude this dispersion discussion and I would like to briefly talk about how this dispersion is important when we look at electromagnetic wave interaction with any matter for that purpose. So, when light is propagating through a medium, so the medium is characterized by its dielectric constant, its polarization and its magnetization all those properties there.

So, basically you have atoms inside the system that is looking for this energy that is available, you can take it out or it can interact with the electromagnetic energy inside the system and they could couple, so it can take out energy and it can change the property of the light itself using the energy that it absorbs or intrinsic properties itself. So, there is lot of interesting light matter interaction happens. So, one of the basic material property that we need to understand is what is the origin of this refractive index in itself?

So, we just said it is, it can be characterized by refractive index n and we also saw how the refractive index affects the material but where is this refractive index coming, from where is this resistance to flow of light is coming from and this is coming from the microscopic nature of the material itself. Let us look at how one can view or conceptualize this refractive index of the material, so that is where we would understand the light and matter interaction

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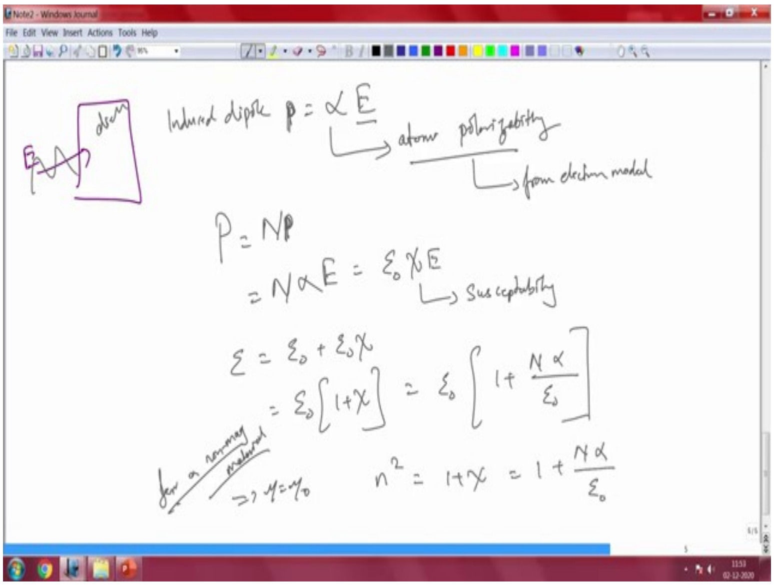

Induced dipole $p = \alpha \underline{E}$ → atomic polarizability → from electron model

$$P = Np$$
$$= N\alpha E = \varepsilon_0 \chi E$$ → Susceptibility

$$\varepsilon = \varepsilon_0 + \varepsilon_0 \chi$$
$$= \varepsilon_0 [1+\chi] = \varepsilon_0 \left[1 + \frac{N\alpha}{\varepsilon_0}\right]$$

for a nonmag material $\Rightarrow \mu = \mu_0$

$$n^2 = 1 + \chi = 1 + \frac{N\alpha}{\varepsilon_0}$$

So, let us look at you know what happens if you take a material and you put a certain electric field into the system and when you put this electric field and it is a dielectric, and what this this electric field or energy is going to do is to create a dipole, so there is an induced dipole what you call, so there is an induced dipole p that is represented, that depends on the electric field that you apply. But then there is polarizability of that electric field, this is a very important characteristic. This is atomic polarizability, so this is derived from the classical electron model.

So, you must have come across this in your solid state physics courses, so the polarizability how much you can polarize this material depends on this α and the electric field E. So, the polarization, the net polarization depends on the number density. So, number of dipoles in the system to the induced dipole. So, in other words Nα, so this is small p let us say, so I do not want you to get confused, so this is small p and this is capital P. So, this is what you have for the polarization or in other words one can do this as susceptibility as well. So, this is nothing but susceptibility.

So, $\varepsilon_0 \chi$E is your polarization. And what is **ε**? **ε** is nothing but $\varepsilon_0 + \varepsilon_0 \chi$ and if you take this ε_0¿1 + χ¿ is what you have. And from the earlier equation we could write 1 + Nα/ε_0, for a nonmagnetic material let us say. For a nonmagnetic material what happens, your $\boldsymbol{\mu}$ becomes μ_0, so then one can write the refractive index n^2 is nothing but 1 + χ, in other words 1 + Nα/ε_0 .

So, now your refractive index now depends on the number density that you have inside the medium or the number of atoms that you can polarize inside the medium and the polarizability, the ability to polarize something, there are some material that you will not be able to polarize let us say. So, what is the polarizability of this material? We will quickly see what this alpha is just going forward and then epsilon other free space dielectric constant all.

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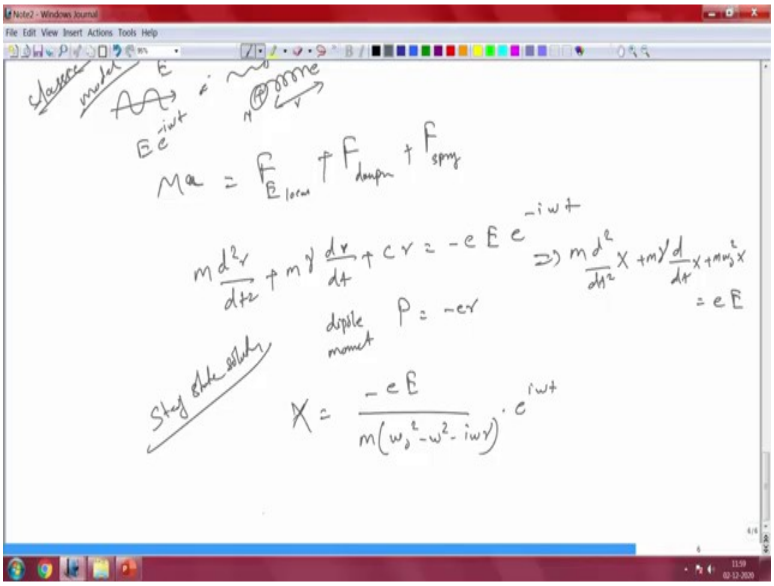

$E e^{-i\omega t}$

$$Ma = F_{E\,local} + F_{dampn} + F_{spring}$$

$$m\frac{d^2r}{dt^2} + m\gamma\frac{dr}{dt} + cr = -eEe^{-i\omega t} \Rightarrow m\frac{d^2}{dt^2}x + m\gamma\frac{d}{dt}x + m\omega_0^2 x = eE$$

$$P = -er$$

$$X = \frac{-eE}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \cdot e^{i\omega t}$$

So, let us look at a classic electron model. Some of you might have already done this, since we are reviewing our past electromagnetic knowledge, let me just also review this classic electron model from where we understand the susceptibility, how the, what is the origin of susceptibility and also how we relate the refractive index to the material resonances and so on. So, let us look at the classic electron model.

In a classic electron model what we are doing is we take an atom and then we are inducing this atom or let us say we are illuminating this atom with a uniform electric field. This electric field of this form, so the electron here, so you are putting into an atom, but then there is a electron associated with this, so you have the nucleus and then there is an electron sitting somewhere.

So, when you apply an electric field, because the electrons react to this external field, so there is a work done on this electrons. So, the electrons are bound to this nucleus through the coulombic force. So, you have a force that is reacting. So, it is keeping the electron within the atom here, but now you are having an external energy given to this electron, so this electron is going to take this energy and trying to oscillate. So, it is going to move around and that is going to create an additional energy.

So, now there is a restraining force that is going to pull this back, so there is an external field, but then there is a force to pull the electron into the influence of this. So, what we can do is we can easily figure out the force balance here to see how one can write this. So, there is a force coming from the electrical field, so that is E electric field in the local, so not the global one. So, there is a local electric field plus there is a force from damping, because that is the restraining force, and there is a force from the spring and this is all equals to m a, so mass times acceleration, so the mass of the electrons and the acceleration that this electron is moving.

So, we can simply write this, let us say if r is the distance that we have so this is the nucleus, this is r's the distance, $m\frac{d^2r}{dt^2}+m\gamma+m\gamma\frac{dr}{dt}$. This is the damping $m\frac{d^2r}{dt^2}+m\gamma\frac{dr}{dt}+cr=-eEe^{-i\omega t}$, so this is e that is the local field. So, the dipole moment of the system, so what is the dipole moment of a system, very small system, so this is basically P= -er. So, this is your dipole moment of the system.

So, now what one could do is we can simplify this and this is all second order PDE solution that we all know, a steady state solution for this, so let us say the steady state solution for something like this would look relatively easy. Let us say you have r which is nothing but

$$r=\frac{-eEe^{i\omega t}}{m\left(\omega_0^2-\omega^2-i\omega\gamma\right)}$$

So, how did I get $\boldsymbol{\omega}$ here is through this transformation where we write $\frac{md^2x}{dt^2}+m\gamma\frac{dx}{dt}+m\omega_0^2x=eE$. So, if that is the case, then I can write this as x ¿ $\frac{-eEe^{i\omega t}}{m\left(\omega_0^2-\omega^2-i\omega\gamma\right)}$ and this is the solution that we have. So, we know the induced dipoles, so now we have a solution for x, the induced dipole

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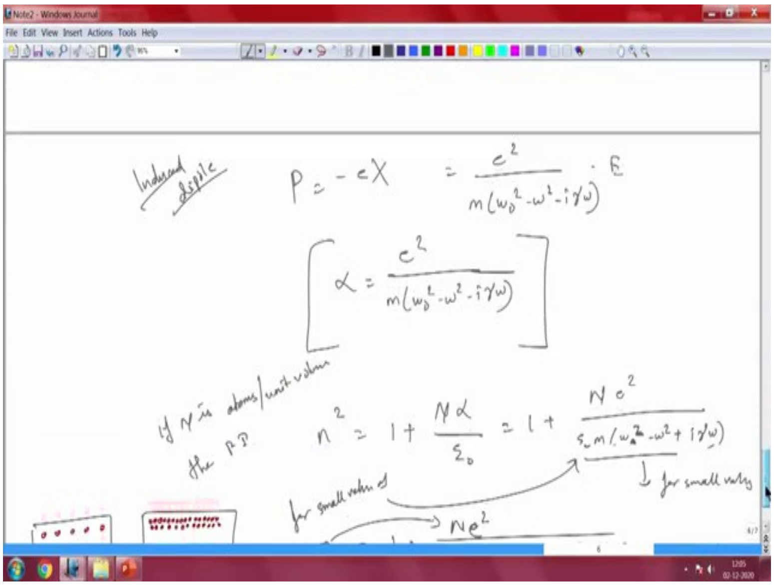

Induced dipole

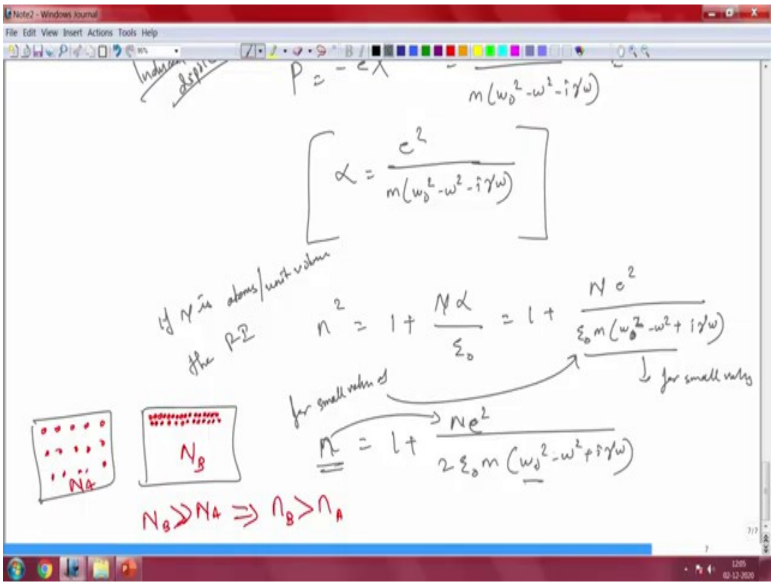
$P = -eX = \frac{e^2}{m(\omega_0^2-\omega^2-i\gamma\omega)} \cdot E$

$\left[\alpha = \frac{e^2}{m(\omega_0^2-\omega^2-i\gamma\omega)} \right]$

If N is atoms/unit volume the RI $n^2 = 1 + \frac{N\alpha}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0^2-\omega^2+i\gamma\omega)}$ ↓ for small value

for small value of

Ne^2


Induced dipole

$P = -eX$

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If N is atoms/unit volume the RI $n^2 = 1 + \frac{N\alpha}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0^2-\omega^2+i\gamma\omega)}$ ↓ for small value

for small value of

$n = 1 + \frac{Ne^2}{2\varepsilon_0 m(\omega_0^2-\omega^2+i\gamma\omega)}$

N_A N_B

$N_B \gg N_A \Rightarrow n_B > n_A$

So, let us move to the induced dipole. So, the induced dipole P = -ex here and that is nothing but P = -ex = ¿ $\frac{-e^2 E}{m(\omega_0^2-\omega^2-i\gamma\omega)}$ now. So, this is our induced dipole, so from the induced dipole we could calculate this alpha. So, this is nothing but alpha without the electric field there, so the alpha is nothing but $\frac{e^2}{m(\omega_0^2-\omega^2-i\gamma\omega)}$.

So, this is our alpha term and N of course, will come to that N little later, but now you can already see that where we are getting at, so if N is atoms per unit volume, so we said number density, then the refractive index, so the refractive index we saw here so n^2 this term, $n^2 = 1$

+ Nα/ε_0. So, let us just substitute this. That means $n^2 = 1 + N\alpha/\varepsilon_0 = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0^2 - \omega^2 - i\gamma\omega)}$, so this is what you get when you represent this n^2 as a function of this.

Let us look at this a little bit more carefully, the denominator term here, it is a solution that we already know of, but there is a condition that when these oscillations are rather small. So,, if this is, let us say very small value. This is for small values and then it does not make to look at n, so you could look at just n^2 instead of n^2 you look at n, so this is for small values of this. Then you can write this as $n = 1 + \frac{Ne^2}{2\varepsilon_0 m(\omega_0^2 - \omega^2 - i\gamma\omega)}$.

So, now you can see the refractive index, so the refractive index depends on the material properties. So, what is this material property? So, it depends on the atomic, the number density. How many atoms are there per unit volume, it is directly proportional, so what that means is when there are more number of atoms per unit volume, so I take two material system, so where the atoms are sparsely situated, so there is lot of distance between them. So, that means so the number density N is very low.

But then I could take another material system where there is a very tight packing of these atoms. That means for a given unit volume the number of atoms are more, so if I say this is material N_b and this is N_a, so $N_b >> N_a$, so the number density is higher. So, what this implies is the $n_b > n_a$, so a material which is denser will have higher refractive index. You can go back and check, cross check this with different materials.

Look at their number of atoms per unit volume with respect to the refractive index, you will see this happen and the other thing is the frequency of oscillation. So, this omega is nothing but the frequency at which your electrons are oscillating, so there is an oscillating frequency there, so you are refractive index also depends on those oscillation, so if you know the number density and the frequency of oscillation, you will be able to understand what is the refractive index.

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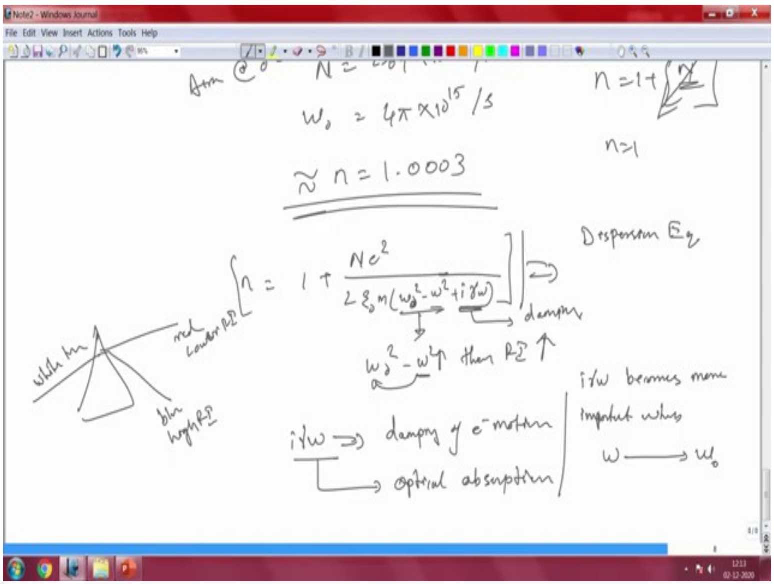

So, let us look at what is the, what should be the refractive index of air. So, air at 0 degree c, so the number density probably is a popular number that you all know $N = 2.69 \times 10^{25}$ per meter cube, so this is the number density and the resonance frequency is $4\pi \times 10^{15}$ /s. So, this is your oscillation of this. So, when you plug this number n, this is actually comes around n = 1.0003.

So, we say free space refractive index is 1, the vacuum refractive index is 1, but then air has its own number of atoms inside the system. So, when that happens then your n is 1. So, when the number density is 0, so n = 1 + {N} this some quantity here, so which depends on n. So, when there are no atoms present in the system, then n is equal to 1, but air will have molecules present in it, it could be nitrogen molecules are there, water molecules are there. So, you will have these molecules there and there is a density to it.

And also the air density varies based on moisture and so on, so if you look at air in Bangalore and air in Chennai, air in Delhi, they are all different, the reason is it could be pollution, it is also local weather and so on, so the refractive index of air will change based on the number density here. So, this is this is how you relate the material property with the refractive index of the material. So, that brings us to understanding the dispersion now, so let us look at the dispersion. We understood that it depends on the frequency of oscillation and everything, let us look at how the refractive index creates this dispersion. So, let us look at the dispersion in this medium.

So, we all know that the refractive index is represented by this form, so mass of this electron and then you have this oscillations of the electron that is having. So, this basically we call this as a dispersion equation. So, when omega increases, so the $\boldsymbol{\omega}_o$ is the fundamental frequency of the oscillating electron and then $\boldsymbol{\omega}$ is the external frequency that you apply let us say that is a electric field frequency that you are giving as an input.

So, that is what we saw here, so we are giving some external electric field $\boldsymbol{\omega}$ a while your $\boldsymbol{\omega}_o$ is your fundamental frequency of the system. So, let us look at that, did I do anything special here, no, that is good. I think probably there is a sign issue here, so this this should be plus, did I miss it anywhere, yeah, so this is again, this is a plus here. Pardon me for that, for some of you might have already noted it, if you have not so just make that correction.

So, once we have the dispersion equation then we understand, try to understand what happens to the material when the input frequency is changed. So, let us try to look at that, so when the omega here, so we are looking at this, so $\omega_0^2 - \omega^2$. So, when I am increasing this $\boldsymbol{\omega}$ and it becomes very close to $\boldsymbol{\omega}_o$, so what will happen?

So, when you when you have this particular term very close to $\boldsymbol{\omega}_o$ then the refractive index will shoot up, what you are trying to do is you make this denominator term going very close to 0, so provided that the mass of electron is also small and then the damping is also pretty small. Let us say when the $\boldsymbol{\omega}$ is very close to $\boldsymbol{\omega}_o$, the refractive index increases.

So, this is one of the reasons why in a prism your blue light bends more than the red light, so when you take a prism, when you have white light, so your blue light comes out here, your red will come out here, so blue light sees higher refractive index compared to red light. Red light say lower refractive index, it is relative, it is not low reflective index compared to blue light. The reason for that is this, blue light and red light they have different frequencies.

So, when the frequency is very close to $\boldsymbol{\omega}_o$, so $\boldsymbol{\omega}_o$ is a natural frequency of oscillation, when they are very close to it, then you see that your refractive index actually increases, so what happens to the imaginary term. So, the imaginary term here, so the imaginary term is nothing but the damping term. So, this $i\boldsymbol{\gamma\omega}$ is nothing but damping of the electron motion. It is trying to restore it and this is what causes absorption.

So, when the light is propagating through the medium, at certain frequencies you will see that you start absorbing and that is again an interesting way to look at this the dispersion equation. When omega becomes very close to omega nought that is when you see the effect of the imaginary part, so the imaginary damping is smaller. So, the imaginary part becomes more important, so $i\boldsymbol{\gamma\omega}$ becomes more important when $\boldsymbol{\omega}$ reaches $\boldsymbol{\omega}_o$, when close to the resonance or close to the resonant frequency of this device or resonant frequency of your material, then you see damping take over so that is when you see absorption starting to happen.

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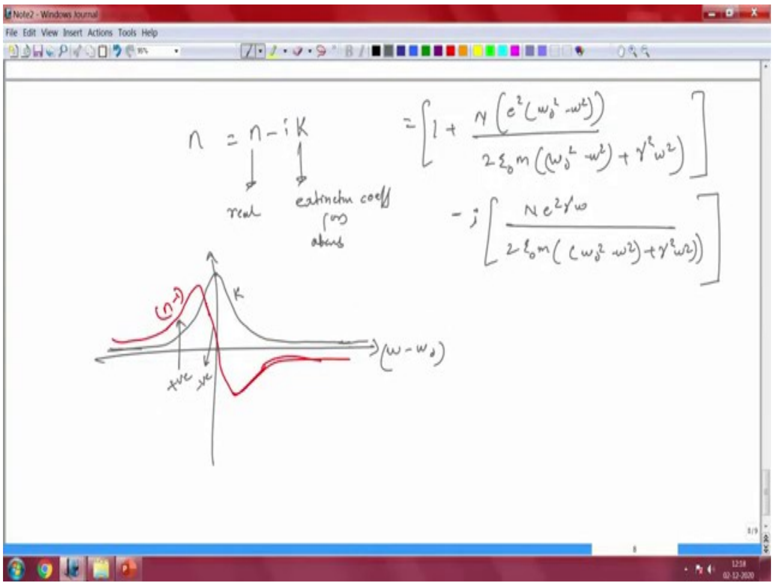

$n = n - iK$

$$= \left[1 + \frac{N\left(e^2(\omega_0^2 - \omega^2)\right)}{2\varepsilon_0 m\left((\omega_0^2 - \omega^2) + \gamma^2\omega^2\right)}\right] - j\left[\frac{Ne^2\gamma\omega}{2\varepsilon_0 m\left((\omega_0^2 - \omega^2) + \gamma^2\omega^2\right)}\right]$$

So, you can take this dispersion equation and then you can split the real and imaginary parts, if you do that you will get something like this minus. So, this is the real part and then the imaginary part starts here. So, now you can see the real part of the refractive index and the imaginary part of the refractive index very clearly. So, you can actually try to plot this one, when you try to see when omega becomes very close to omega nought.

So, this whole term becomes close to 0, it is reaching 0 and on the other hand the imaginary term here when $\boldsymbol{\omega}$ reaches $\boldsymbol{\omega}_o$, this is reaching infinity very large number here, so this term becomes 0 and all you get is $1/\boldsymbol{\gamma}^2$. So, not necessarily infinity, the number is pretty large, because it is $1/\boldsymbol{\gamma}^2$. So, when $\boldsymbol{\omega}$ reaches $\boldsymbol{\omega}_o$ you are getting a very large value of k while n becomes small.

So, if you were to draw this as a function of $\boldsymbol{\omega} - \boldsymbol{\omega}_o$ and this is how it would look like and this is your resonance, so this is what you see, your k will become very large at resonance, so when $\boldsymbol{\omega}$ reaches $\boldsymbol{\omega}_o$. So, this is where you have the resonance and then the refractive index n. So, this is k and your refractive index would do something interesting, so it will, so this is n - 1 let us say, so this is how it looks.

As you can see at resonance your refractive index is rather low, it is, but then your k value is very high, but then outside the resonance you will not see much, so you do not see the effect of k at all, so it is rather low, it is dominated by the refractive index and here again you can see two slopes. So, you see a positive slope and then you see a negative slope.

If you remember we talked about dispersion $\frac{dn}{d\lambda}$, we talked about dispersion and we talked about anomalous dispersion and normal dispersion in the material that is actually originating from this nature of your refractive index curves. So, that brings us to the end of our discussion on how the material properties is dictating the refractive index.

So, what is the link between these two? So, material they got their own atoms and electrons are vibrating at certain resonant frequency and when you put an electric field at a certain frequency into the system then the electrons are going to look at it that is, basically the polarization that we talked about. So, they are going to take this energy and then they are going to oscillate even more so that energy it is taking.

So, there is a relation between the frequency that you put in and the frequency of oscillation and when these frequencies are very close to each other then there is a lot of absorption happening, it is not, that is the reason why we say refractive index is complex. It is not a real number alone. So, it has a real part and then the imaginary part. So the real part tells you about the number density that you have, the density of the material, primarily it is dictated by that. But then the absorption strongly depends on how far you are from the resonance or resonance of your vibrating electrons.

And there are different type of resonances, so in this case we only looked at the electron resonance, but you also have resonance from the phonons, there is a lattice vibration that happens and also we have absorption from the electronic levels. So, there are different absorption phenomenas or resonance condition inside the material, what we today discussed in this lecture is only the resonance coming from the electrons there that is oscillating.

There are different resonances, so as you they will all have different frequencies, so that is the reason why when you look at the absorption spectrum of a material you will see peaks coming at different position of frequencies and these frequencies are representative of different processes of absorptions. There are ionic absorption, electronic and phononic absorption.

So, this is all different absorption phenomena that makes the refractive index complex and later on in the course we will see how this complex refractive index is affecting light propagation through wave guides. If you have how light is getting absorbed as it propagates through the medium.

So, with this we have covered or have done a review of the basic things that we need in order to progress in the course of how a wave is represented and how the material properties affect the waves and how the material property in itself can be understood from the microscopic scale. With this we have completed our review, in the coming classes we will move towards understanding photonic integrated circuits, the components and all the integrities. Thank you very much for listening.