Photonic Integrated Circuit
Professor Shankar Kumar Selvaraja
Centre for Nano Science and Engineering
Indian Institute of Science Bengaluru
Lecture 17
Field Equation

Hello everyone. So, let us look at the wave equation or rather from wave equation to how the field equation looks like. So, the last lecture we looked at how we can understand light's propagation and how one can confine and how the light actually gets decayed so outside the waveguide structure. So, now we need to put a proper mathematics around this understanding of light decay that is the evanescent field and the oscillating field inside. So, we need to look at the wave equation for that and find solutions to the structures that we have using Maxwell's equations. So, let us dive in.

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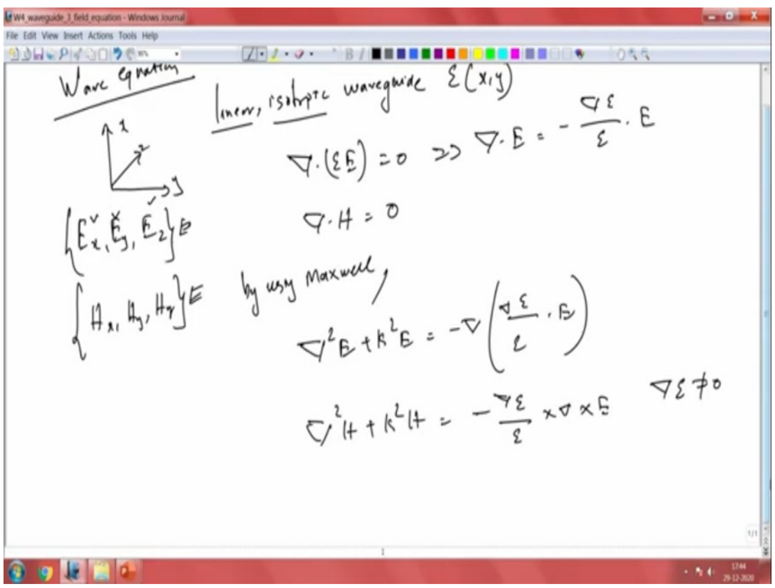

So, a wave equation could be approached by looking at different components that we have in the system. So, we have y, x and z. So, we will have field E_x, E_y, E_z these are all possible fields for E and you will have H_x, H_x, H_z. So, basically a common approach to find these fields is to solve the wave equation with a boundary condition. So, let us look at a linear isotropic waveguide. So, linear isotropic is very important to understand because this has implication on the permittivity. So, we want to take this as a isotropic medium, so your dielectric constant is independent, spatially independent.

So, there are 2 laws we are going to, there are 2 gauss law we are going to use for E and H and we can write that taking those 2 gauss law we can write this $\nabla . \varepsilon E=0$, $\nabla . E=\frac{-\nabla \varepsilon}{\varepsilon} E$ and $\nabla . H=0$. So, $\nabla . E$ will not vanish in general because epsilon x y is spatially dependent. So, that is something we should keep in mind. So, now let us use the Maxwell, we will have something of this kind, this is and for H field, so this is what we have once you apply the Maxwell into this.

So, there are 3 components here E_x, E_y and E_z for the electric field. They are generally coupled together because your delta epsilon is non-zero in a waveguide series. So, for the same reason H_x, H_y and H_z are also coupled. So, all these components are coupled as I mentioned and this as well. So, one could then that is the vector characteristic of these mode fields in a waveguide or strongly dependent on the geometry and the refractive index profile of your waveguide. The reason for that is epsilon x y. It strongly depends on the dielectric constant how it is the represented in the structure.

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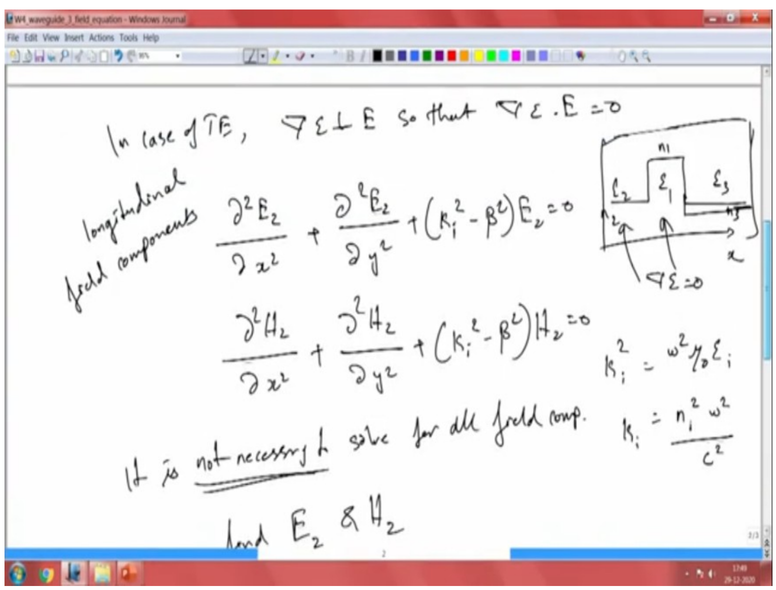

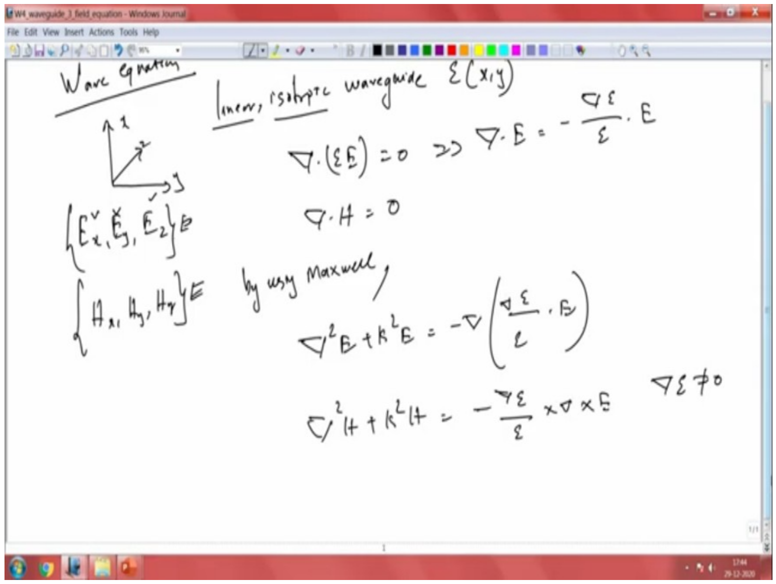

So, for a TE mode in case of a TE mode let us say we have delta epsilon perpendicular to E so that $\nabla . \varepsilon E = 0$. So, each component of the electric field of a TE mode satisfies a homogeneous scalar differential equation. So, the magnetic field component of TE modes are still coupled, so they are all coupled as well all. So, with this let us look at a very simple understanding of a step index waveguide or so a wave equation for a step index profile. So, step index is something that we already saw where you have n_2 and n_1. So, this is a function of x.

So, you have 2 different refractive indices or 3 different refractive indices depends on how you want to do this. So, now we can write a homogeneous wave equation separately for these different regions. So, you will have epsilon 1 here epsilon 2 here and for all practical purpose you could also have n_3 here if you want. So, we can put it little bit lower if you want. So, we can write this there because the constant epsilon becomes 0 here so within each region.

So, within a region delta epsilon is 0 within each region but then between the region it is not. So, if you look at the whole structure altogether, your delta epsilon is non-zero but inside the medium a delta epsilon is 0 because it is uniform medium. So, assuming E and H are harmonic guided waves then we can write the longitudinal component as $\frac{\partial^2 E_z}{\partial x^2}$. So, this is nothing but longitudinal field component plus $\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k_i^2 - \beta^2) E_z = 0$.

So, k_i represent whichever medium you are in whether in n_1, n_2 or n_3. So, similarly we can do it for H field as well. So, $\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+(k_i^2-\beta^2)H_z=0$. So, your xy field is where you have your fields and that is why we do it this way and here $k_i=0$squared is nothing but $k_i=\omega^2 \mu_o \varepsilon_i$, which is nothing but $k_i=\omega^2 \mu_o \varepsilon_i=\frac{n_i^2 \omega^2}{c^2}$, it is a constant within each region. So, if you are talking about region n_1 and this is a constant if it is n_2 this is this is a constant based on the refractive index.

So, this homogeneous equation is same as the same form written for each of these field components you can do it for x y and then H_x and H_y and it is not necessary to solve for each and every component. It is important, it is not necessary to solve for all field components. So, it is not required so it is not necessary to solve for all the field components that we have. It is good enough if we solve or if we find E_z and H_z. So, this is good enough by doing we can actually expand and then find the other components.

So, the more field (()) (9:51) can be obtained by solving just 2 equations. So, whatever we have here, we do not need to have the 6 equations that one can expand for the 3 components, here E_x, E_y, E_z and H_x, H_y and H_z. So, we do not have to do that. So, we can just keep it this way where just 2 components are good enough in order to extract the other components and how do we get that is by using the boundary condition these interfaces between the different regions based on that we should be able to do that.

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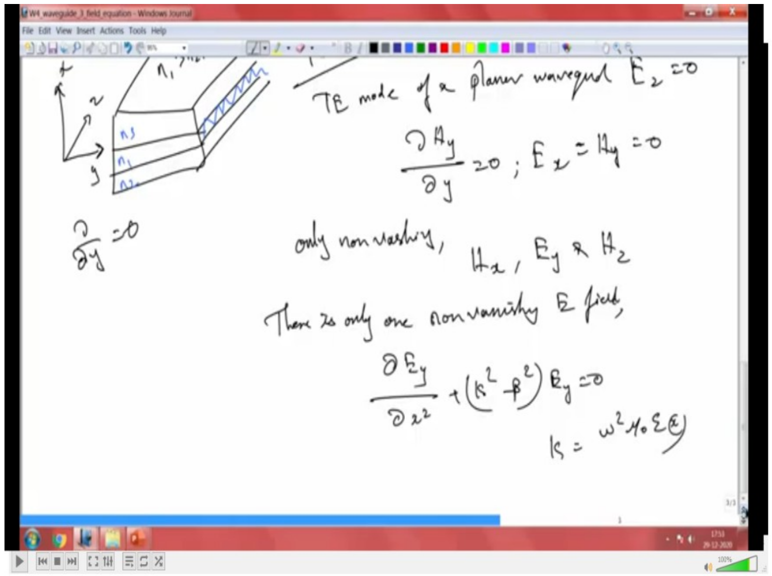

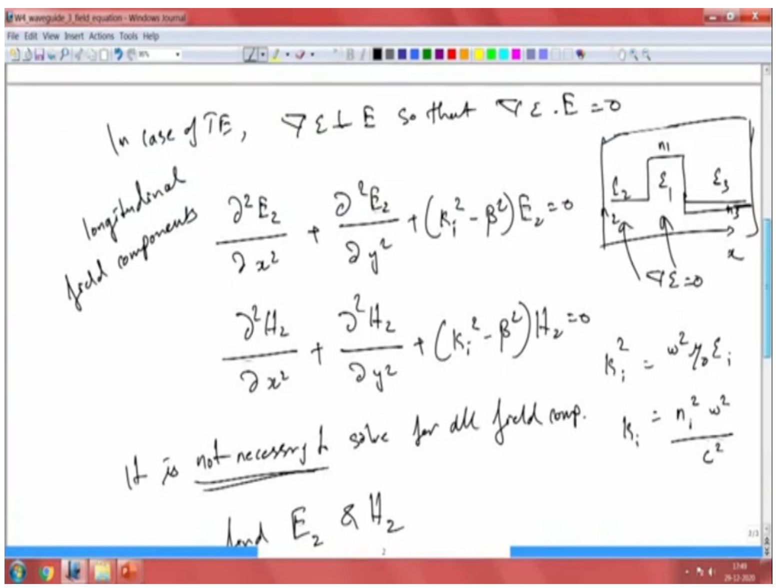

So, let us look at how one can use this approach to find the solution of this electric field or magnetic field in a very simple planar waveguide. Let us look at a very simple planar waveguide here. We have a waveguide that has refractive index n_1 here, n_3 and n_2 so light is propagating through this and the coordinate system is similar to what we have been discussing so far. So the direction of propagation is z and we have field in x and y direction. So, here n_1 is greater than n_2 and n_3, so there is something given. So, we call this as a simple slab waveguide.

So, for this let us look at both TE and TM. So, you will have TE mode and you will also have TM mode. So, we could consider $\frac{\partial}{\partial y}=0$ because the refractive index profile change across y direction is nil. There is no change in the refractive index along y. So, your $\frac{\partial}{\partial y}=0$. So, there is no change there. So, this will actually reduce the complexity that we are going to have here. So, this is very useful starting up with a very simple planar waveguide. So, for TE wave let us look at how the TE is going to be, for a TE planar in a planar waveguide.

So, let me so where E_z will be 0 all so such that the form relation, the relation here for transverse component to the longitudinal component is that$E_x = H_y = 0$ because $\frac{\partial H_z}{\partial y}=0$. So, your $\frac{\partial H_y}{\partial y}=0$ because field is continuous and your E_x and H_y will be 0 as well here. So, this is the implication here. So, the only non-vanishing part or field are H_x, E_y and H_z. So, look at it.

So, this is something we already noted so $\frac{\partial}{\partial y}=0$. So, $\frac{\partial H_y}{\partial y}=0$ in this case and that will result in E_x and H_y being 0. So, because of this implication the only non-vanishing part we have is H_x, E_y and H_z. So, because there is only 1 non-vanishing electric field, so there is only 1 non vanishing E field and that is your E_y and this becomes much more simplified like this and your k here is

$k=\omega^2 \mu_o \varepsilon_x$.

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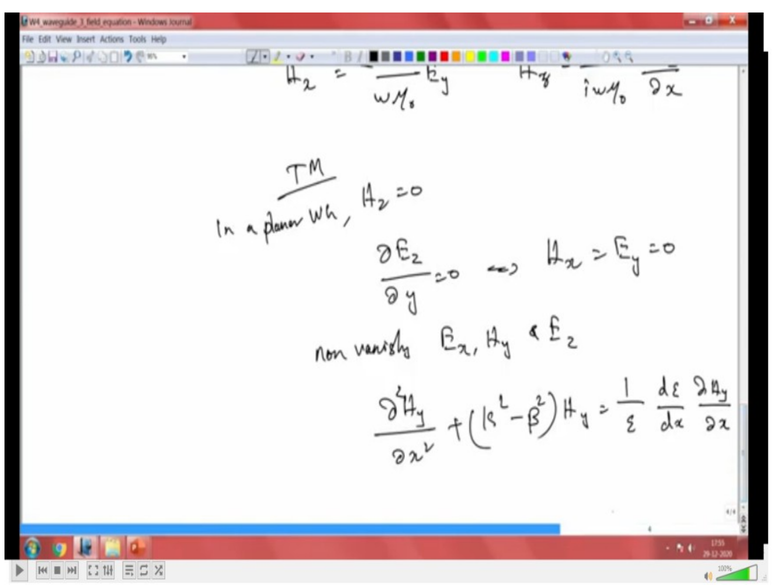

And you can get from this you can get our $H_x = \frac{-\beta}{\omega \mu_o} E_y$ and $H_z = \frac{\beta}{i\omega \mu_o} \frac{\partial E_y}{\partial x}$. So, now we can do a similar approach for TM waveguides as well, so, phi TM mode as well. So, in terms of TM your H_z field is 0. This is only for planar waveguide, in a planar waveguide your Hz is 0 and since your field is 0 this implies H_x and E_y are also going to 0 and now since you have only 1 H field that is non-vanishing, so, here non-vanishing fields are E_x, H_y and E_z. So, now we can write your H field as, so this is actually your H field.

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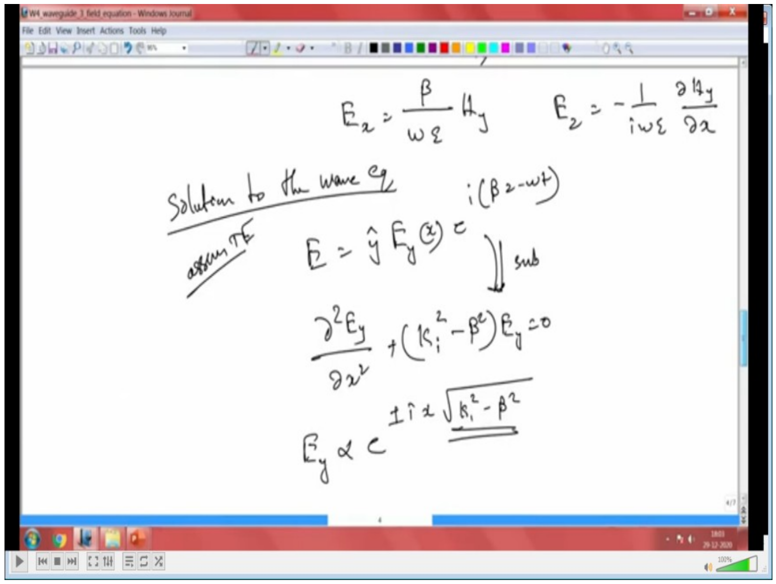

Now, from this H field we know we can get E_x field which is $E_x = \frac{-\beta}{\omega\varepsilon} H_y$ then we can also get Ez field which is $E_z = \frac{-1}{i\omega\varepsilon}\frac{\partial H_y}{\partial x}$ and, so you can apply this into over earlier equations that we saw we should be able to get these fields done. So, now having these fields defined, so this is how they are going to look but we have to still find the modes modal solutions here, how the fields are going to look like. So, this is just the wave equation. We do not know the solutions for this. So, we know we have formulated the wave equation but we need to find the solution to this wave equation.

So, let us look at how the solution could be. So, the solution to the wave equation. So let us assume that this is the how the solution will be. So, for a TE field, unit vector here so field is along x $E = \hat{y} E_y(x) e^{i(\beta z - \omega t)}$. So, this is very simple form. So, when you substitute this into our wave equation here, so what is the wave equation $\frac{\partial^2 E_y}{\partial x^2} + (k_i^2 - \beta^2) E_y = 0$.

So, we need to substitute this substitute this in the wave equation and this we can solve this the second order differential equation and your solution for this will be equal to $E_y \propto e^{+¿ix\sqrt{k_i^2-\beta^2}¿}$. So, this how your field is going to look like and the solutions are going to be sinusoidal in nature that you can see from here. So, it will be sinusoidal depending on the magnitude of k. So, let us look at the nature of this solution.

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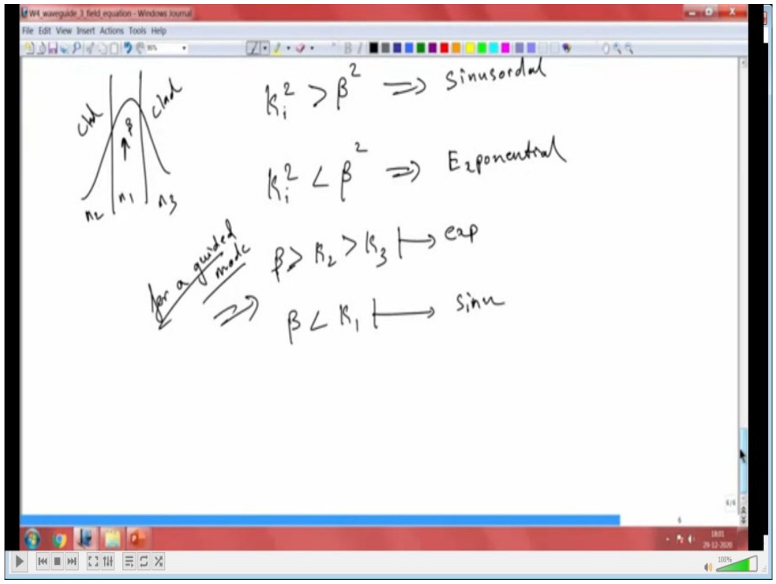

So, we have to look at this factor and k_i^2 whether it is $k_i^2 > \beta^2$, $k_i^2 < \beta^2$. So, these two are two possibilities. So, when $k_i^2 > \beta^2$ this will result in the sinusoidal solution and if $k_i^2 < \beta^2$ e.this will result in exponential result. So, if a mode or if a solution needs to be propagated through the, through a waveguide we call it that guided mode, if you want to have a mode that is guided the cladding region should be exponential that means we saw earlier so something like this.

So, in the cladding region you want the solution to be having exponential solution while in the core region it should have sinusoidal solution. So, how to achieve that? So we have this let us say this is n_1 and n_2 and n_3 . let us say. So, in this case you want for a guided mode. For a guided mode you want the beta that is propagating through this should be greater than k_2 and k_3.

So, this is what you want for an exponential solution but now $\beta^2 < k_i^2$ to have a sinusoidal solution. So, this is the condition that you want to have when you are having such a waveguide structure. So, this is exponential half this is sinusoidal. So, let us let us look at the field component what is the general form that you can you can define for this different region.

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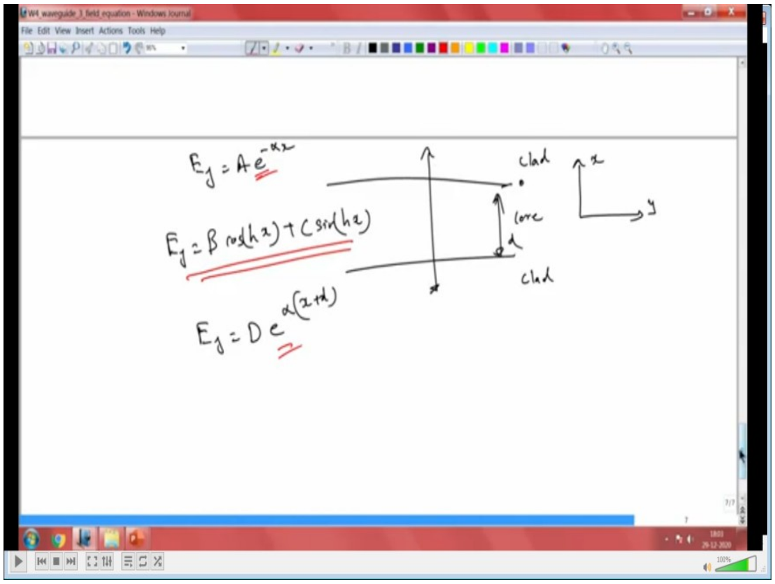

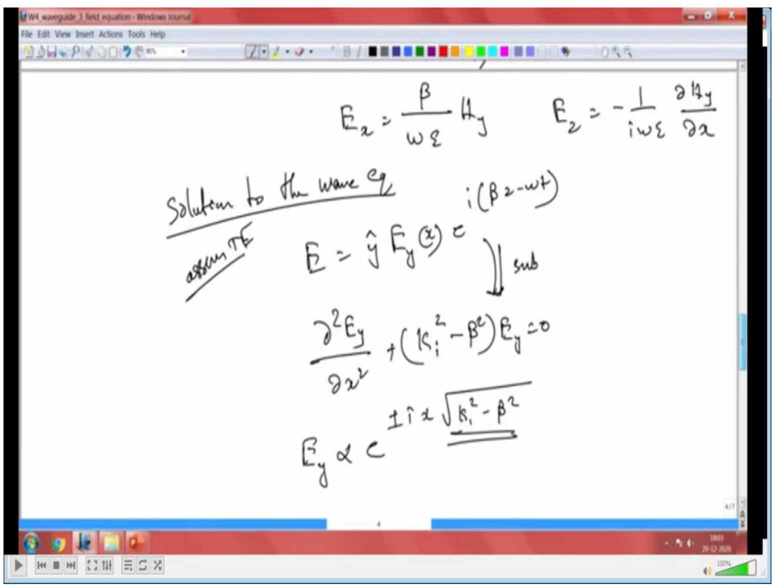

So, let us take a simple waveguide structure here and we have a core and we have clad. So, we define a simple core here this is x and let us say this is y. So, we are going to look at how the field is going to look like. So, in the core your field is the something of this kind, while in the cladding let us say some thickness, so you have some thickness d and here this is d and so on and here the field will look something like $e^{-\alpha x}$. So, you can see the oscillating field is at the center and while the field outside are decaying.

So, we could go back to our earlier discussion in the last class I believe where we looked at the phace matching condition in order to achieve the similar kind of results here, so here again from the detailed wave equation and solving the wave equation we also arrive at the same. So, all these constant a b c d, so these are all constants that will determine the oscillation strength and so on. So the boundary condition is definitely useful in understanding this and based on this you can find both for TE modes and also for TM modes in order to arrive at the same kind of solution that we had.

So, this is this is rather important to understand how we achieved oscillating solution or what are all the conditions to achieve oscillating solution you can keep a close eye on the approach here. This is coming from assumption that E field is only in x y direction. So, each field is distributed when the light is, when the field is in the x y plane and it is propagating along z plane. So, that that is a reasonable assumption that we have made and holds good in many circumstances all.

So, with that we have come to an end to this this particular topic in finding the field nature so how the field is going to be inside the waveguide and outside the waveguide. We had 2 different approaches and here we looked at the wave equation and then solving the wave equation and we were able to achieve it. And we could also do a similar kind of approach by just simply looking at the wave vectors and how the phase matching happens and based on that we should be able to do that. In the next class we will look at some of the important parameters that we use to characterize the waveguides. So, with that thank you very much for listening.