Mathematical Methods and Techniques in Signal Processing - 1
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Lecture – 72
Convergence of Fourier series for all square integrable periodic functions

Yeah let us begin with this theorem.

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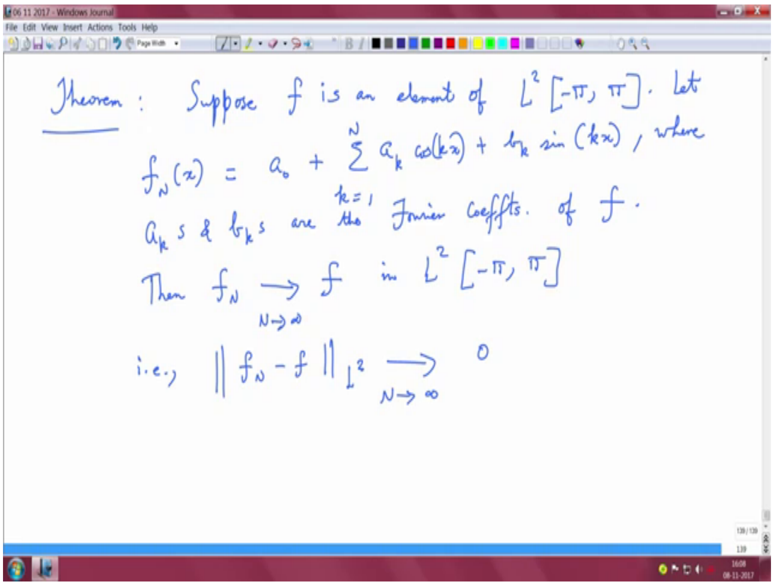

Suppose f is an element of this space that is all square integrable functions minus pi to pi. let f suffix N of x be given by this expansion a naught plus summation k equals 1 to, a k cos kx plus b k sine kx, where a k and b k are the Fourier coefficients of this function f which is a square integrable function.

Then f of x N heads to f in L 2 over the internal minus pi to pi as N goes to infinity, which means the norm between fN and f the integral norm this heads to 0 as N goes to infinity. So this is what the statement of the theorem is consider as square integrable function, let fN be the Fourier expansion then this expansion heads to this function in the L 2 sense that means, L 2 norm of f N minus f heads to 0 heads to 0 as N goes to infinity of course, if you are little careful about it you might sense that f this function is L 2 is in L 2.

We have to say that in some where we can approximate this function by a piecewise smooth periodic function, if you somehow get it in that form then we can approximate that function the original function by a piecewise smooth periodic function and approximate the piecewise smooth periodic function uniformly by the Fourier representation.

There are 2 two parts to this right I mean all your discontinuities can happen though it is square integrable, you may have discontinuities and you will have to extend them carefully right, so we will see these aspects carefully,

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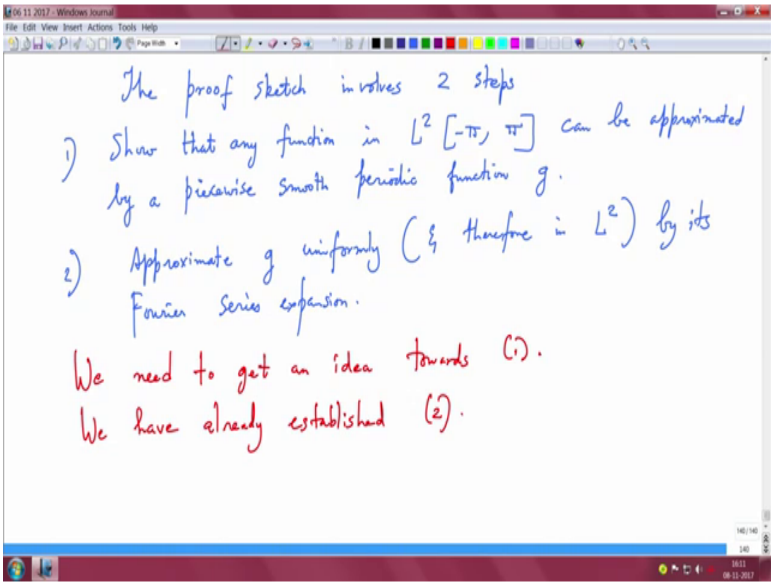

But let us get a feel for the sketch of the proof the proof sketch involves 2 steps, first show that any function in L 2 over the interval minus pi to plus pi can be approximated by a piecewise smooth periodic function g. And then approximate g uniformly and we saw the theorem that if you have if it converges uniformly therefore, it converges in L 2 by its Fourier series. So we need to get an idea towards 1.

We have already established 2 right the second part we have already established suppose you it converges uniformly then it converges in L 2 right. We have established this result already, so let us get the idea towards 1.

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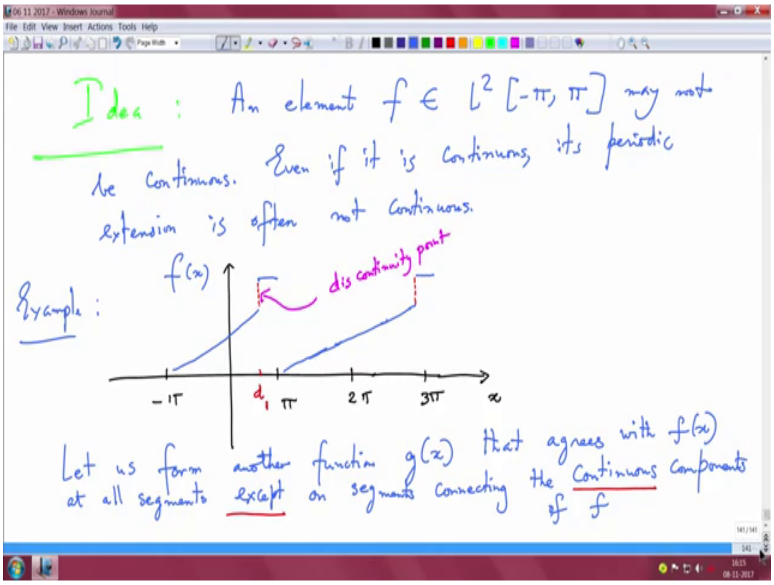

An element f belonging to this space which is a space of all square integrable functions, may not be continuous even if it is continuous its periodic extension is often not continuous, there are plenty of such cases a good example is f of x equals x over the interval minus pi to pi right if you if f of x equals we saw the points of discontinuity.

Now let us just take a sketch of this we have minus pi 2 pi 3 pi and so on. So have something like this say and this is this is a discontinuity point is a jump, similarly take a periodic extension you have a same problem. So let us form, another function g of x that agrees with f of x at all segments except on segments connecting the continuous components of f it is very important.

So let us from another function g of x that agrees with f of x at all segments except on segments connecting the continuous components of f right. So these are this is a continuous component this is a continuous component at these points g of x is going to be differing from f of x that is the whole idea right let us see what we could do.

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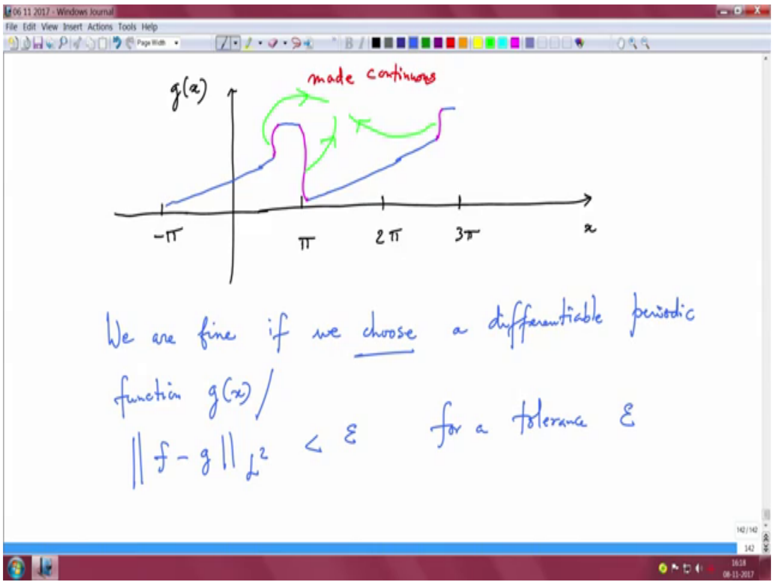

So revisiting this like a spy pi is 2 pi its 3 pi; now here I am connecting them and I am making them continuous ok. So I think you can see I can I can I can I am I am putting this I am joining these segments I am and I am making them continuous.

We are fine if we choose is very important a differentiable periodic function. g of x such that the norm between f and g in the L 2 sense is less than epsilon for a tolerance epsilon right; this is the first step right you try to make it you choose a periodic function differentiable as well such that in the L 2 sense f minus g this norm right in L 2 sense is within epsilon correct I can choose such a function.

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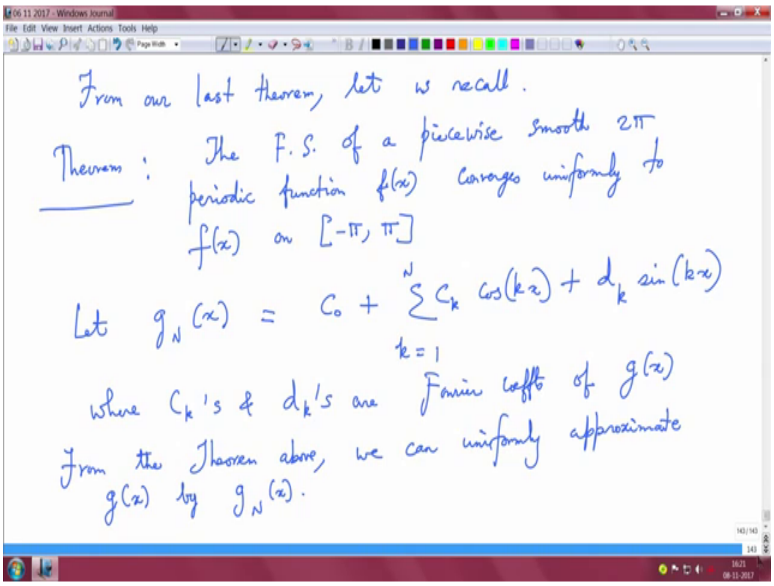

Now we this we have to recall from our last theorem.from our last theorem it is one of the previous theorems let us recall the following, the Fourier series of a piecewise smooth 2 pi periodic function, f of x converges uniformly to f of x on the interval minus pi to plus pi.

So let gN of x b equals C naught the summation k equal to 1 to N Ck cos kx plus dk sine kx where cks and dks or Fourier coefficients of g of x right. We know how to extend the function f of x to g of x right that we choose a differentiable periodic function and extend the function original function f which was just square integrable, but possibly discontinuous at certain points right.

Now from the theorem above we can uniformly approximate g of x by some g N of x right and how do we do that,?

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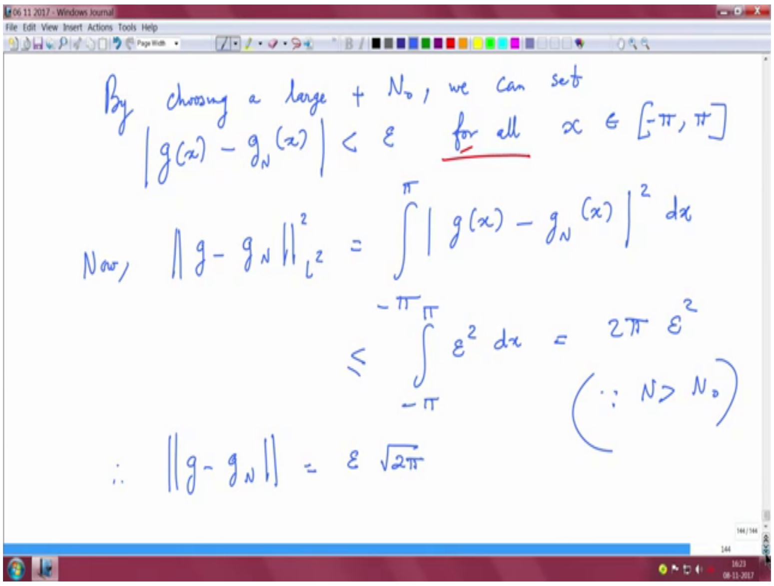

By choosing a large positive N naught. we can set I mean with the Fourier expansion g of x minus gN of x the absolute value of this is less than epsilon for all x in the interval minus pi to plus pi this is very important because it is uniform.

Now the norm of g N minus g square in the L 2 sense is integral minus pi to plus pi absolute value of g of x minus g N of x square dx, and this is minus pi to plus pi this is epsilon square dx which is 2 pi epsilon square that is for N greater than N naught.

Now therefore the norm of g minus g N is basically bounded by I mean if you choose this to be equal to fN of these little tolerance norm square is within tolerance to pipe system, square this norm is if you choose some epsilon you can meet it with epsilon times root 2 pi right straightforward.

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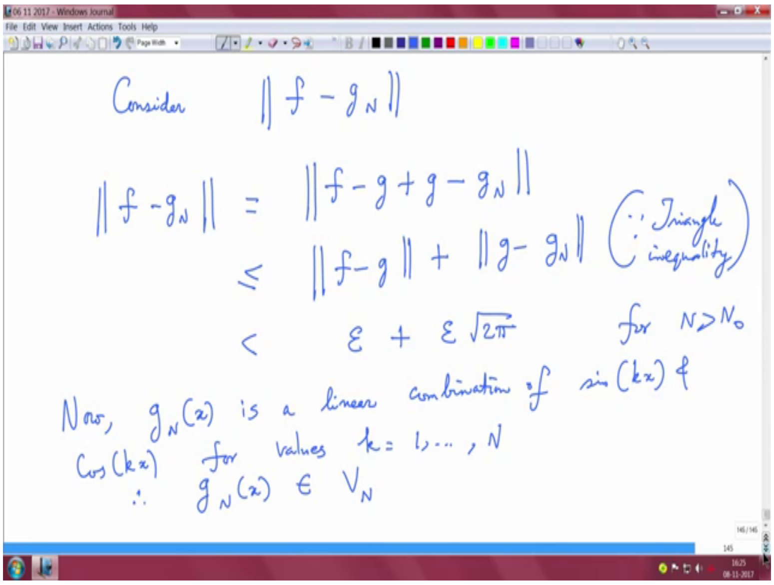

Now let us consider the norm of f minus g N that is we had our original function f which is in L 2, but which possibly had discontinuity points etcetera right this norm I can write it as follows f minus g N is basically I add and subtract g into it is familiar trick that one does.

Now we apply the triangle inequality this is f minus g norm plus norm g minus g N this is because triangle inequality and this we said is less than I just put a less than here less than epsilon this is less than epsilon times root of 2 pi for N greater than N naught.

Now gN of x is a linear combination of sine kx and cos kx for values k equals 1. So on till capital N, so therefore gN of x belongs to the space V N, now it is easy to interpret this because from an earlier lemma f suffix N is a closest element from the space V N to f in the L 2 sense right.

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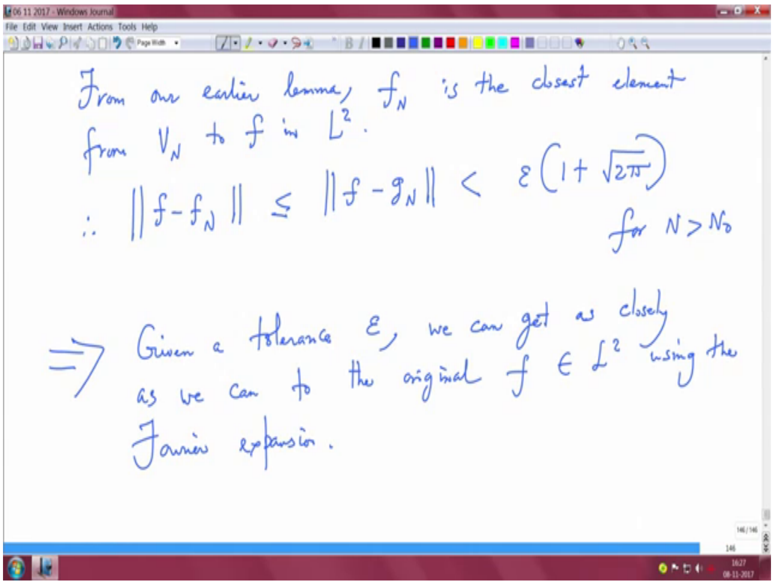

So from.our earlier lemma earlier result f suffix N is the closest element from this space VN to this function f in the L 2 sense. So therefore, if you look at the norm f minus f N f suffix capital N this is less than or equal to norm f minus gN this is less than epsilon times 1 plus root of 2 pi, which means given a tolerance epsilon, we can get as closely as we can to the original function belonging to L 2 using the Fourier expansion.

So this is an important result I think this is sort of summarizing the various modes of convergence and basically trying to tell that if you are given a function which is possibly suppose you are not able to achieve point wise convergence or uniform convergence basically look at the weaker form of convergence which is basically in the L 2 sense and for you to get towards L 2.

So take a function f which is belonging to L 2 it may have discontinuities, but extend that function using a differentiable function right and connecting those points of discontinuities through a differentiable function and periodically extend that right. And you can figure out such a g which is close to f within a tolerance epsilon and then use uniform. If you can do a uniform approximation then you can show that that result will converge in the L 2 sense and then we are home for good right and how do we approximate the function g itself is a construction process which is not discussed in the proof of this result, but if you can do that then the (Refer Time: 23:58) it is valid ok.

So this gives us an idea that Fourier expansion is quite generic enough we can figure out we studied the convergence aspects at a point of discontinuity at points where it was continuous it was no problem, but we had subtle issues where we had to link the additional we had to bring the additional a kernel and look at some subtle aspects where those points were a differentiable right, and then we looked at the L 2 convergence aspects and I think we have sort of summarized the convergence aspects of the Fourier series expansion and this is a very useful tool for expanding signals in the Fourier space right, and I think you can think about a similar expansion in the wavelet sense you have all these aspects.

That you have to deal with I mean I remember in 1 of the theorems which I left it as a as a theorem without a proof that you can expand this in terms of the direct some spaces in the limiting form, and when you have to look at the limiting form then you have to look at all these subtle aspects at points of continuities at points of discontinuities etcetera. You can sort of take a lot of analysis in this work and think about in the wavelet expansion as well right.

I think you now get a very clear picture as to how to expand functions using bases and what are the subtle aspects; particularly you are in trouble if you have jump points or discontinuity points and I think this is where you have to spend little bit of thought to carefully see in what sense it converges ok. So with this we are done with this module on Fourier expansion our next lecture would be on kl ok, so we can stop here.