Mathematical Methods and Techniques in Signal Processing – 1
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Lecture – 71
Convergence in norm of Fourier series

So, let us get started with the lecture. One of the motivational questions we have to ask is can we connect the various issues we studied regarding convergence and interpret this as projecting a signal onto the onto the basis right I mean this is this is this is where we are sort of leading towards I mean in terms of the Fourier expansion, right.

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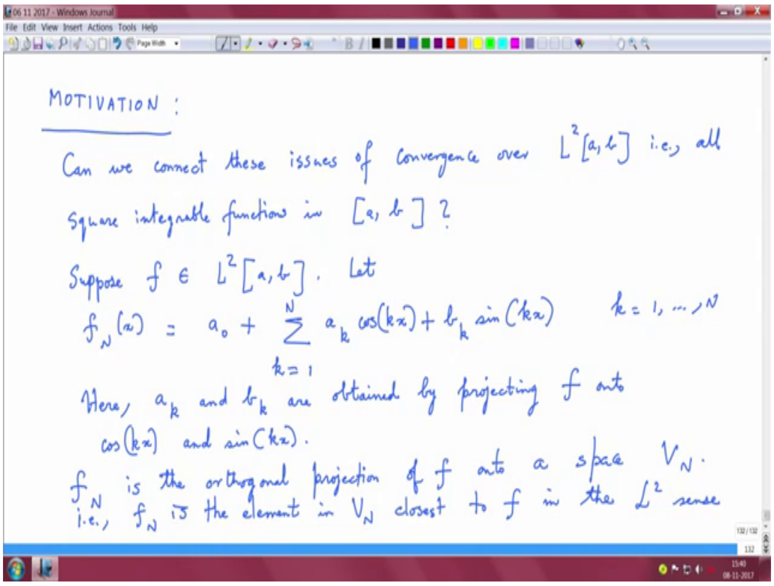

So, the question is can we connect these issues of convergence over L 2. Now, suppose f is a function that belongs to the square integrable functions I mean I am specifying the interval let f N of x equals a naught plus sigma k equals one to capital N a suffix k cos k x plus b suffix k sin k x, k goes from 1 to N and we see this in the summation, ok.

Now, here a k and b k are obtained by projecting f this is a function this is the square integrable function on to cos kx and sin kx now we can interpret this as f N is the orthogonal projection of f onto the space V N that is f suffix N is the element in V N at in this in this space V N closest to f in the L 2 sense.

So, I will say maybe this is a space V N, right. So, what it means is now, we are given some function which is square integrable function that is you take the square modulus of the square integrated over that interval and that is finite right and consider those functions and f N is basically f suffix N is orthogonal projection if you write it in terms of the Fourier series representation that is basically the orthogonal projection of square integrable function onto a space V suffix N right and f N is an element in V suffix N closest to this function in the L 2 sense right.

That is if you look at the norm of f minus f N under the L 2 sense, right that that should be the closest I mean or that is if you want to minimize that metric, right. In the L 2 sense then the expansion happens to be of this form.

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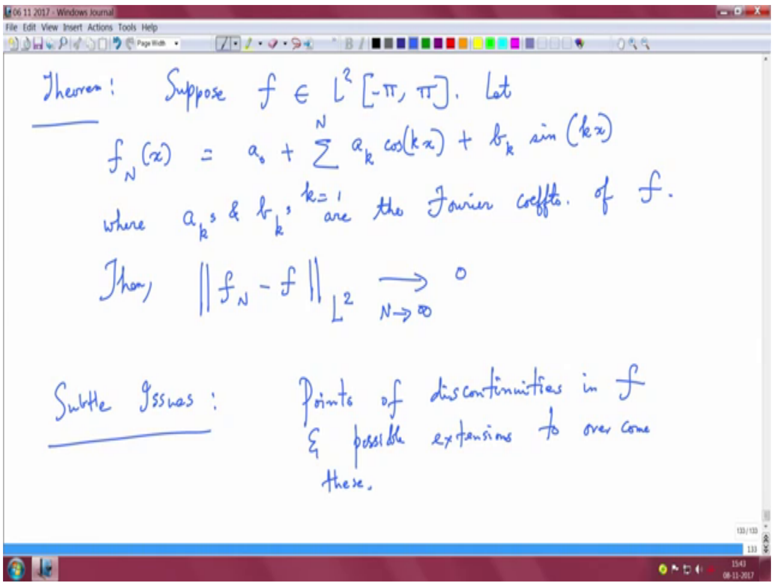

Now, let us formally put this here. Suppose f belongs to L square minus pi to pi, let f suffix N of x be equal to a naught plus summation k equals 1 to capital N a suffix k cos kx plus b suffix k sin kx there a k and b k, at a k's and b k's are the Fourier coefficients of f.

Then if you look at this norm f N minus f under the L 2 sense, right this heads to 0 as N goes to infinity. So, the proof is basically involved and we will discuss this proof later, but I think you get a sense of some geometry here right you consider some square integrable function. Let f N be in this form then under the L 2 sense this sets to 0 as N goes to infinity right. So, I think the subtle issue is what if you have points of discontinuity, right.

The subtle issues points of discontinuities in f and possible extensions to overcome these we will discuss this proof with a geometric interpretation, but before we discuss this theorem we will have to prove one more theorem and I will state that result and then we will connect all the pieces together. So, my trial of pedagogy here is more there is a problem at hand we try to look at the problem what it requires then, we set up certain lemmas that is required and then we will nest it within a theorem, right.

Let us see that is the whole idea I mean you could start very formally with definitions in lemmas and theorems and if I go in that route maybe you will be lost a little bit in getting the intuitive feel as to how to go about things. So, slightly different than the conventional approach

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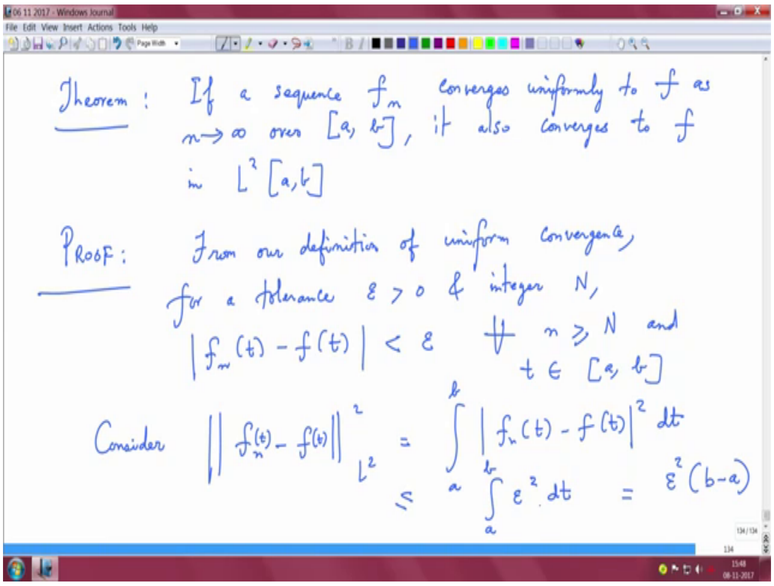

So, we will revisit this theorem, but bear in mind. So, let us prove an easier theorem and then we will use this result into the other theorem if a sequence f suffix N converges uniformly to f as N goes to infinity over the interval a comma b, it also converges to f in the L 2 sense, this is an interesting result. If there is uniform convergence then it also means it is convergent in the L 2 sense. So, let us prove this result. It is a very easy proof, will we will get the idea very clearly.

From our definition of uniform convergence for a tolerance epsilon which is greater than 0 some positive tolerance and integer capital N absolute value of f n of t minus f of t is within epsilon for all integers small n greater than or equal to capital N and t belongs to a interval a comma b, right. This is our definition of uniform convergence for all primes t within this interval. This function is bounded within epsilon and you have to recall the other form of convergence where epsilon depends upon tolerance depends upon the point. So, this is uniform convergence.

So, it is not dependent on the point t now let us consider the L 2 norm of this I have to put it t here just to be a little careful, ok. This is basically integral a to b from our definition mod f n of t minus f of t square dt right this is by definition. Now this assuming this property is less than or equal to epsilon, right so, therefore, this norm is less than or equal to integral a to b epsilon square dt and this can be simplified as epsilon square b minus a, right and see how easily we were able to plug it assuming uniform convergence.

If it was point wise then we would have difficulty because all of which for a for every time I would have to choose some epsilon which depends on the point and then have to look at some series a summation of various such tolerances and I may I might not be able to simplify it in this form, right. So, this is easy now with uniform conversions.

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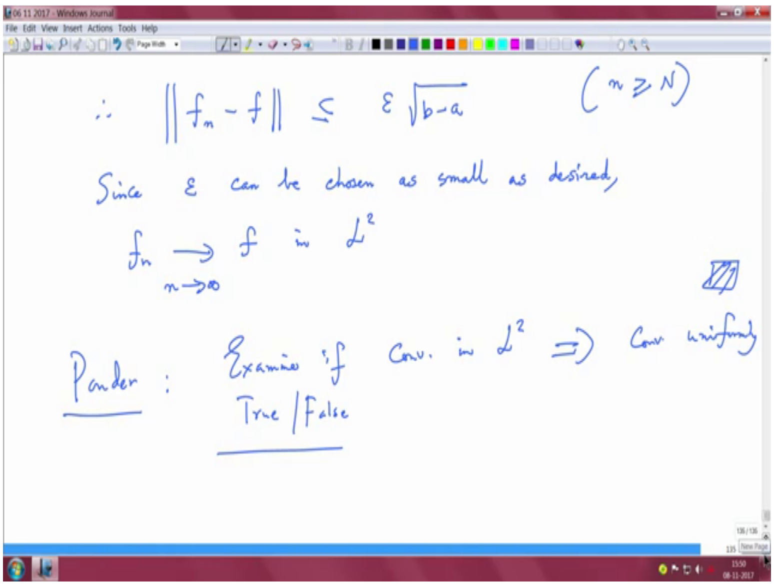

Now, this implies the distance between f n and f is epsilon times square root b minus a all right and this is for all small n greater than or equal to capital N. So, since epsilon can be chosen as small as desired f n heads to f as n goes to infinity in L 2. So, what does this mean; that means, if you have a sequence which is uniformly converging it is convergent in the L 2 sense, right. I want you to ponder the other way around examine if convergence in L 2 implies convergence uniformly.

So, there is a homework exercise. So, I think you can construct a function and show one way even. It is enough if you can give a counter example if not you will have to prove this result if you think it is true.

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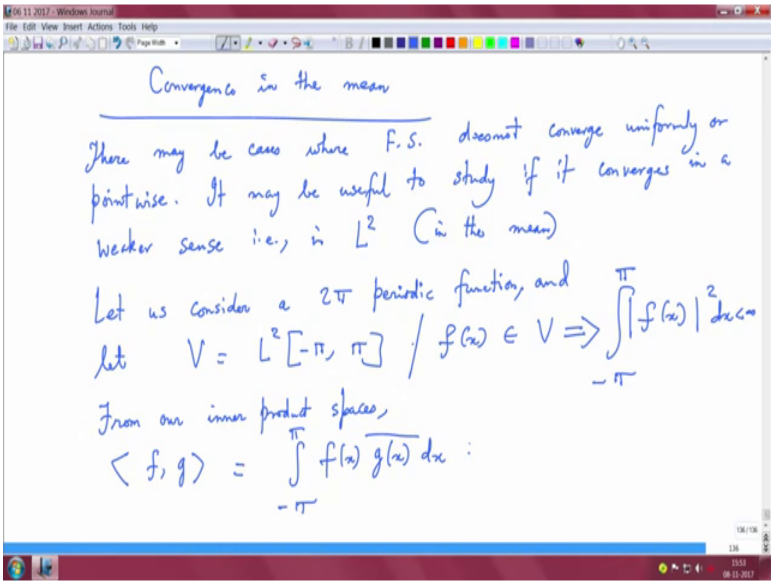

Now, let us slightly interpret convergence in the mean there may be cases, where Fourier series does not converge uniformly or point wise. It may be useful to study if it converges in a weaker sense that is in L 2 in the mean. So, I already gave you a hint that this is a weaker notion of convergence. So, therefore, you know sort of an answer to the previous question.

Now, let us consider a 2 pi periodic function and let V be the space of all square integrable functions from minus pi to pi which means is if f of x belongs to this space it implies that integral minus pi to pi modulus of f of x is less than infinity.

Now, from our inner product spaces the inner product between two functions f and g is given by this quantity, where you look at the conjugate of g if it is complices Hermitian inner product, ok.

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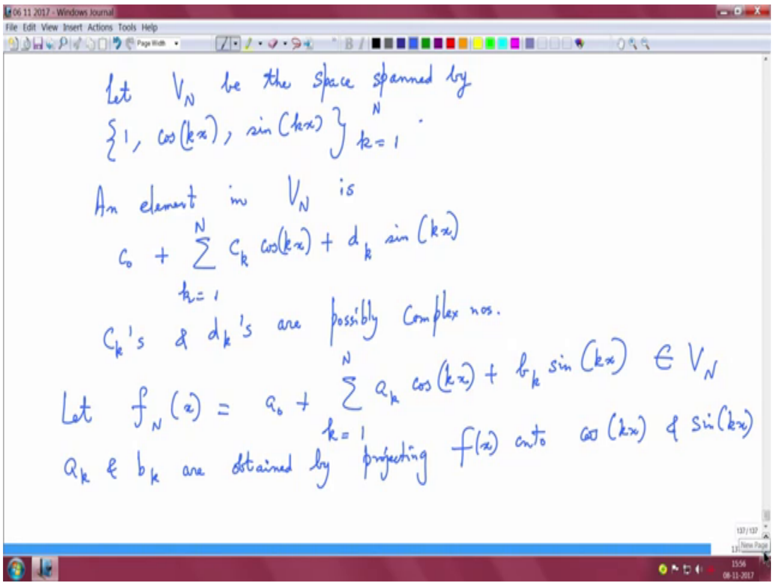

Let V suffix N be this space spanned by 1 cos kx and sin kx, k equals 1 to N an element in this space V N is C naught plus sum k equals 1 to capital N, C k cos kx plus d k sin kx, right. This is our notion of the Fourier expansion C k's and d k's are possibly complex numbers.

Now, if you say some f N of x is say a naught plus summation k equals 1 to N a k cos kx plus b k sin k x belonging to this space a k and b k are obtained by projecting f of x onto cos kx and sin kx, right. So, I think what we have to understand is we have recall a lemma that we did in module one that is the basics of a single geometry, right.

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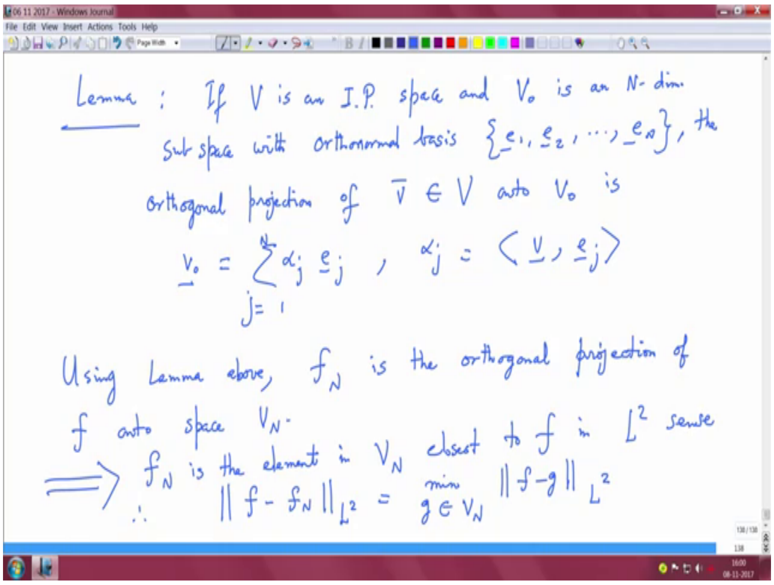

So, I will state this result as a lemma if V is an inner product space and V naught is an N dimensional subspace with orthonormal basis set e 1 e 2 dot dot dot e suffix N, the orthogonal projection of some vector V belonging to the space V onto V naught is this vector V naught, this is space V naught which is summation j equals 1 to N alpha j e j with alpha j is the inner product between this vector V and this basis e j.

So, we proved this result and it is exactly the interpretation of the flourier expansion right. At the end of graduate level you might have just taken the function just figured out that you want to multiply with cosines and sines and somehow compute those coefficients and therefore, you have a Fourier representation and you would interpret that this is your amplitude spectrum right, you know you look at the complexes in a form, complex form of the Fourier expansion and then interpret that as amplitude spectral density.

And then you know what portion of the spectrum contains energy what does not contain energy and you would have interpreted that in your electrical engineering, but this is all the mathematics behind that. It is just another way to understand and interpret the representation in a linear algebraic form and then in what sense you take the inner products there are faculties, right. So, I think this feature should be very clear in your mind. So, let us use the lemma here.

So, using lemma above f suffix N is the orthogonal projection of f onto space V N. So, this would imply that f suffix N is the element in V suffix N in the space closest to f in L 2 sense. Therefore, f minus f N in the L 2 sense is minimum over all g belonging to V N the norm of f minus g in the L 2 sense, right. This is the f N is the element this f N which is the Fourier expansion is the element in this case V N which is closest to the function in the L 2 sense,.

And, we want to bring in this L 2 notion because it is a weak slightly weak form of convergence, ok. Ideally, we would like to stick with point wise convergence or uniform convergence, but sometimes it may not be possible we may have to look into the weaker forms and at least the L 2 sense, we can basically describe these things in a subtle way, ok. So, now, we are sort of ready to revisit the earlier theorem that I stated and we will discuss the proof of the result.

We can stop here and I will begin with the next.