Mathematical Methods and Techniques in Signal Processing -1
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 70
Uniform convergence of Fourier series of piecewise smooth periodic function

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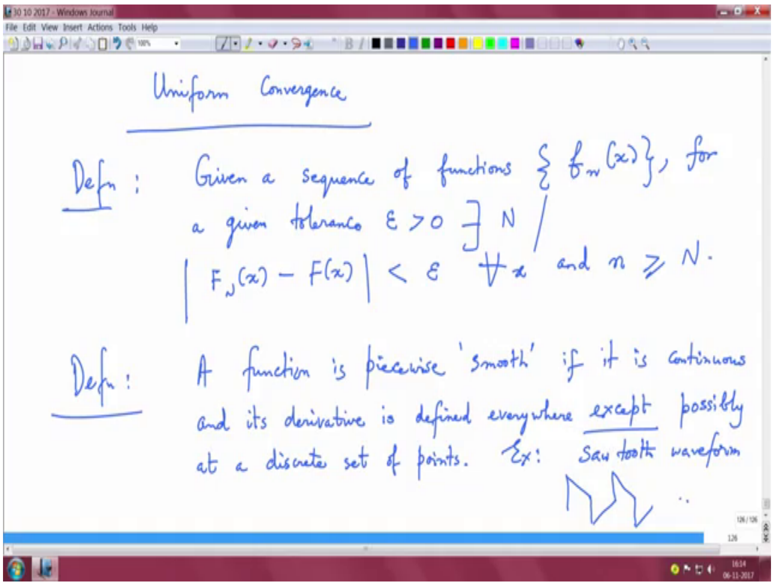

Now, I should have basically covered uniform convergence and point wise convergence, but I deliberately avoided this. But if you go through analysis lectures basically we will start with sequences and then you will start with what point wise convergence is and that leading into uniform convergence and implications and so, on. But, here we will just stress upon only certain ideas from analysis which are useful for us to discuss the convergence aspects of Fourier series ok.

One of the important concepts is uniform convergence so, let me define what this means. Given a sequence of functions f small n subscript small n x, for a given tolerance epsilon greater than 0 some positive tolerance. There exists capital N such that absolute value of F N of x minus F of x is less than epsilon for every x and all small n greater than or equal to capital N.

And this is basically what you reform convergences, I give you some function, I give you a tolerance epsilon, I give you some number capital N, it could be a million, it could be a billion some number. Then I take f of x Fourier series expansion, I take f n of x truncated to n partial sum and this is within epsilon for every value of x and all small n greater than or equal to capital N right. That is; that means, if I take the sum it basically is bounded, this is what I mean here is this what I mean here. And if the tolerance depends upon the point, then it is point wise conversions.

So, uniform is a more sort of a more general statement right. So, you can think about epsilon which you know which I can give you a tolerance. And the tolerance if it is dependent upon the point; that means, every if you evaluate the function and you look at the approximation the tolerance if it depends upon the point; that means, some point x naught I give you some tolerant tolerance epsilon, epsilon which depends on x naught, for some other it is absolutely depending on x1 so, on and so, forth.

Then it is for a very specific set of points. Here it is not dependent on any point, I give you some tolerance for all points x this is bounded with an epsilon ok, the absolute value is bounded within epsilon. So, there is another definition which is I will state a function is piecewise smooth, if it is continuous and it is derivative is defined everywhere except importantly except possibly at a discrete set of points. An example of this is the saw tooth stall saw too saw tooth waveform ok.

A function is piecewise smooth if it is continuous and it is derivative is defined everywhere except possibly at a discrete set of points at some points. So, saw tooth waveform is one of them I mean it is imagine this is a saw tooth form. Now, there is a result so, what does it mean? The smoothness property is very important because if it is it depends upon the number of derivatives.

If it is nth order smooth let us take the function f of x equals x, first derivative is a constant x, the second derivative is 0. It is vanishing take f of x equals x square, second derivative it exists right second derivative first derivative is 2 2 x, second derivative is 2 constant right. 2 derivatives exist after which it just vanishes.

So, if a function is nth or smooth if all the n derivatives exists and you take the next derivative it just vanishes next higher order derivative it just vanishes. So, when we talk about smooth functions it should be really differentiable many many times. Now you think about the Gaussian function, how many derivatives does it have? Many; So, this Gaussian kernel it is used useful in many many applications in signal processing because it is smooth, keep on taking the derivatives it is smooth to that extent right. This is one of the important ideas in signal processing.

And I think in the wavelet also the hour for example is not a nth order smooth function, just take the immediate derivative it just there is trouble at the points. And of course, we can investigate the time frequency uncertainty at those points; it is an interesting exercise.

But, I think what you have to bear in mind is since it is not smooth if you want a smoother approximation, you may need conditions in creating your wavelet functions etcetera. So, this is one of the important considerations in signal processing ok.

So, a little bit of digression, but I think you have to get the idea of what smoothness is.

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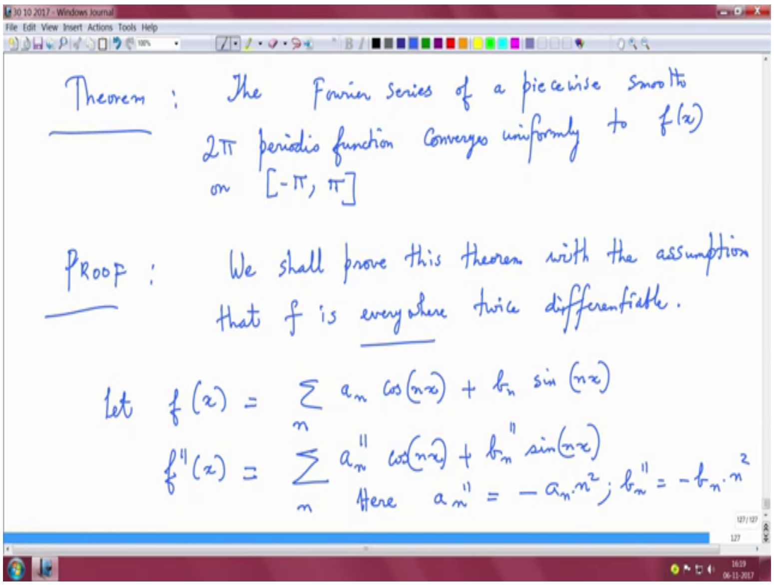

Now, we have a theorem, the Fourier series of a piecewise smooth 2 pi periodic function converges uniformly to f of x on the interval minus pi to pi. I mean f of x equals x is not smooth everywhere right. I mean it is exhibiting discontinuities when you try to periodically extend it does not Gibbs phenomena we saw that.

Now, imagine that we have periodic is it piecewise smooth 2 pi periodic function and the Fourier series of that function converges uniformly to f of x on the interval minus pi to pi you have to prove this result.

So, we should prove this theorem with the assumption that f is everywhere twice differentiable, we will see the condition why this twice differentiability needs to be satisfied ok. So, let us start with the function, let f of x be summation over n a n cosnx plus b n sin n x. Now, I take a derivative 2 times, this one thing nice about this trigonometric functions cos becomes minus sign when you take a derivative right, sign when you take a derivative it becomes cos.

So, taking twice derivative is not so difficult for us. So, if this is giving us a n double dash cosnx plus b n double dash sign n x, then we can relate a n double dash and b n double dash in terms of a n and b n ok. So, here a n double dash is minus a n times n square and b n double dash is minus these suffix n times n square. Not too difficult just take a derivative n pulls out take another derivative n pulls out, cos becomes minus sin sin becomes causes a negative sign, there it is a straight forward.

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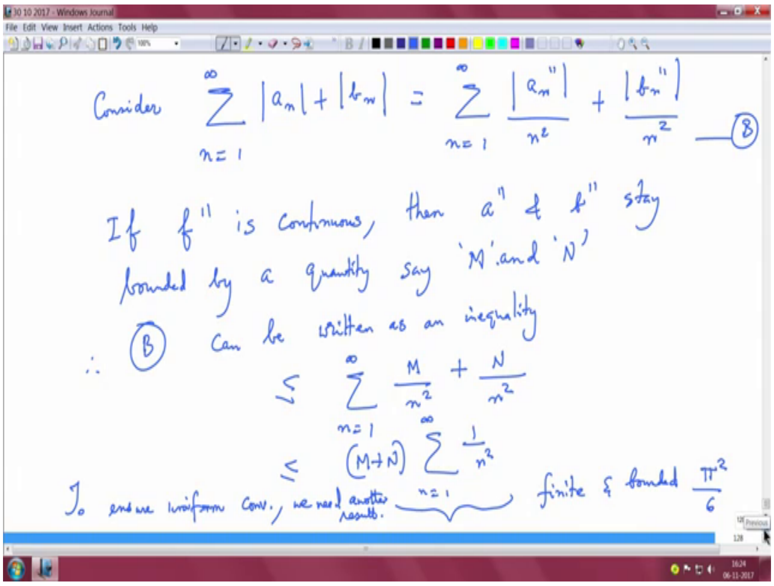

Now, consider summation n equals 1 to infinity absolute value of a n plus absolute value of p n. Now, this is equal to summation n equals 1 to infinity absolute value of a and double dash upon n square plus absolute value of b n double dash upon n square right because, we have the link between a n and b n. Now, if f 2 dash is continuous then a double dash and b double dash stay bounded by a quantity let us say this is some M.

So, therefore, the will I use A somewhere earlier? Maybe I can call this A set of equation, this is I call A. B can be written or simplified as this is bounded, this is less than or equal to summation n equals 1 to infinity say M upon n square. Let us say this is M and n if you want or you can bring in 1 quantity which is M which is max of M comma N you could do that as well, but let us just get it into this form.

So, this is basically M plus N times sum n equals 1 to infinity 1 upon n square and we know this is finite and bounded and given by pi square upon 6, it is not too difficult to prove this result. So, now, if M and N are finite, this is finite.

So, everything is finite therefore, you can say that this is not heading to infinity and this is this is basically bounded ok. So, is this is this clear? Now, to ensure uniform convergence we need another result and we will prove that in a in a lemma ok.

So, this is one part of it, if you look at the statement of this theorem it says it converges uniformly to f of x, piecewise smooth. So, first we said is we want to start with assumption it is twice differentiable and if it is twice differentiable we said if this is this function is continuous if f 2 dash is continuous.

Then their coefficients are bounded by some quantities M and N and then we replace that bound through an inequality we say that the sum of our absolute value of the coefficients is bounded. That is what we have established that that is; that means, it is finite it is strictly less than infinity ok.

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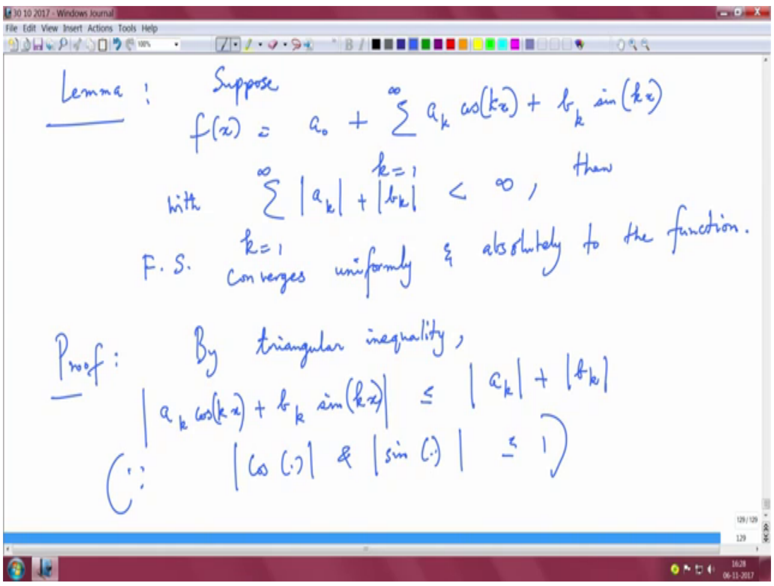

Let us get into the technical details of another lemma which is useful towards the main result. Suppose f of x is a naught plus summa k equals 1 to infinity k suffix k cos kx plus b suffix k sin kx with k equals 1 to infinity absolute value of a k plus absolute value of b k, this being strictly less than infinity. See, this is this condition is being satisfied in the previous result right, we took the absolute value of the coefficients and that is finite. If this result is true then Fourier series converges uniformly an absolutely to the function.

So, basically now in the first part of the previous result we have not proved it completely, we have just established this result we need to start with this condition here then the correct that the coefficients are bounded and then establish the uniform convergence property.

So, therefore, this lemma is nested inside the theorem ok. I could have done a lemma first and I could have proved something, but I want you to get a feed of the flow of the theorem. So, therefore, I am nesting the lemma literally inside the theorem so that you can understand how this works. So, we will start with the proof.

Let us start with our familiar inequality which is a triangular inequality, you take absolute value of a k a cos kx plus b k sin kx. This is less than or equal to absolute value of a k plus absolute value of a b k because, cosins and sin the absolute value of these functions is less than or equal to 1 ok. This is the first thing that you would do.

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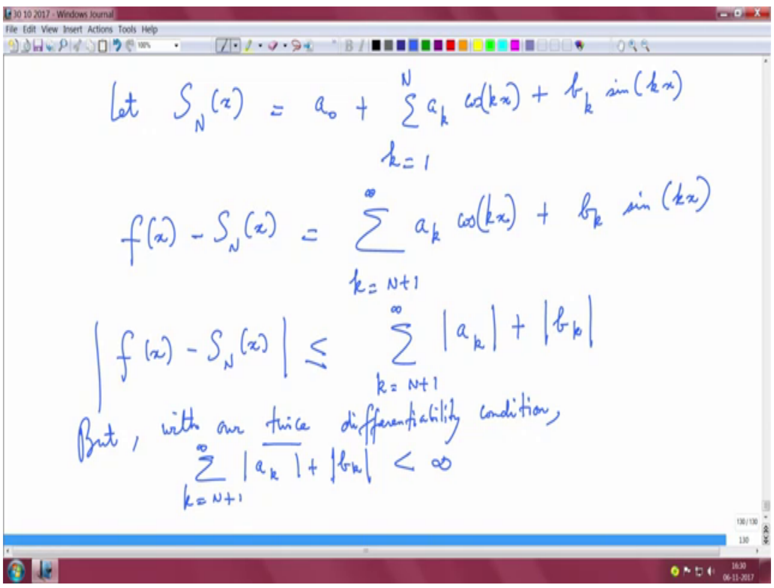

Now, let S suffix N of x be a naught plus summa k equals 1 to capital N a suffix k cos kx plus piece of xk sin kx.

Let us look at this partial sum. Now, f of x minus S N of x is summation k equals N plus 1 to infinity a k cos kx plus b k sin kx. Right, if you just remove this from k equals 1 to N, you are left with the pile from N plus 1 to infinity that you need to take a sum.

Now absolute value of f of x minus S suffix N of x, this is now less than or equal to summation k equals N plus 1 to infinity absolute value of a k plus absolute value of b k. But, with our twice differentiability condition summation k equals N plus 1 to infinity mod a k plus mod b k is less than infinity right? We prove that result because, that constant was heading to pi square way.

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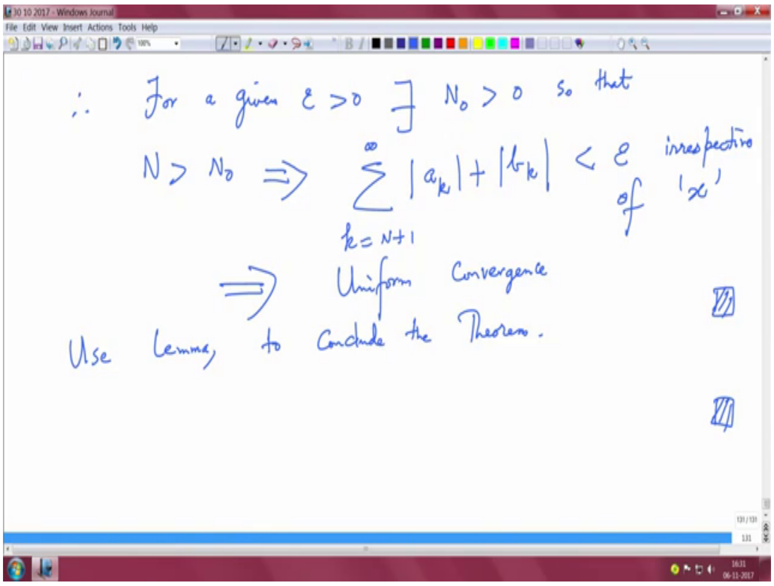

Now, this is easy for us so for a given epsilon greater than 0, there exists some capital N naught which is greater than 0. So, that for N greater than N naught this implies summation N plus 1 to infinity k equals N plus 1 to infinity absolute value of a k plus absolute value of b k, this is less than epsilon irrespective of x.

This is very very important. Irrespective of x this is strictly less than epsilon because, you could choose your M and N and all these things carefully. So, this establishes uniform convergence. So, I would just close the proof of this lemma here, I would also close the proof of this theorem saying use lemma to conclude the theorem ok.

So, this is a very important result, I think it is a significant theorem it starts here the theorem starts here and it it ends after the lemma is proved and then as a consequence of the proof of the lemma. So, the Fourier series of a piecewise smooth 2 pi periodic function converges uniformly to f of x on the internal minus pi to plus pi.

So, we will stop here.