Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 69
Convergence of Fourier series for piecewise differentiable periodic functions

So in today's lecture let us consider convergence at a point of discontinuity right we proved in the last class that there is convergence at points where it is continuous. And we arrived at conditions linking the dish slab kernel, and the function g right that that satisfies at the points of continuity right.

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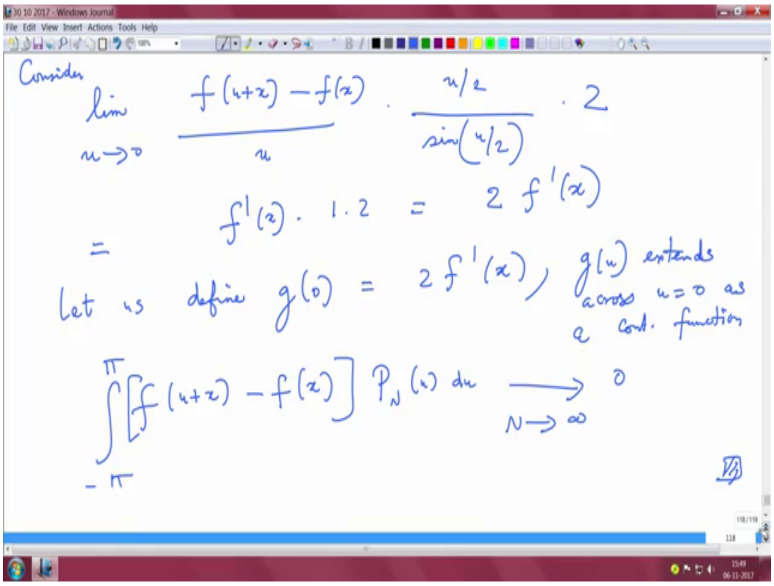

I mean we have that precise equation I can just pinpoint to you. So, if you said this g of 0 to two times f dash of x right then they can extend this function g of u across u equal 0, as a continuous function and this is a very important step to show that there is convergence.

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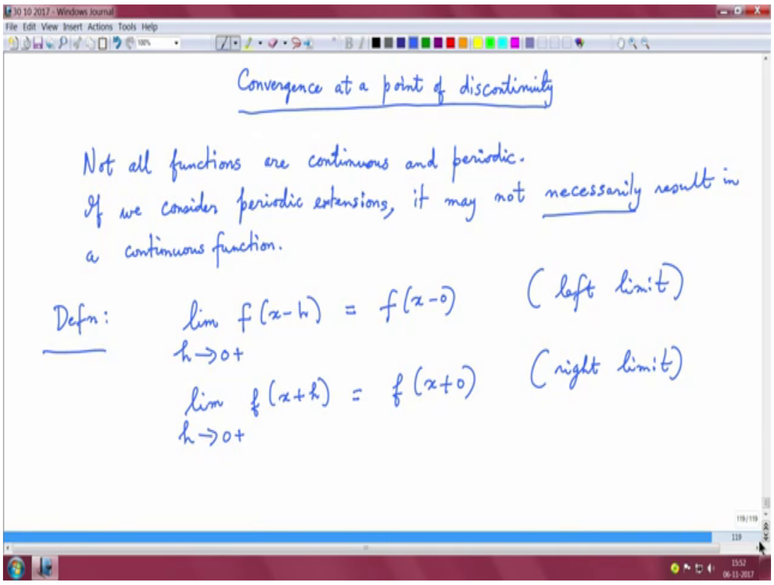

Now, let us look at convergence at a point of discontinuity. So, the first in the foremost thing that comes into your mind is not all functions or continuous and periodic. And the other part is if we consider periodic extensions it may not necessarily result in a continuous function.

There are plenty of such cases right what we did f of x equals x was one such case, we start with the following definition we know from basic calculus limit f of x minus h h tending to 0 plus is f of x minus 0, this is the left limit.

Similarly we have limit x tending to 0 plus f of x plus h is f of x plus 0 this is the right limit. When the function is continuous the left and the right limits must be the same and it has to evaluate to the function at that particular point right, but it may not be the case, but we will see the importance of these left and right limits. Similarly we can recall the differentiability conditions.

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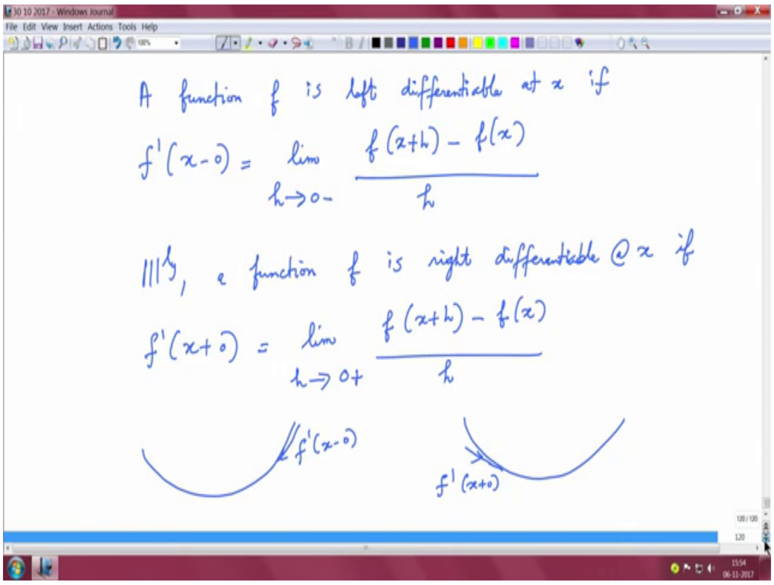

A function f is left differentiable at x, if f dash of x minus 0 is limit, h tending to 0 minus f of x plus h minus f of x divided by h this is the left differentiable condition.

Similarly, a function f is right differentiable at x if a f dash of x plus 0, is given by the limit h going to 0 plus f of x plus h minus f of x divided by h. So, you can think about if you are going like this, imagine your computing the tangent and you are going like this then this is basically derivative of x minus 0. On the other hand you have this function like this and you are approaching like this is x plus 0 right. So, this is a sort of air the pictorial notion that you have to get when you actually evaluate these this limits.

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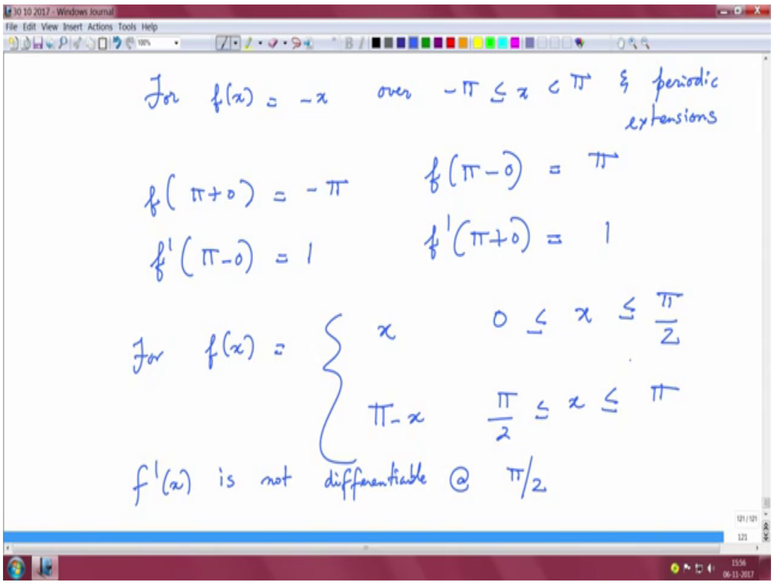

Now, for f of x equals minus x over minus pi less than or equal to x less than pi and periodic extensions f of pi plus 0 f of pi plus 0 is minus pi f of pi minus 0 is pi. If we take f dash of pi minus 0 is 1 f dash of pi plus 0 this is 1.

Now if you consider f of x is x for 0 less than or equal to x less than or equal to pi by 2 and pi minus x between pi upon 2 between x lying between pi upon 2 and pi f dash of x is not differentiable at pi upon 2. You can see the slope how things are changing when you are approaching just around pi by 2, whether you are just going to the left side and right side you take the derivative compute the left and right derivatives you will conclude that it is not the limits do not agree.

Now, once we get this kind of feel I mean you have a function it is not necessarily continuous, but we take an extension of the function and it there are discontinued continuity is because of the periodic extension. What can we say about the convergence at such a point or such points of discontinuities ok.

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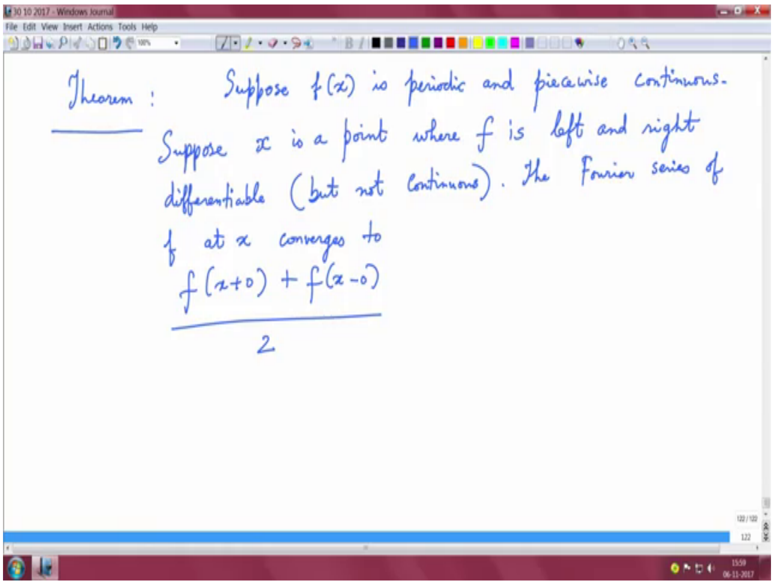

So, this leads us into a theorem the statement of the theorem is as follows. Suppose f of x is periodic, and piecewise continuous suppose x is a point where f is left and right differentiable, but not continuous.

So, look at the subtle point there is a left derivative left derivative the right derivative, but it is not continuous. The Fourier series of f at x converges to f of x plus 0 plus f of x minus 0 upon 2 i it is basically the mean of the function on either side mean a mean of the function evaluated at the left and the right values ok. So, let us see the proof of this theorem.

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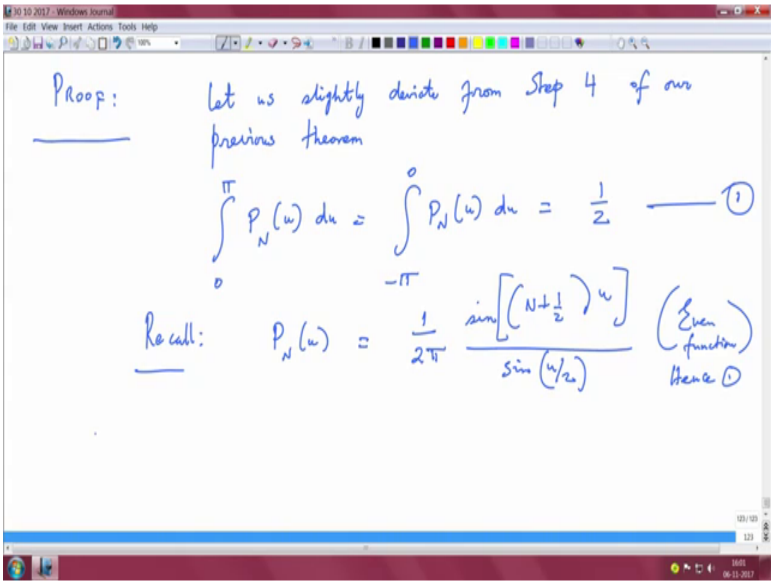

Let us slightly deviate from step 4 of our previous theorem. So, if you recall step 4 I think we said the cardinal integrates to 1 right from minus pi to pi that is the result. So, we will slightly deviate from that point. So, integrals it is a symmetric kernel.

So, integral 0 to pi P N of u d u is integral minus pi to 0 P N of you d u and this is 1 upon 2, let us call this equation 1 right. Just for your recall P N of u is 1 upon 2 pi sin N plus half u put this brace here upon sin u divided by 2 right these vibe are getting this result it is an even function. Therefore, hence this is equation.

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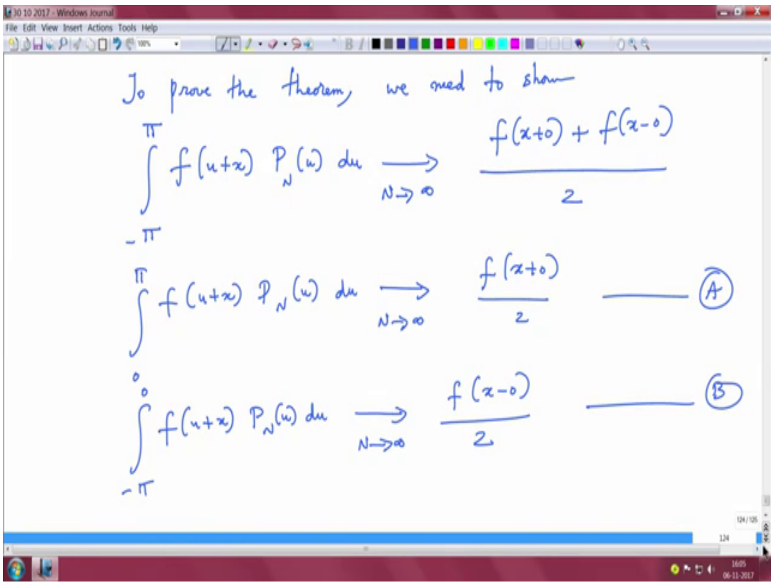

Now what will we need? What do we need? To prove the theorem, we need to show integral minus pi to plus pi f of u plus x times P N of u d u heads as N goes to infinity to this equation which is f of x plus 0 plus f of x minus 0 upon 2, so we already have a hint here. So, break this limit from minus pi to plus pi as from a minus pi to 0 and then 0 to pi ok. And then it is easy for you to see through this.

Now, integral 0 to pi, f of u plus x P N of u d u this heads to f of x plus 0 upon 2 as N goes to infinity. Just taking one portion right; and this is basically integral this is basically half of the whole kernel and therefore, this is just basically x plus 0.

Now, this is done symmetrically for the limit minus pi to 0, f of u plus x P N of u d u as N goes to infinity heads to f of x minus 0 upon 2 right. Why you would do it exactly the same way right you just put this x plus 0 here, mine you know exactly the way we did and show that that is basically heading to 0 right, using the same arguments that we did.

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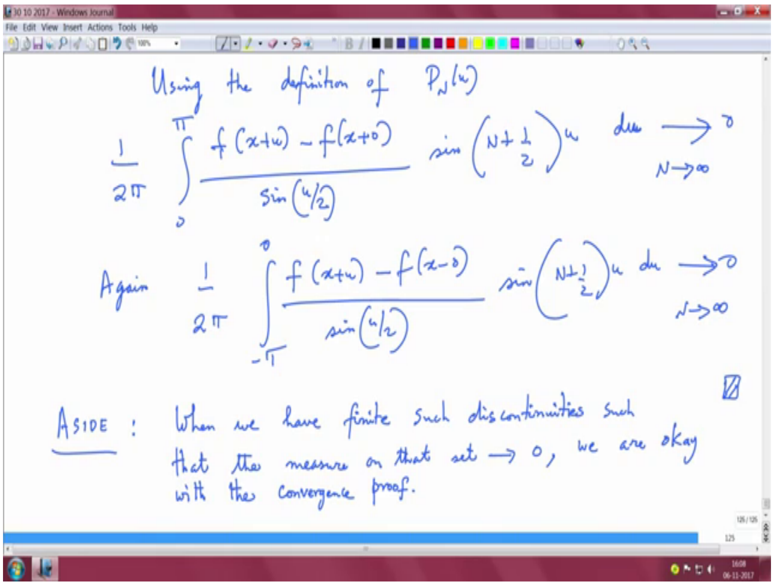

Using the definition of P N of u 1 upon 2 pi integral 0 to pi f of x plus u minus f of x plus 0 upon sin u by 2 this is basically the g of u function form sin of n plus one half u d u this hits to 0. And that is how we conclude that this is if you take this function out, you have this equation here which is I call this A, I call this B. This is where you get A from.

Now again 1 upon 2 pi integral 0 to pi the sets to 0. So, therefore, this theorem is established right, so you could put the limit here basically through minus pi to plus y just the symmetric. So, you take a limit from minus pi to 0 and here you take a limit from 0 to pi this establishes this result ok. Now one of the key questions that would be bugging your mind is what is the measure on the set of all such discontinuities right?

So, when we have finite or countably finite such, these continuities such that the measure on that set basically it is to 0. If this happens we are we are we are with the convergence proof right, this is very important result.

And I will not be discussing a lot of the aspects regarding measures and these analysis ideas because as out of the scope of this course, but I strongly encourage you to listen through the MOOC courses on analysis in math and you will see all these details coming up basically start from set theory and that is the reason why you do not want to have uncountably infinite number of discontinuities.

So, every point you are looking into is discontinuous or the measure on that set is basically not heading to 0, then you have trouble because it does not get to 0; these reasons you do not be able to establish ok. So, this is a an important, but subtle detail.