Mathematical Methods and Techniques in Signal Processing - I
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Lecture - 68
Convergence of Fourier series at a point of continuity

Ok. So, having proved the previous theorem, so, we will investigate slightly into the convergence at the at a point of continuity ok.

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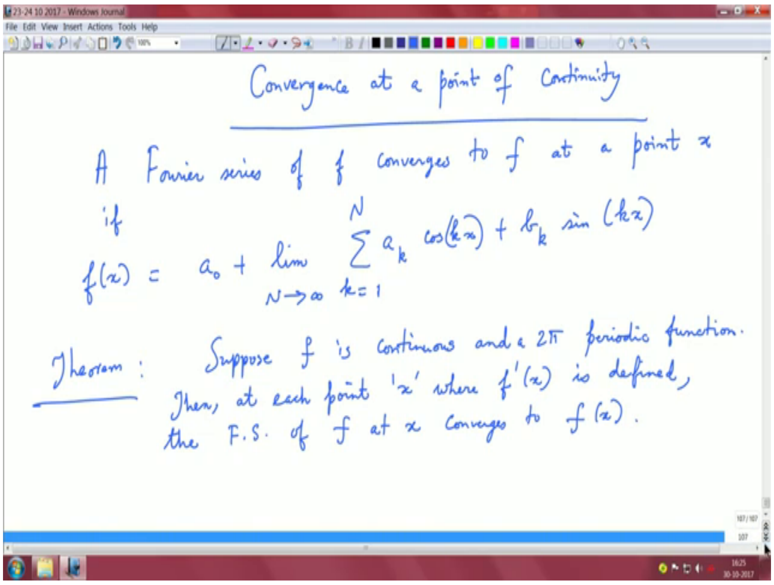

So, a Fourier series of some function f converges to f at a point x, if f of x is written in the form a naught plus limit N going to infinity summation k equals 1 to capital N a k cos k x plus b k sine k x. So, there are various notions of convergence whether it is point wise convergence, or uniform convergence etcetera right. So, I would basically refer you to analysis lectures for these various notions.

Now, what is the saying that I can take a series infinite series expansion and that series expansion is very close to the true function that is it converges to this function as N goes to infinity right capital N goes to infinity. So, with this we can state a theorem. Suppose f is continuous and a 2 pi periodic function, it is continuous and it is 2 pi periodic, then at each point x, there f dash of x that is derivative of x is defined the Fourier series of f at x so, at every point x you have a an expansion right at x converges to f of x. So, suppose f is continuous and 2 pi periodic, then at each point x where the derivative of the function is defined, the Fourier series of f at x converges to the true function f of x. So, we will prove this result there are many steps in the proof of this result and it is a long way long theorem.

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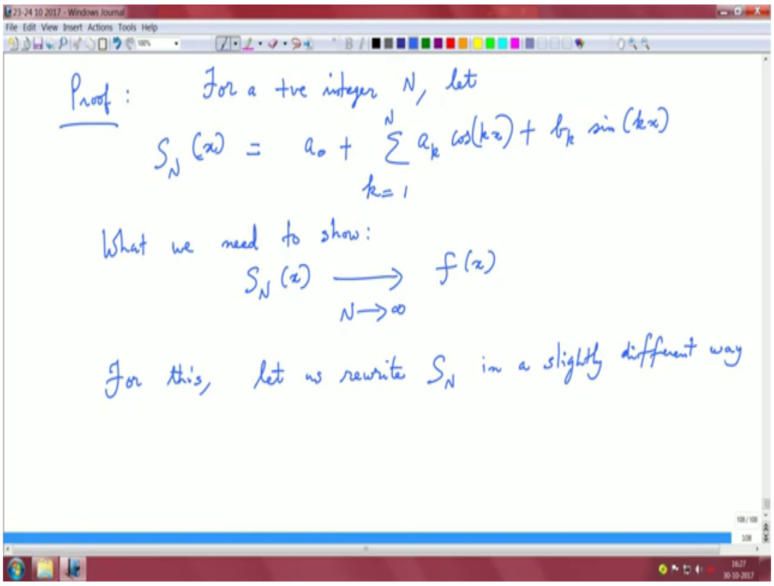

But I hope you appreciate the details in this in this result. [nose] Now, for a positive integer capital N let us define the sum S N S N of x is a naught plus summation k equals 1 to capital N a k cos k x plus b k k sin k x we saw this summation. Now what we need, what we need to show is very important right, we have to precisely put it in mathematical terms S N of x to f of x as N goes to infinity right, this is what we need to show. Now for this let us rewrite S N in a slightly different way ok. So, we do not see this through a sequence of steps.

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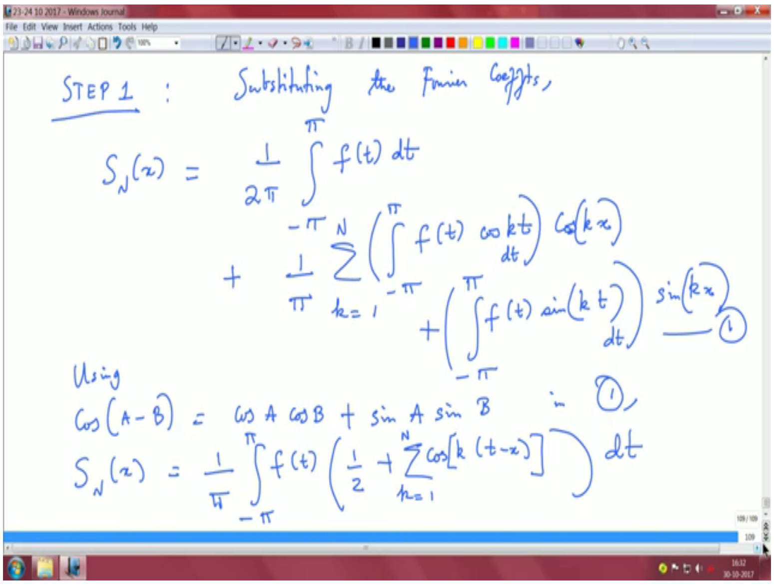

So, step 1 so, we substitute the Fourier coefficients and, when we do that we can write S N of x a naught is 1 upon 2 pi integral minus pi 2 pi f of t dt and, then a N is 1 upon pi integral minus pi to pi f of t cos k t dt. So, basically I write as summation k equals 1 to capital N, this is a series sum f t cos k t times cos k x. You see deliberately I have gotten this t and x because, the definite integral right. So, I can I can always write it in this form.

So, this 1 upon pi could be here I just pulled it outside, but you can see this is basically your A N right and, then your B N is you can put this here itself, this is basically integral minus pi to pi f of t sin k t sin k x and I have to put a dt here, whenever I write this in integral a dt.

So, now we can pull so, we can we can we can pull all the terms together right and, I can I can say I have a cos k t cos k x, then there is a sin k t sin k x right, I can pull all the terms together. So, I can say S N of x. So, and I can use this formula cos A minus B is cos A cos B, this is the compound angle formula cos A cos B plus sin A sine B using this, I call this equation 1, we can simplify S N of x this is 1 upon pi integral minus pi to pi f of t there is a half factor outside.

So, 1 1 by 2 pi integral minus pi to pi f t dt is this term. So, I will put a dt here ok. So, 1 upon pi f of t is half factor that dt is this term ok. Now I have f of t outside I can pull this out right, I have a cos k t cos k x sin k t sin k x and, I can write this as summation k equals 1 to capital N cos k into t minus x right. The same Fourier series is written like this, slightly different way ok, this is sort of straightforward.

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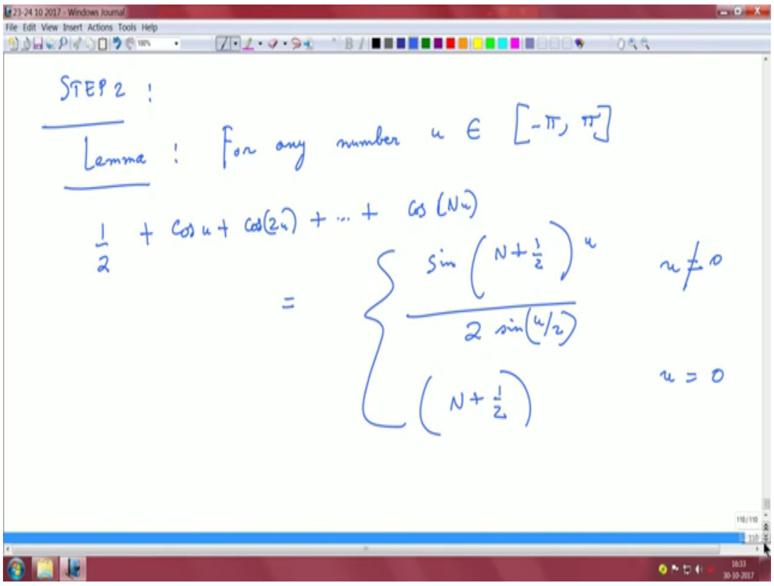

Now, in step 2. We have the following lemma, but writing is a little more unconventional I could start with sequences of definitions lemmas and theorems and I say yes invoke that lemma here, but I do not want to do this way because, I want you to get a feel of how you want to progress in proving the main theorem and, a step could be potentially a lemma, or it could be just some portion of a derivation. Now for any number u belonging to minus pi to plus pi 1 half plus cos u plus cos 2 u plus 1 cos N times u, this is basically sin N plus half u over 2 sin u upon 2 for u naught equal to 0 and equals N plus half, when u equals 0.

Then u equals 0 you can just apply L'Hospital rule and you can; obviously, see that this is sort of valid to this equation. Let us or if you just put u equals 0 directly, you have N terms here N plus half both the ways, you can immediately check the veracity of this result for this condition. Let us look at this part this is also very easy to straight forward just a trigonometric sum.

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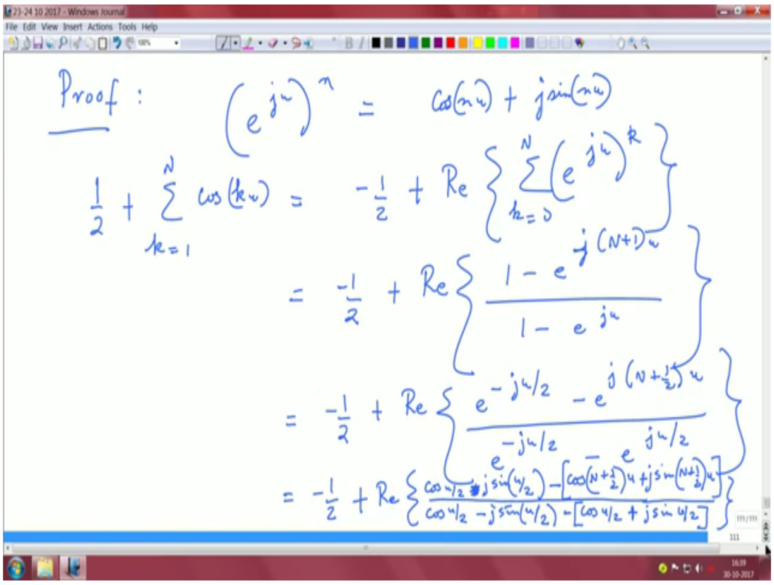

So, whenever you see this series sums you have to get the Euler formula in your mind ok. So, e power j u power n is cos n u plus j sin n u. Now consider 1 half plus summation k equals 1 to capital N cos k u, now when k equals if I want to replace k equals 1 to N by k equals 0 to N, then what I do is I put m minus half factor out plus real part of the summation equals 0 to capital N e power j u power k this is not really difficult to see write k equals 0 this is 1.

So, 1 minus half is half you have half here and the rest is basically take the real part you are going to get the cos no big deal right, this is minus half plus real part, now this is a geometric sum. So, you can write this in this form over j times N plus 1 u divided by 1 minus e power j u right. The common ratio is e power j u and the first term is 1. So, A into 1 minus R power N over m A into 1 minus R power N, or 1 upon 1 minus R R power N plus 1 over 1 1 minus R ok, you applied the geometric series sum here is what you this is what your are going to get.

Now, you simplify this further it is minus half that is a real part, I will get e power minus j u upon 2 minus e power j N plus half you, if I just multiply and divide by e power minus j u upon 2 for both numerator and denominator I get it in this form ok. So, what do I get here? So, I get a minus half plus a real part now e power minus j theta is cos u upon 2 minus j sin u upon 2 that minus and minus will cancel it is going to be a plus right cos theta ok. This is a minus here minus j sin u upon 2 minus, this is a cos N plus half u minus j sin N plus half u ok.

Now this is again cos u upon 2 minus j sin u upon 2 minus. So, there is some big minus here and then, I just have to put a brace here appropriately and there is a plus here ok. Now, here it is cos u upon 2 plus j sin u upon 2 it is just routine routine algebra right. If you are very shrewd you will notice that just group the j terms appropriately here, right and then just it will cancel out. So, this cos u by 2 minus cos u upon 2 will cancel out. So, the denominator u will be landing up with minus 2 j sin u upon 2 right.

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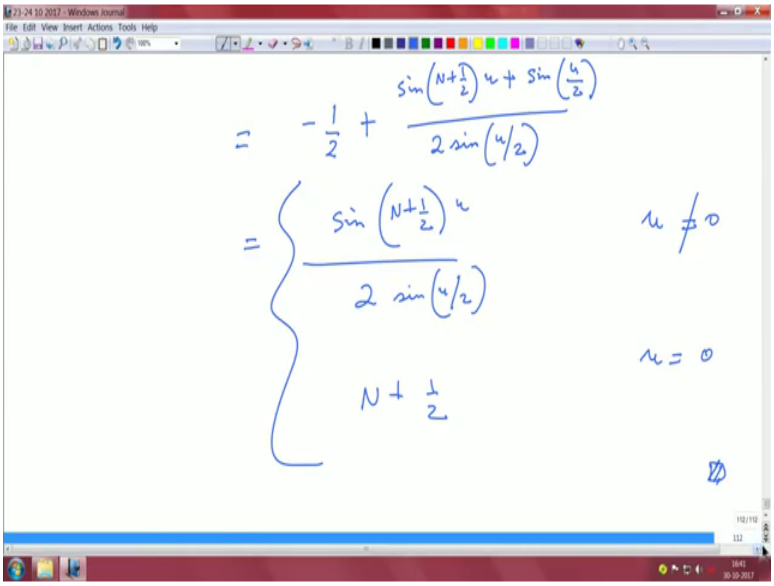

So, you have it is simplified this carefully, it is minus 1 half plus 2 sin u upon 2 and in the numerator, it is sin N plus half u plus sin u upon 2 and how do I get this, there is a minus j, there is a minus j, this is minus 2 j sin u upon 2 right because, it is cos u by 2 and minus cos u by 2 will cancel out and, then you will here have cos u upon 2 minus something. And then you grouped j terms you will have a minus j times sin u upon 2 plus sin n plus half u right in the j and j will cancel the negative will cancel out and, then you will end up with taking the real part, you will land up with this equation here.

It is sort of straightforward and, this you could simplify right because, the sin u upon 2 and this will cancel out this half and half will cancel out. So, therefore you will land up with sin n plus half u divided by 2 sin u upon 2 and, this is when u is not equal to 0, when u equal to 0 you apply L'Hospital rule and you will land up with N plus half because, denominator will cancel out ok.

So, this is indeterminate form this is 0 by 0 form apply L'Hospital rule differentiate. The numerator and denominator and take the limits as u goes to 0 and you land up with N plus half. And this is consistent because, you just add the terms when u equals 0 you have N terms N plus half is what we have. Now, this proves this lemma, but we have just addressed the second part, second step of the proof of this theorem.

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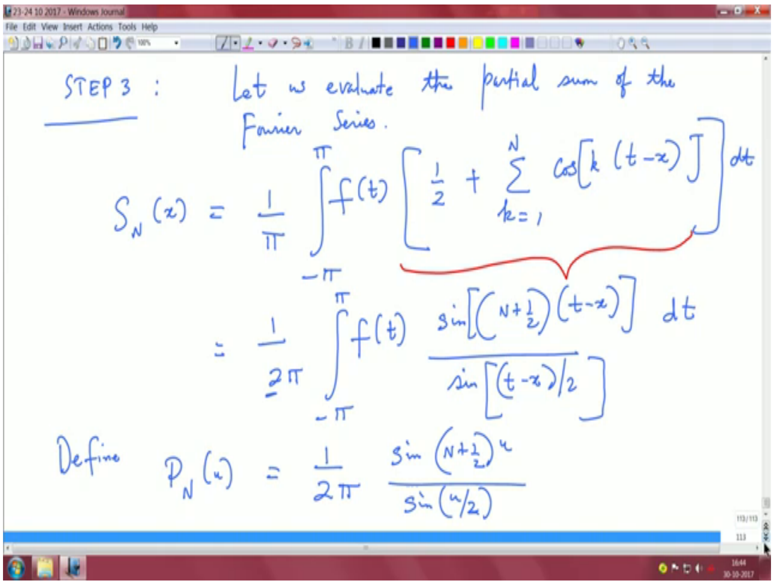

Then we go to step 3. Let us evaluate the partial sum of the Fourier series, we will evaluate the partial sum of the Fourier series. So, now we have S N of x is 1 upon pi integral minus pi to plus pi f of t and we have half plus sum of k equals 1 to capital N cos k times t minus x dt. Now I write this in this form 1 upon 2 pi integral minus pi to plus pi f of t using the previous lemma right, I can simplify this quantity as sin N plus half t minus x divided by sin t minus x upon 2 dt. The 2 factor that appeared I just pulled it here ok, just observe this to here ok. The factor of 1 1 upon 2 here, write that do I am pulling it at this step.

Now, if we define P N of u as 1 upon 2 pi sin N plus half u divided by sin u upon 2, if I define this function P N of you as this form.

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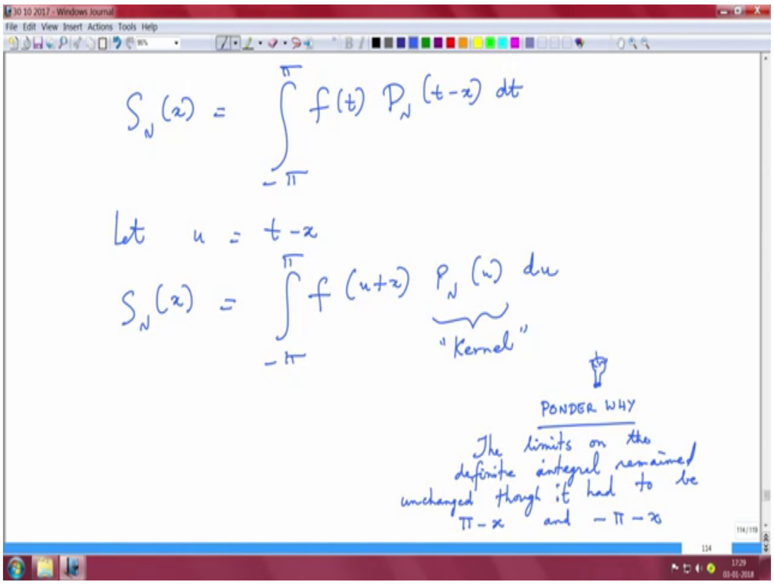

Then I can write S N of x as integral minus pi to plus pi f of t P N t minus x dt right. This is this is our sort of simplified form right, but let us go one step further, let u equals t minus x just a change of variable. So, therefore d u is dt, I can write S N of x is integral minus pi to pi f of u plus x P N of u d u. So, that it is in this form this is like the kernel ok, I ponder why the limits on the definite integral, remained unchanged though it had to be pi minus x and minus pi minus x, when you have to do a change of a variable here.

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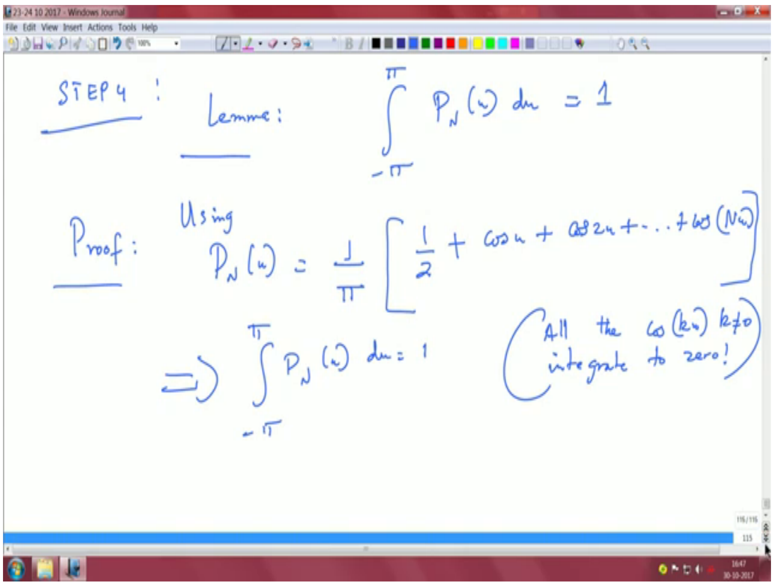

Now, let us go to step 4, there is an interesting property call this as a lemma minus pi to pi it is a trivial trivial result P N of u d u equals 1 that is if you integrate this kernel it gives 1. Now, how would you like prove this result so, basically you write P N of u in the original form 1 upon pi 1 half plus cos u plus cos 2 u plus so, on till cos N times u. Now where you integrate all these functions cos u cos 2 u so, on till cos n u that is that just vanishes, over the cycle. And 1 upon 2 pi you integrate from minus pi to pi it is 1 right, this is basically straightforward no big deal here.

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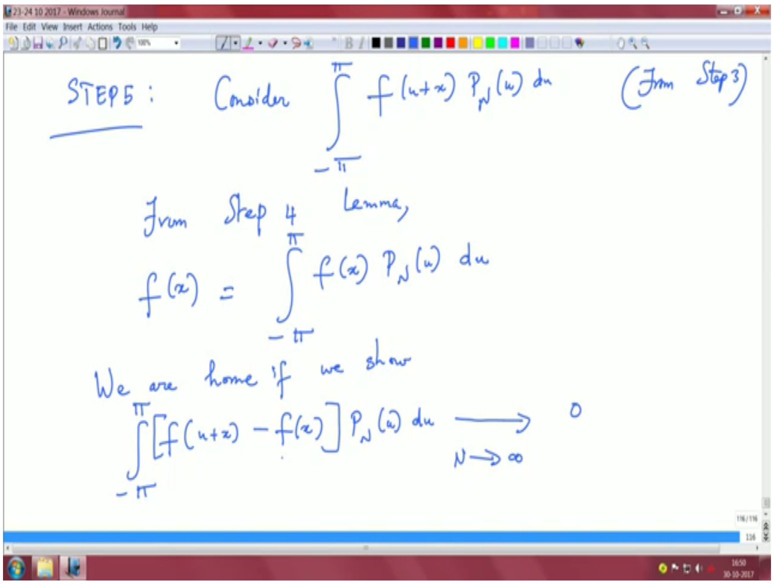

Now we will go one, one last step which is step 5. We start where we were in step 3. So, from the previous lemma right from step 4 lemma I can write f of x is integral minus pi to plus pi f of x P N u du trivial right, I just folded f of x inside to integral minus pi to pi P N of u du is 1. So, it is a very trivial result just observe that I am integrating with respect to u not x right I am integrating with respect to u and not x.

Now, this trick is important. We are home if we show integral minus pi 2 plus pi f of u plus x minus f of x times P N of u du heads to 0 as N goes to infinity, this is the important part. The second part is no big deal because, I am integrating with respect to u and f of x is a constant so, therefore, that is no big deal. This is important because, then I take f of u plus x and then I fold it because now there is no argument to you, which is appearing in the expansion of the function and, I have to show that this heads to 0 as N goes to infinity. Now you appreciate why this kernel and all these things are playing your own right.

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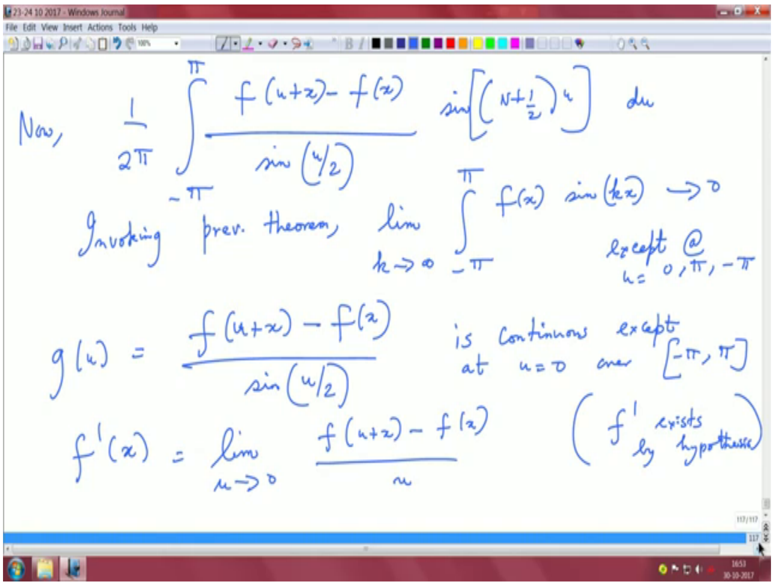

So, now let us simplify this now 1 upon 2 pi is what we have integral minus pi to pi f of u plus x minus f of x divided by there is a negative sign here, divided by sin. So, what I do is I just replace it by the kernel sin N plus half times u du. Now invoking previous theorem, limit k going to infinity integral minus pi to plus pi f of x sin k x heads to 0 as k goes to that is s k goes to infinity right, except at u equals 0 pi minus pi, that is some function I am basically expanding this and when this is basically modulating as N goes large this basically vanishes.

So, now let us see what is happening here. So, let me write this function g of u as f of u plus x minus f of x over sin u upon 2. So, this function is continuous except at 2 equal to 0 over minus pi and pi except at u equals 0 right, when u equals 0 it is in the form 0 by 0 form therefore, it is it is continuous except at this point.

Now the derivative of this function can be written as limit u going to 0 f of u plus x minus f of x over u right, it is the f of x plus u minus f of x over you as you going to infinity that is the derivative of t of t function that is the reason we say f dash exists by hypothesis, where the derivative exists this is where I mentioned because, we will make use of this fact.

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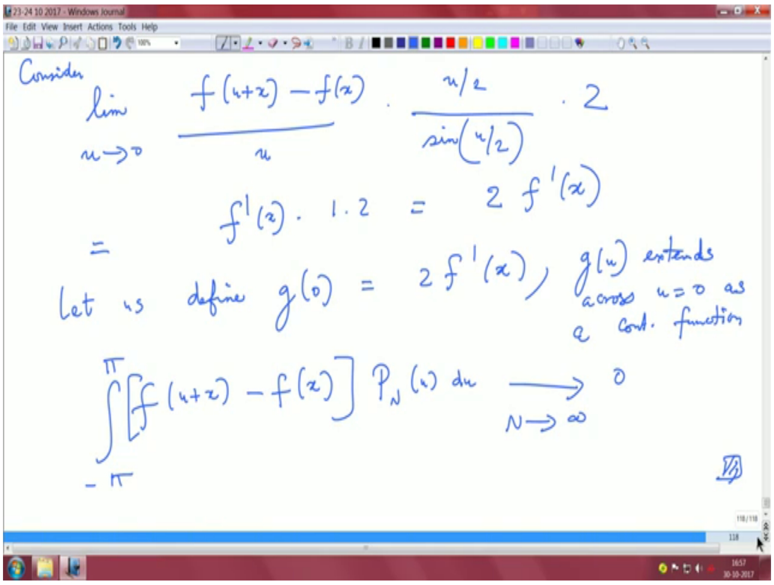

Now consider limit u going to 0 f of u plus x minus f of x divided by u times u upon 2 divided by sin u upon 2 times 2. So, this is by definition this is f dash of x times this is 1 this is 2 this is 2 times f dash of x. Now let us define g of 0 because, g of u is what is that function write g of u is that function which we defined like this and, we want to look at that limit as u goes to 0 because that is the point which we are interested in because, it was discontinuous point.

So, if you define g of 0 as 2 times f dash of x at those points x g of u extends across u equals 0, as a continuous function right at that point it is discontinuous, if I put this in this form it extends as a continuous function right. So, therefore we can we can conclude that this is at all the points that is u equals 0 and the rest of the points, integral minus pi 2 plus pi f of u plus x minus f of x times P N u du this heads to 0 as N goes to infinity right, we close this theorem here ok.

So, there are 2 subtle points, we had to note 1 is u being equal to 0 and when u is not equal to 0 and, when u not equal to 0 then we have no problem because, we invoke the previous theorem and therefore, that that integral basically vanishes right because you have some term here which is of this form times some frequency large frequency here, and that basically dives to 0 and we invoke the previous theorem right. So therefore, this is true but when u is equal to 0 we have a trouble because we have a 0 by 0 form so, we have to consider it carefully and using the derivative definition for the function f of x, we show that if this is possible and we extend g of u as a continuous function with u of 0 point then still we are and therefore, this expansion the Fourier series expansion in the limit form it converges to the function itself ok. So, in the next lecture we will discuss the convergence at a point of discontinuity and, then we will subsequently build our notions of various other forms of convergence of the Fourier series we will stop here.