Mathematical Methods and Techniques in Signal Processing – I
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Lecture- 67
Fourier series and notions of convergence

So, in the previous lecture we finished the ideas behind time frequency representation. So, in this lecture we will talk about Fourier series various notions of convergence, properties and so on. So, we will go slightly into some details concerning the Fourier series ok.

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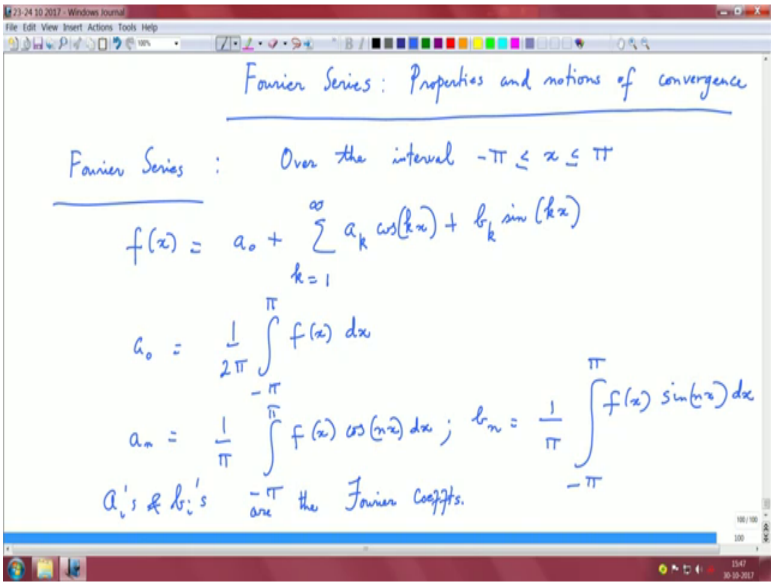

Now, just to recall most of you would have done this over the interval minus pi to pi you can expand f of x as a 0 plus summation k equals one to infinity ak cos kx plus b k sin k x ok.

So, this is a general representation for m series a naught is your dc term,which is one upon 2 pi integral minus pi to pi f of x dx and these aks are basically, the Fourier coefficients a n is given by 1 upon pi integral minus pi to plus pi f of x cos n x d x and b n is basically the inner product of f of x with sin nx right and a is bis are the Fourier coefficients.

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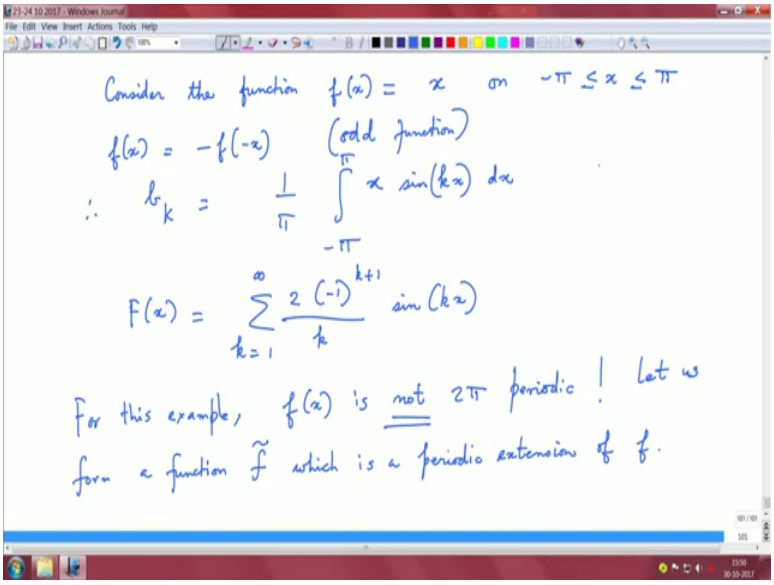

Now, let us delve a little carefully consider the function f of x equals x on minus pi to pi. So, this function is clearly an order odd function right, it is straightforward it is an odd function therefore, you do not have these ais and you can compute these bis. So, let us say the b k coefficient is given by this formula.

And if you did carefully integration by parts and compute at b k you can say that F of x is summation k equals 1 to infinity 2 times minus 1 power k plus 1, upon k sin k x this is what you get as an expansion right. Which most of you would have done in your undergraduate in electrical sciences, which of our electronics communication electric in whichever stream electrical sciences you would have done this.

Now, for this example F of x is not 2 pi periodical. So, let us form a function f tilde, which is a periodic extension of f ok. So, if you just sketch this function.

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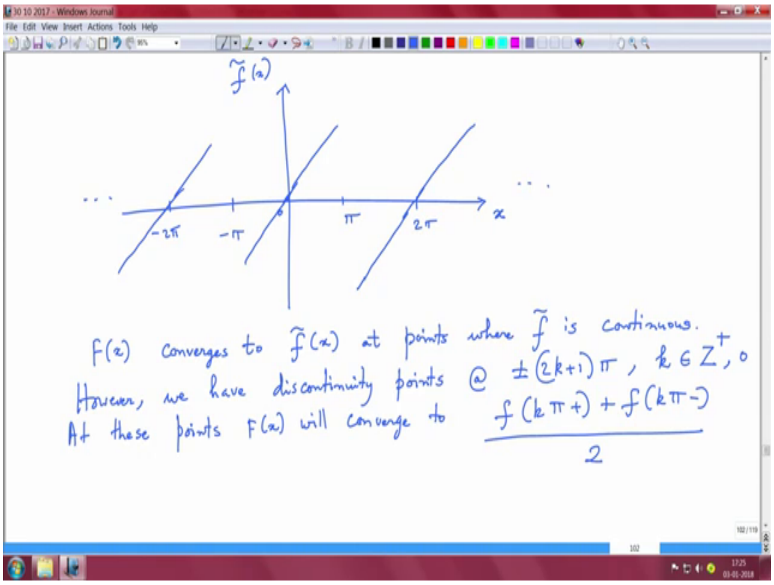

You, get these get the functions for 0 or then extended by 2 pi either side, if you can just write this as right. Now, observe that there are these jump points like plus minus k pi are points of discontinuities or jump form of discontinuity. So, F of x converges to f tilde x at points, where f tilde is continuous.

However, we have discontinuity points at plus minus k pi, where k is some integer right, and at those points at these will have to prove this result anyway we will go into the details and depth through the subsequent lectures. At these points F of x will converge to f of k pi plus f of k pi minus; that means, you take the left hand limit right hand limit and you take the mean value.

Of course, when we observe this figure the discontinuity points are at odd multiples of k here. So, therefore, I would just say that this is going to be at plus minus 2 k plus 1 times pi and k belongs to set of integers positive integers and you have to include 0 as well. And we would have basically studied this in our undergraduate curriculum ok.

So, now let us look at the finite sum case.

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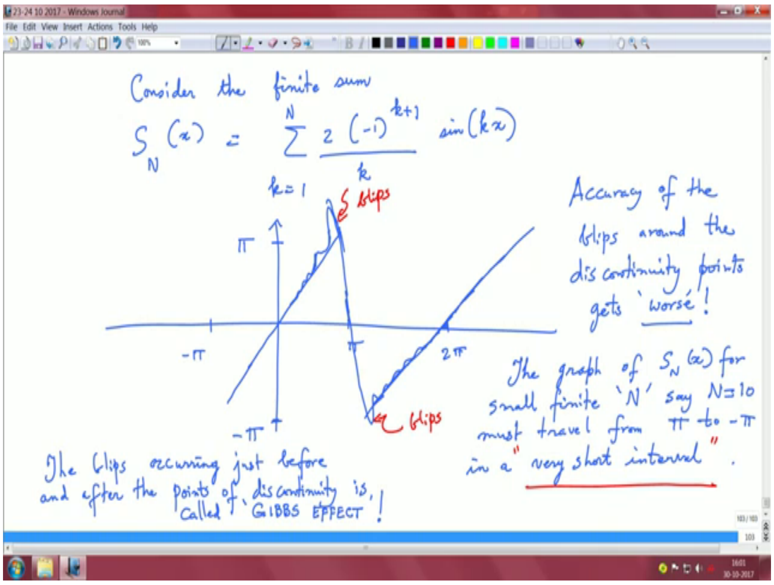

So, N instead of being infinite capital N being infinite is now finite; that means, I take the first capital N terms in the Fourier series expansion and then I compute.

Now, there is a weird thing that one would notice as a function of x if you plot the function right, ideally it has to be like this. Now, at this point there is a sharp jump right. So, if you plot this and graph this function S N of x is capital N of x for different values of N you will see that there will be some blips, at this point and then the signal just jumps through this point I mean from at this point pi there is some positive value it has and from some positive value it has suddenly jumped to a negative value the thing is really sharp right.

That means, the graph of this function must travel from if you just look at what this value is right I mean f of x equals x. So, therefore, this point is pi this point is minus pi on the ordinates right. So, from pi to minus pi it has to just jump in a very small interval and this you can sort of imagine like a waterfall or it is called the waterfall effect.

So, imagine you have a river it is going straight and then as you approach a waterfall I do not think anybody would want to just go through the process of going through the boat on the on the river with leading to a waterfall, because you can see the rapids right as the rapid increase in exactly the same fluctuation phenomena, which is happening here there is rapids and then there is a huge fall right.

So, we would like to investigate to so, this is because of a truncation and this point of discontinuity. So, at this point of discontinuity you are trying to represent this summation and this is not accurate limit is not defined at this point there is a sharp jump right. So, therefore, at this point of discontinuity the accuracy of these blips, it really gets worse right I mean as; that means, if you sketch 20 100 1000 so on for N, you can see that there is a variation in this height and then the oscillations and this phenomenon is called blips effect.

So, I call this as blips accuracy of the blips around the discontinuity points gets worse. So, what it means the graph of S N of x for small finite N capital N say N equals I just take 10 terms right, must travel from pi to minus pi in a very short interval. So, this is something like the you know like imagine you have a very small orifice and you just have to just go through that point of discontinuity right. So, the blips occurring just before and after the points of discontinuity is called Gibbs effect.

This is a very important phenomena I do not know I mean you have to really exercises you have to write a program compute this. And then observe, what is happening right and an artist's way of visualization of the Gibbs phenomenon is imagining, that you are going through a river rapid and then there is and if you are finding the oscillations are becoming larger you better beware that there is a steep fall expected right it is it is like that.

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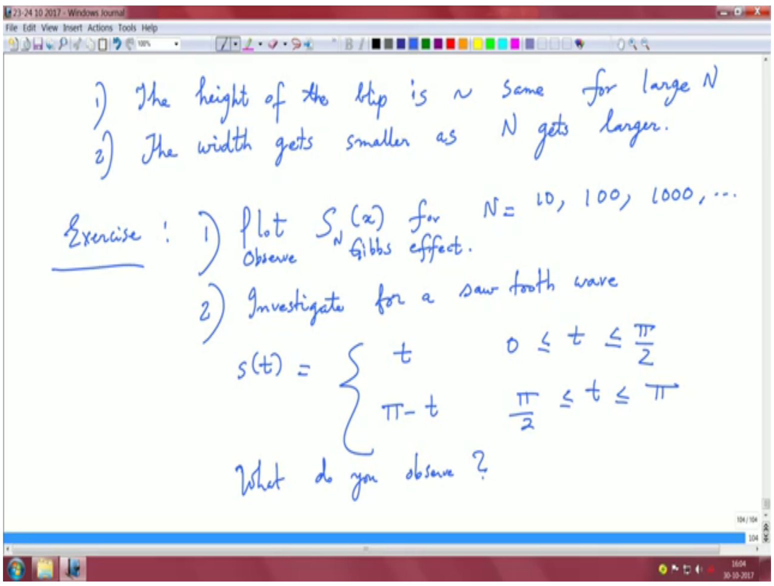

Now, a few things to note the height of the blip is approximately the same for large N and the width gets smaller as N gets larger. So, this is a through observations actually I will give you a homework when you will actually prove that these results are indeed correct.

So, what I would like to do is like you to do is plot S N of x for N equals 10 say 100 1000 dot dot and sketch this observe Gibbs effect, then I would also want you to investigate for a saw tooth wave, which is given by S of t equals t 0 less than or equal to t less than or equal to pi upon 2 and pi minus t from pi upon 2 less than or equal to t less than or equal to pi right. I give you this function observe what do you observe and what do you what can you conclude from this. So, this is just a homework exercise for you.

So, it is just not computing the Fourier series is what we have done? I mean we have to understand many subtle details in this series expansion right can we approximate all functions are there any conditions, what does the function take at the point of continuity? What does it take at the point of discontinuity right? So, on and so forth and these subtle issues are what we will be dealing with through the next set of lectures ok.

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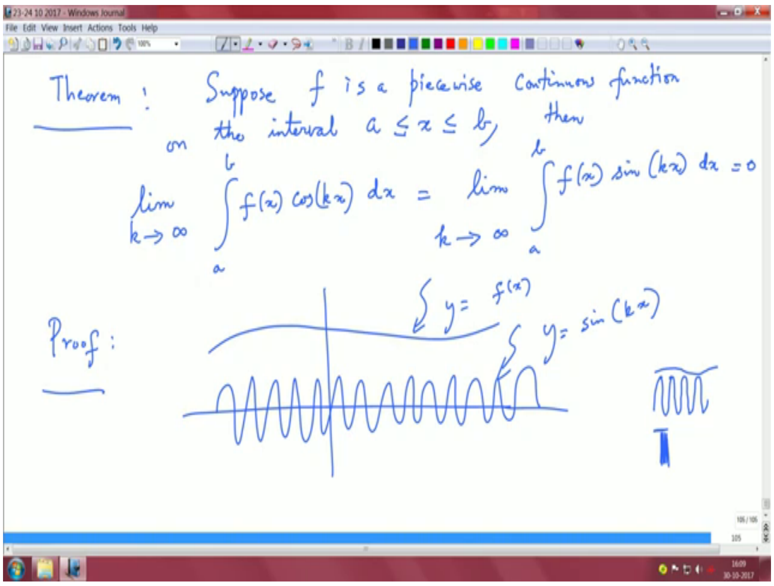

So, we will first start with a theorem suppose f is a piecewise continuous function on the interval a less than or equal to x less than or equal to b, then limit k going to infinity integral a to b f of x cos kx dx is limit k going to infinity integral a to b f of x sin kx dx and this is equal to 0, f is a piecewise continuous function.

So, what does this mean? So, a piecewise continuous function has only finite number of discontinuities right, at those piecewise points. And therefore, they are only a finite number of Fourier coefficients larger than a certain absolute value given a certain positive number right, I give you certain positive number only a finite number of Fourier coefficients, which are bigger than this number and basically this implies you can do data compression for these type of functions or signals I use them interchangeably right.

So, let us see the proof of this result. So, you can imagine that you have a high frequency wave this is almost like, my sin wave right and I have some function that I am sketching this is y equals f of x and this is like my y equals sin kx ok.

Now, as the frequency increases right f of x is nearly constant between 2 adjacent periods of sin kx and cos kx right. So, what is happening is if I take imagine this really very high frequency and I have a function here some function I increase this frequency a lot right you can say that this between these 2 peaks it is a nearly constant right I mean this is really not a very high frequency, but I imagine terahertz if tera is not sufficient for you just bring in some huge number for frequency; that means, it is almost you really cannot see it, it is as if it is piecewise flat I mean this is this is even high frequency probably right, if you just put a function around that you probably see it is it is constant right.

Now, this is the idea. Now, we take that high frequency and then we you know you take the inner product of that with this function and let the frequency go to infinity you have to show that this result is 0. I mean you can it can imagine why this is true, because over a cycle the value is constant and the swing basically gives you a positive value and a negative value it should hopefully sort of cancel out, but they will actually prove this result, well as the frequency k increases f of x is nearly constant on 2 adjacent periods of sin kx and cos kx.

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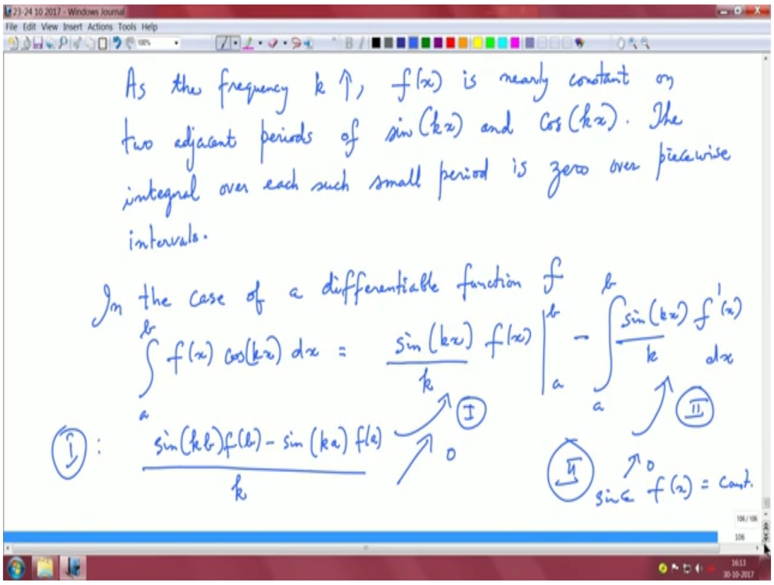

The integral over each such small period is 0 over piecewise intervals. Now, in the case of a differentiable function f let us assume that this function is differentiable. Now integral a to b f of x cos kx dx I am considering this integral this inner product here.

So, this is sin kx I am integrating this by parts first function integral of the second function minus integral of the second function derivative of the first function since, I need to take the derivative of the first function I am introducing that it is a differentiable function ok. So, this also gives you a hint sort of how mathematicians work is also playing around with these quantities right, I mean integrating the second function from a to b minus integral from a to b of this form.

Now, let us look at each of these terms and call it 1 I call it 2. So, now, turn 1 is basically sin k be f of b minus sin k a f of a divided by k. Now, f of x is a and nearly constant and therefore, the sign is bounded function. So, therefore, this basically heads to 0 heads to 0. And in the second part right so, you have to use the fact that when k goes large f of b is f of a right, and then apply this rule here it is straight forward. And then in this part the derivative of this function is going to be 0, because it is a constant function. So, this vanishes. So, 2 is also heads to 0, because f of x is constant when this is really large, because of this observation.

So, with this we are ready to discuss the notion of convergence at a point of continuity ok.

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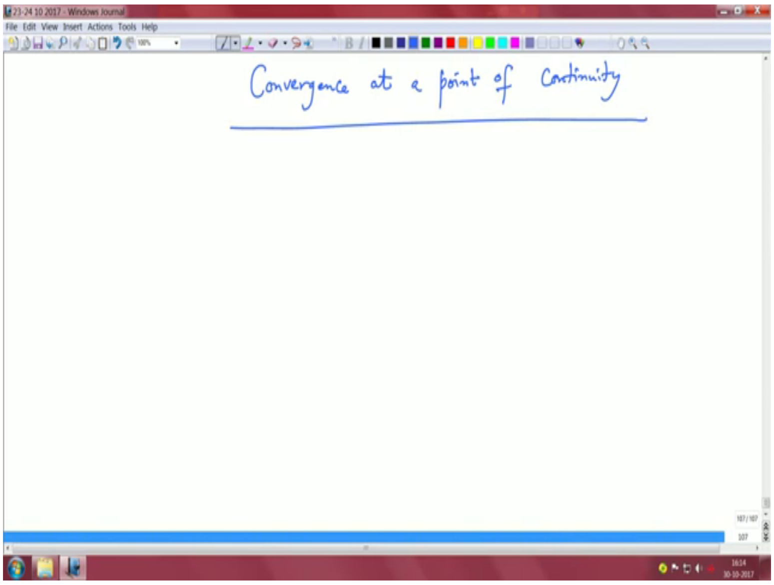

So, this is a just going back, we this concludes I mean you can take the sin you can take the cos it is the same thing it does not really matter. So, therefore, this concludes the proof. So, let us stop here if you have any questions we will take it and then we will get on to the next.