Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 66
Basic Analysis: Convergence of Sequence of Functions

Let us today discuss some notions and properties on convergence of functions. So, we discussed the properties of what continuity means in what uniform continuous functions are. Let us extend this notion to convergence and then connect convergence with continuity, ok.

So, before we delve into the details why this is useful because since we are studying Fourier series some of the properties we want acclimatize ourselves with notions of convergence. So, keeping this in mind let us get started.

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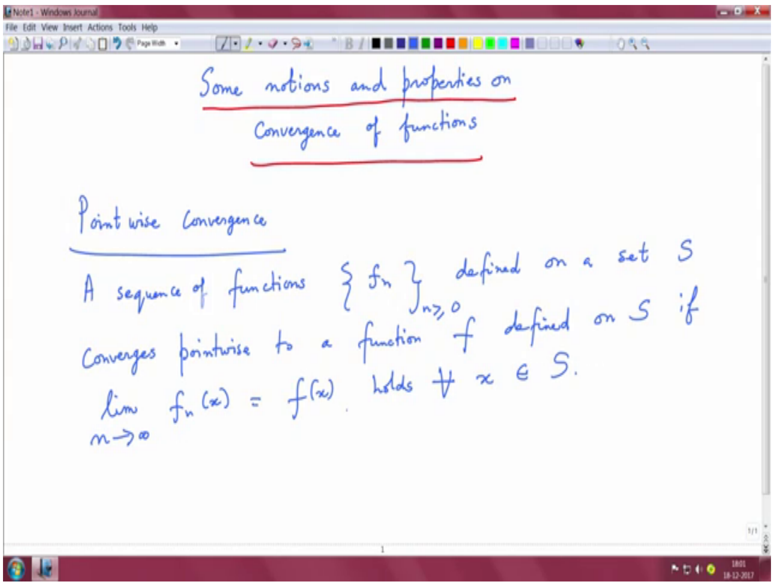

So, first is point wise convergence. A sequence of functions you can interchangeably use functions you can use signals defined on a set some S converges point wise to a function f defined on S if limit as n goes to infinity f n of x equals f of x holds for all x belonging to this set.

So, we will go a little deeper into what this definition means right. So, you have a sequence of functions that are defined on a set S and that converges point wise to a function f defined on S if this condition holds.

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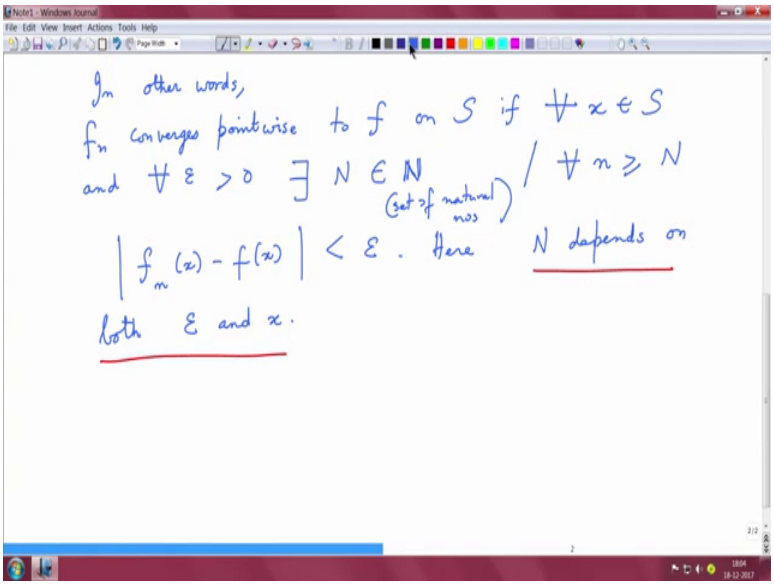

Now, in other words what this means is f n converges point wise to f on this set S, if for every x belonging to this set S and for every epsilon greater than 0 there exists some capital N that belongs to the set of natural numbers. This script n is basically set of natural numbers such that for every small n which is greater than or equal to capital N the absolute value of f suffix n small nx minus f of x is less than epsilon and here N, capital N depends on both epsilon and x.

So, f n there is a sequence of functions converges point wise to some function on that set S if we pick some x belonging to that set S and then for every epsilon that is greater than 0 there exists some capital N a natural number such that for every small n greater than or equal to capital N this function is within epsilon of the limiting function right f n x minus f of x absolute value is less than epsilon. So, n here depends upon both epsilon and x. So, pick the initial point x pick epsilon the capital N depends upon both epsilon and x.

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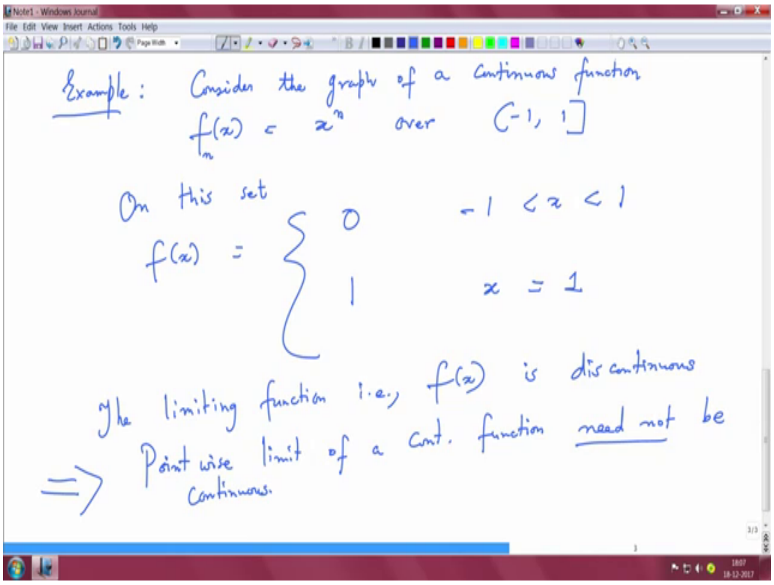

Now, let us see an example. Consider the graph of a continuous function f of x equals x power n over this interval minus 1 to plus 1 this is semi open I mean it says open on the origin on the negative side and it is closed from the positive side 1 is included and minus 1 is not included. So on this set that is over the semi open interval minus 1 to 1 f of x equals 0 for minus 1 less than x less than 1 and 1 when x equal to 1 right, as n goes to infinity because these are bounded between minus 1 and one 1 minus 1 and plus 1 not included in this range basically decays to 0 and then when x equals 1 because 1 is part of the set 1 power n is basically 1.

Now, if you observe this function f of x the limiting function right I mean this should be if nx here, f n x is x power n. Now, if you observe the limiting function right the limiting function that is f of x is discontinuous and this was part of your homework exercise that you know if you have a jump like this right it is a discontinuous function I gave you a homework exercise last time and this is basically this continuous function. So, what it implies is the point wise limit of a continuous function need not be continuous right, we may start with a continuous function like f n x which is x power n and then if we look at the limiting value of this function this function need not be continuous.

So, there are many pathological cases that you can see with point wise limits if you look at the point wise limit of a sequence of differentiable functions they need not be differentiable. Similarly the point wise limit of a sequence of integrable functions need not be integral, right.

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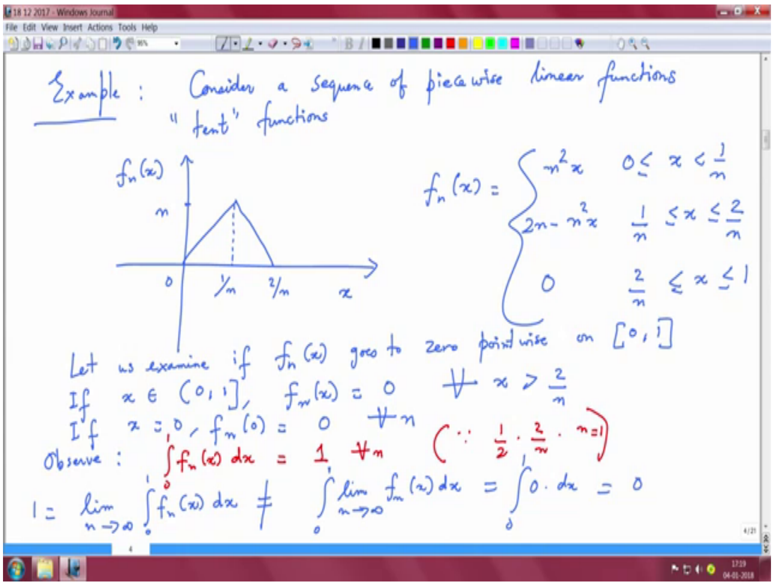

Another example would help us here to understand what we are talking about with these limits. Consider a sequence of piecewise linear functions and these are called tent functions because I will sketch this graph how they look.

So, this is x this is you can call a triangular function retained function because it looks like a tent here this is 1 upon n, this is 2 upon n and this is n, this is f and x, and it is piecewise linear as you can observe is an up ramp and this is a down ramp. So, you can describe this function as n square x in the interval 0 less than or equal to x less than 1 upon n, it is 2 n minus n square x in the interval 1 upon n less than or equal to x less than or equal to 2 upon n and it is 0 for 2 upon n less than or equal to x less than or equal to 1.

So, now, let us investigate into how f n of x dives point wise over the interval 0 to 1, ok. So, let us examine, let us examine if f n of x goes to 0 point wise on the interval 0 to 1 it is basically closed on both sides. Now, if x belongs to this semi closed interval then f n of x equals 0 for all x greater than 2 upon n because you can choose your n and x depending on n such that this is a 0 function. If x equals 0 f n of 0 equals 0 for all n because you have n square x you plug x equals 0 then for all such n this is basically 0 therefore, this function f n of x goes to the 0 function point wise over this interval 0 to 1.

Now, we have to investigate the following. Observe the look at the integral of the function f n of x and look at the limit of the integral of this function and look at the integral of the limiting function and see if what we can infer from these two cases. So, one thing that you need to observe integral from 0 to 1 f n of x, dx equals 1 for all n this is because you are looking at the area of the tenth function here which is half times the base is 2 upon n times the height is n right and this evaluates to 1. Now, the limit as n goes to infinity of the integral of this function this is basically 1. So, 1 is basically limit of limit as n goes to infinity integral of this function because this integral is 1, this is let us examine if it is equal or not equal.

Now, consider the integral of the limit function the limiting function as we discussed earlier is the 0 function because it goes to 0 point wise. So, therefore, if you integrate the 0 is basically which is 0. Therefore, these two limits are not same. So, therefore, the limit of the integral of this function is not equal to the integral of the limiting function. So, this is an important observation that one might make mind while dealing with point wise limits. So, you may wonder if point wise limits are or anything useful at this stage, ok.

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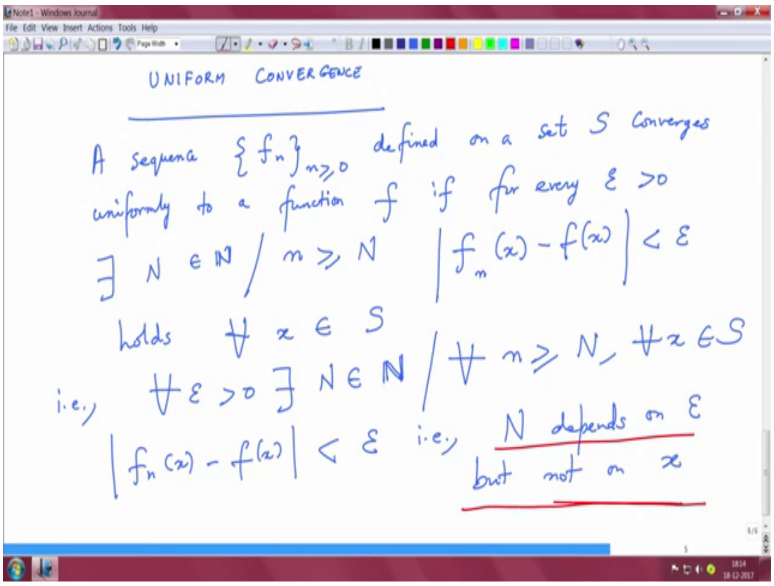

There is a useful notion called uniform convergence, and let us discuss uniform convergence a sequence functions f n defined on a set S converges uniformly to a function f if for every epsilon greater than 0. There exists some number capital N whose belonging to the set of natural numbers such that for small n greater than or equal to capital N absolute value of f n of x minus f of x is less than epsilon holds for all x belonging to this set S. So, you have to look at the order here it is very important, which means for every epsilon greater than 0 there exists some capital N belonging to the set of natural numbers such that for every small n greater than or equal to capital N and for every x belonging to this set S absolute value of f n of x minus f of x is strictly less than epsilon.

That is capital N depends on epsilon, but not on the initial point, x this is a very important difference when we think about uniform convergence versus point wise convergence. That means, for point wise convergence this function is bounded within epsilon and his bound depends; that means, for every point x and a given tolerance epsilon you can you can figure out some integer capital or some natural number n which can satisfy this condition. But if capital N depends only on the tolerance epsilon and not on the initial point it is very important and not on the point then it is uniform convergence and this is a very important idea and we will see this quite often very useful for us.

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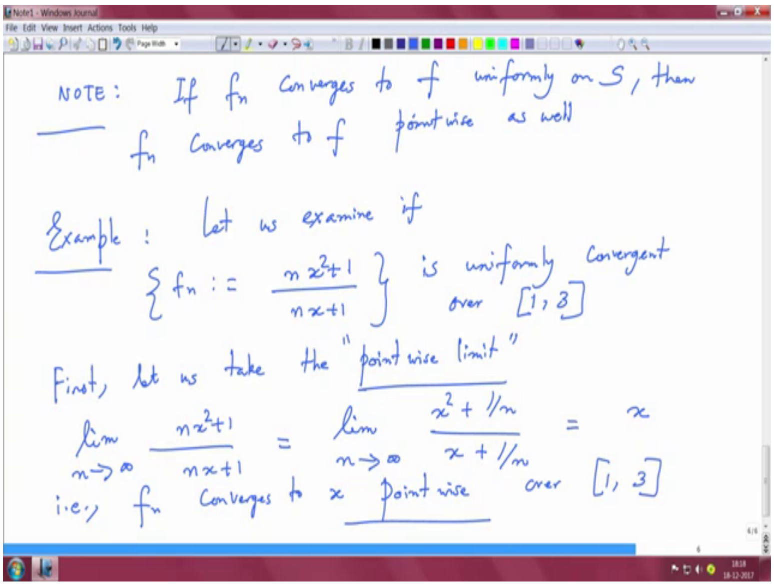

So, we have a note here it is very important if f n converges to f uniformly on S then f n converges to f point wise as well and it is not the other way around. And one of the reasons for studying uniform convergence is basically to study if continuity is inherited from a sequence of functions you might be given a sequence of functions that are continuous and you want to look at a functional series and then you want to see if that helps to you know if there is uniform convergence. And if there is uniform convergence then it implies continuity and is a very important thing and continuous functions are very useful particularly when we deal with certain aspects in the Fourier series. So, we will prove this result as we go through later part of this lecture, but let us revisit quickly an example.

Let us examine if the sequence of functions f n given by say nx square plus 1 upon an x plus 1 is uniformly convergent over the interval 1 comma 3 this is basically the close interval 1 and 3 are included as part of the interval. So, first let us take the point wise limit, ok. Now, limit as n goes to infinity of nx square plus 1 upon nx plus 1 equals limit as n goes to infinity I just pull the n outside which is basically x square plus 1 upon n divided by x plus 1 upon n and this is basically x, right. So, that is f n converges to x point wise over the interval 1 comma 3 right, it basically heads to this ramp function defined over 1 comma 3.

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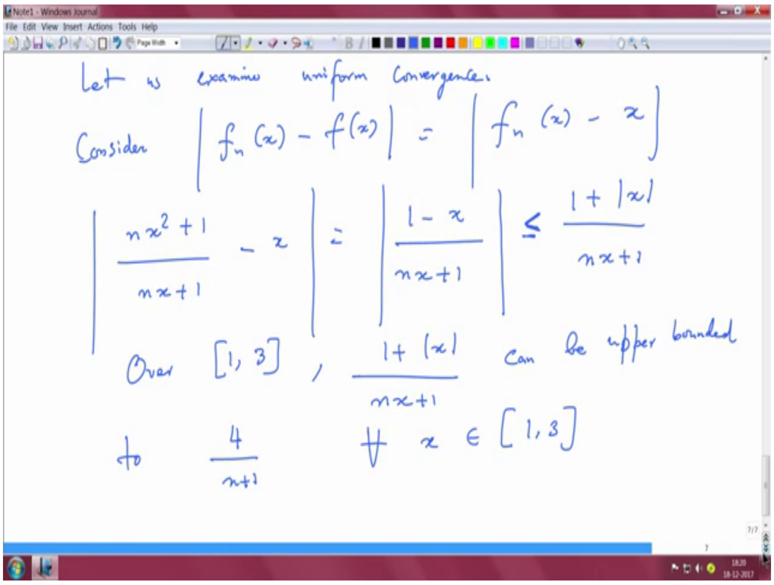

Now, let us examine uniform convergence ok. For this let us consider the absolute value of f n of x minus this limiting function f of x which is f n x minus x, right. So, plug in f n x we get nx square plus 1 upon nx plus 1 minus x which is basically absolute value of 1 minus x. So, just multiply nx plus 1 with x and then you cancel out nx square. So, you get 1 minus x upon nx plus 1 absolute value and then x is positive. So, therefore, the denominator we can say it is an x plus 1 and this is 1 plus mod x this is basically strictly less than or equal to because I would like to say this is mod of 1 minus x is definitely upper bounded by 1 plus mod x right I take the positive quantity here. So, this is true.

So, now, over this interval 1 comma 3 1 plus mod x upon nx plus 1 can be upper bounded to 4 upon n plus 1. I mean if I want this to be maximized the numerator should be more and the denominator should be less. So, the denominator has to be less I choose x to be the lesser value because this is a ramp I choose x equals 1. So, I get n plus 1 and the numerator has to be maximum so therefore, I choose you know mod x x to be 3 here, so therefore, I get 4 in the numerator and the denominator is n plus 1 right. This is for all x belonging to 1 comma 3 you have gotten rid of the x then I compute this quantity.

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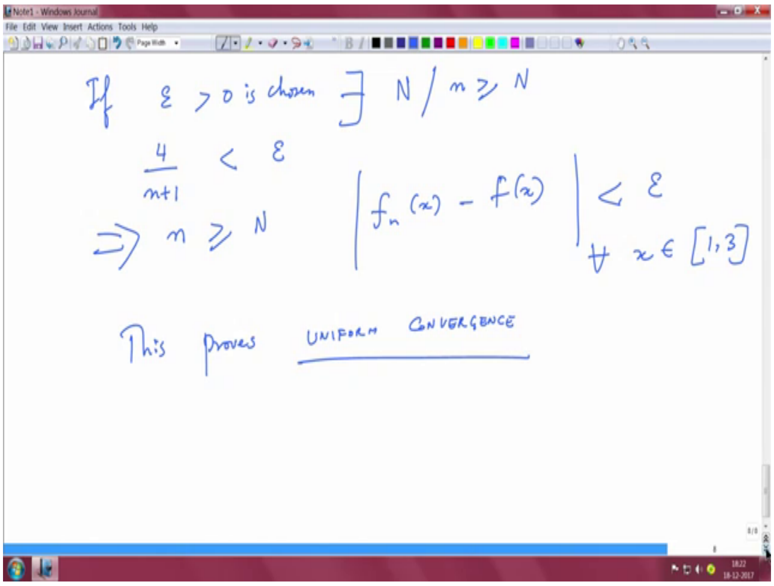

So, now, what it means is if epsilon is greater than 0 some positive quantity chosen. There exists some number capital N which is a natural number such that for all small n greater than or equal to capital N 4 upon n plus 1 is strictly less than epsilon. Which means for every small n greater than or equal to capital N absolute value of f n of x minus f of x is strictly less than epsilon this is over all points belonging to the interval 1 comma 3 this proves uniform convergence.

Now, why do we care upon these ideas in an analysis, right?

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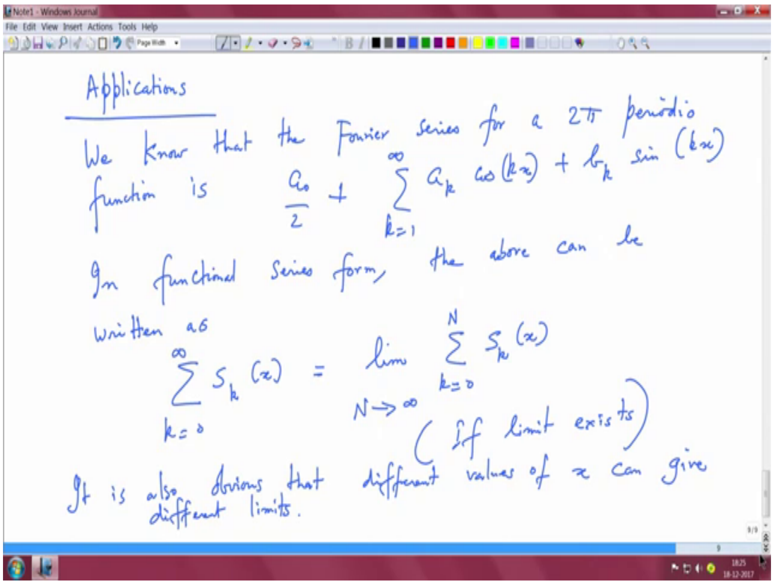

So, let us see the connection in some applications. So, we know that the Fourier series for a 2 pi periodic function is given by a dc term plus the even the cosine and the sin harmonics. Now, in functional series form the above can be written as summa k equals 0 to infinity summation S k of x.

Now, this if you write it in a limiting form is limit capital N going to infinity summation S k of x k equals 0 to capital N that is I take a partial sum from 0 to capital N, I sum and then I take the limit and you see you ask questions if the limit exists right the question is here if limit exists. And for different values of x, I can have different limits right I mean it is also obvious that different values of x can give different limits and this is exactly what you are sort of dealing with in the notions of our conversions, and does it converge to some function, and is it point wise conversions or uniform conversions and this is there we are sort of alluding towards right. We will discuss all these things in detail when we delve into Fourier series, but I am sort of setting up a basic background into basically the notions of convergence as part of this background.

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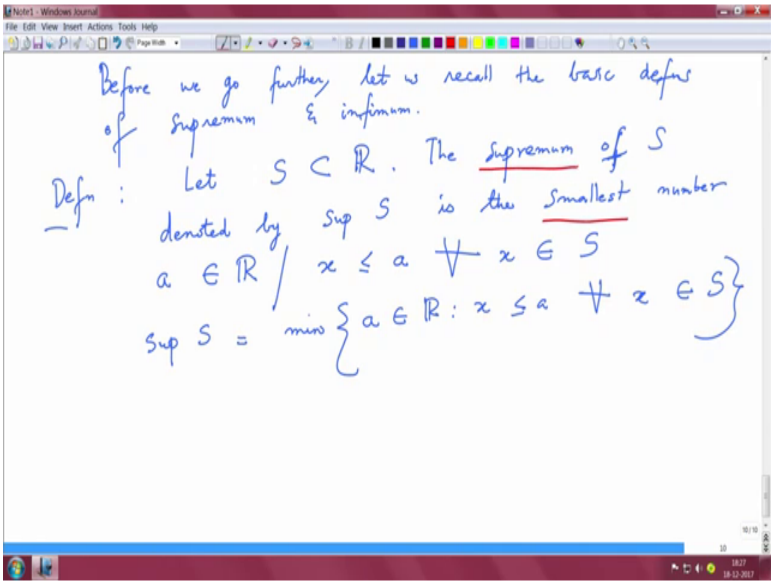

Now, before we go further let us recall the basic definitions of supremum and infimum. You might have done this in your basic math, but we will just recall this definition because we will use this definition subsequently. Let S be a set which is contained in R the supremum of S denoted by sup S is the smallest number a belonging to the set of real numbers such that x is less than or equal to small a for all x belonging to the set S pick an x belonging to this set S which is contained in R and x is less than or equal to a.

Now, supremum of S in the set form is basically minimum over all a belonging to the set of reals such that x is less than or equal to a for every x belonging to the set S right. So, similar to the supremum we have something called the infimum as well I think you can see the connection.

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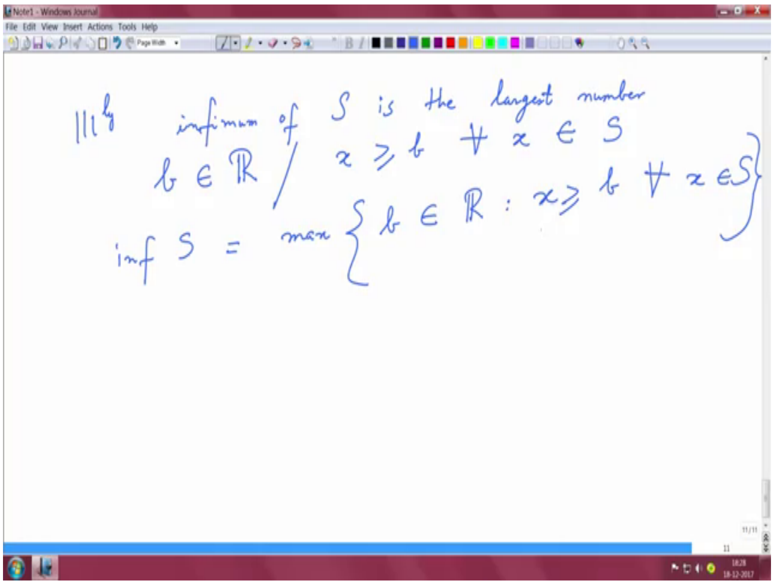

Now, similarly infimum of S is the largest number b belonging to the set of reals such that I am just flipping it x is greater than or equal to b for all x belonging to the set S. So, infimum of S is basically maximum over all b belonging to R set of reals such that x is greater than or equal to b for every x belonging to the set S.

So, pick an x belonging to the set S right and I have an x, I have a b such that b is greater than or equal to x and b belongs to are not necessarily belonging to the set S it could belong to the set S or it may it need not belong to the set S, but b belongs to R.

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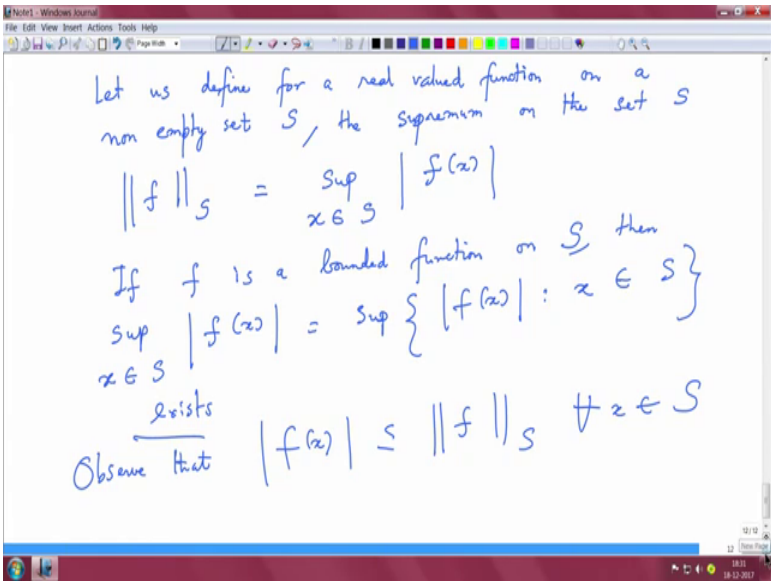

Now, with this let us get slightly a little more into depth here using these notations that we already described. So, let us define for a real valued function on a non-empty set denoted by S the supremum on these set S given by norm of f on the set S is basically supremum x belonging to the set S absolute value of f of x ok.

Now, if f is a bounded function on S then supremum over x belonging to S absolute value of x is basically supremum if you write this in this form absolute value of f of x such that x belongs to S. And this if f is a bounded function on the set S then this supremum exists and you know we have to observed that mod f of x is less than or equal to the supremum of this function on the set S for every x belonging to S and means values that mod f will take is very close to the supremum that is what it means, ok.

So, with this we can slightly think about revisit, revisit revisiting the uniform convergence and let us check if uniform convergence implies point wise convergence.

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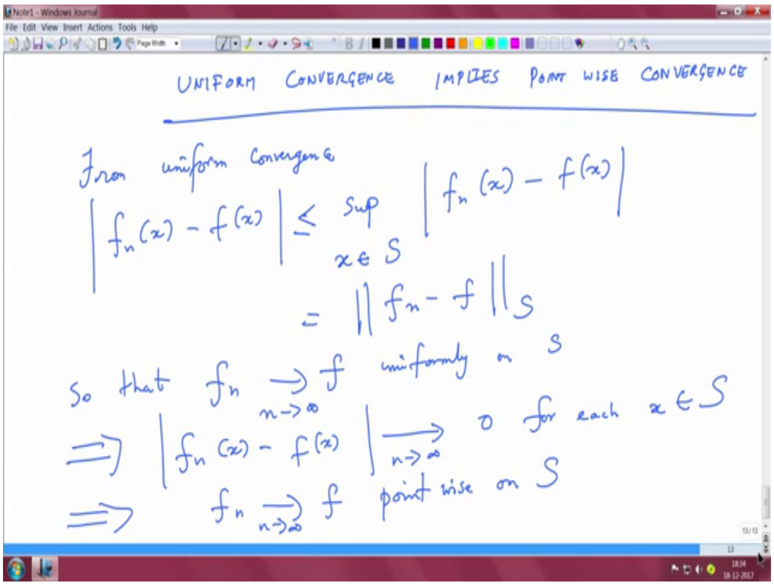

let us check this I think I made a note that uniform convergence implies point wise convergence let us see this. So, from the definition that we have for uniform convergence; that means, the absolute value of f of x, if f n of x minus f of x is within epsilon and that does not it and you can choose some number capital N that does not depend on x, but on epsilon n depends only on epsilon it does not depend on the initial point right. So, absolute value of f n x minus f of x is less than or equal to the supremum for all x belonging to this set S absolute value of f n of x minus f of x which is basically in our notation f n minus f on this set S. So, that f n heads to f uniformly on S as n goes to infinity.

So, this implies absolute value of f n of x minus f of x this heads to 0 for each x belonging to S, this is very important for every x pick in x for each x this f n x minus f of x absolute value heads to 0 right this implies f n heads to f as n goes to infinity point wise on S. So, uniform convergence implies point wise convergence, but point wise may not imply uniform convergence.

Now, there is a link between uniform convergence and continuity and let us establish this result via a theorem.

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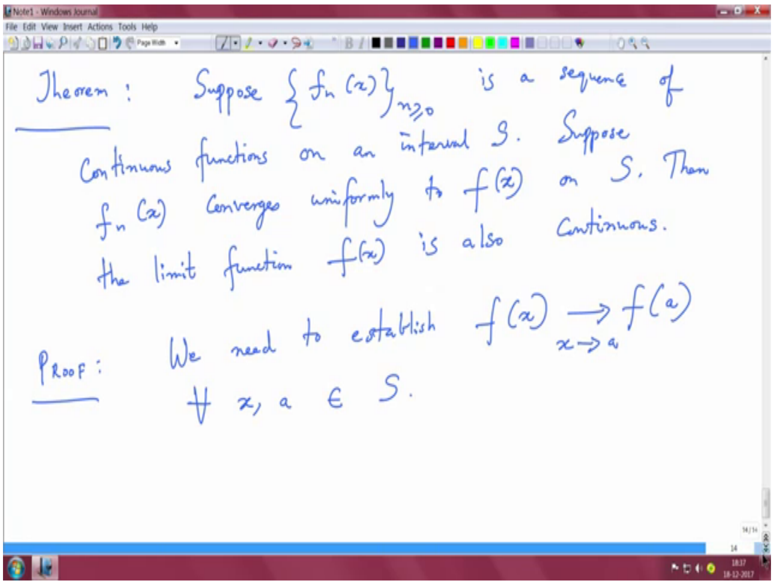

So, I will state the theorem and I will also prove this result. Suppose f n x is a sequence of continuous functions on an interval S suppose f n of x converges uniformly to f of x on S. Then the limit function f of x is also continuous this is a very important very important theorem I have a sequence of continuous functions defined on an interval S and I know that f n of x converges uniformly to f of x on that interval then the limit function is also continuous a very powerful statement. So, if it is not continuous then what are the implications right, I mean it the convergence of a point of discontinuity would have to be invoked and all these subtle notions have to be really revisited. So, therefore, this uniform convergence is a very powerful, powerful idea.

Now, let us try to establish the proof of this result, ok. So, we need to establish that f of x heads to f of a when x heads to a. So, when x heads to a I want establish that f of x goes to f of a for every x and a belonging to this interval ok.

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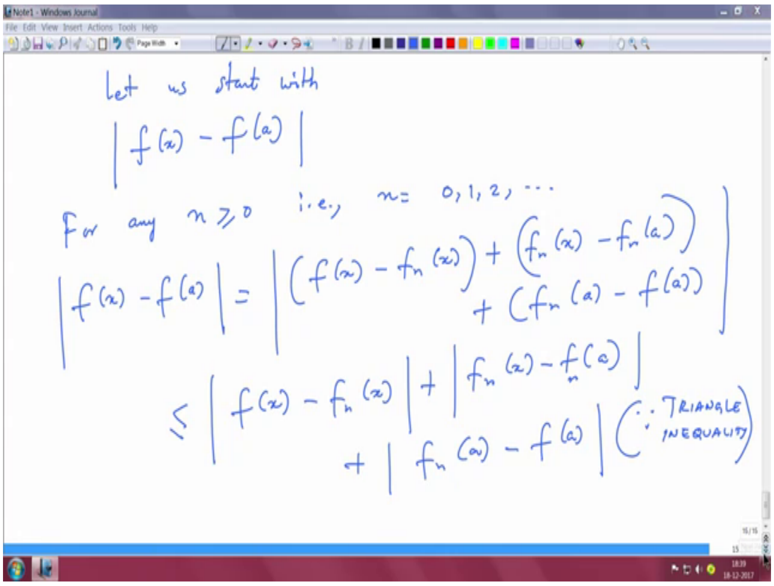

Now, let us start with let us start with absolute value of f of x minus f of a. Now, for any n greater than or equal to 0 that is n equals 0 1 2 dot dot dot absolute value of f of x minus f of a can be written as absolute value of f of x minus f n of x plus I add and subtract f n of x and f n of a very conveniently. And I think if you are seeing the trick here basically you can see that you can invoke triangle inequality for this for this term that I have written here right. So, what I have done is f of x minus a I add and subtract f n x and f n a.

Now, invoking triangle inequality I have this is less than or equal to absolute value of f of x minus f n of x plus absolute value of f n of x and f n of a plus absolute value of f n of a minus f of a. But now, I can bound f of x minus f n of x absolute value this is basically the superior norm of f minus f n same thing holds for this term as well.

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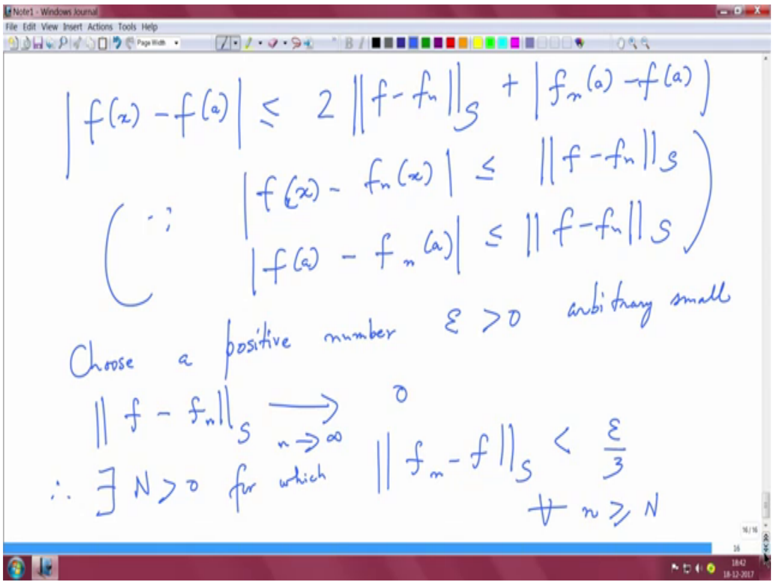

So, I can conveniently state f of x minus f of a is less than or equal to 2 times the norm of f minus f n soup norm absolute value of f n of a minus f of a because mod f of x minus f n of x is less than or equal to norm if minus f n is an absolute value of f of a minus f n of a is less than or equal to f minus f n superior supremum of f minus f n. Now, we are seeing the trick here.

Now, we choose a positive number epsilon which is greater than 0 which is arbitrarily small such that f minus f n heads to 0 as n goes to infinity. Therefore, there exists some natural number n greater than 0 for which this norm f n minus f, I mean I am interchangeably using f n minus f and f minus f n it is basically the same two superior supermom is less than some epsilon by 3. So, let us assume that this is some epsilon by 3 for every small n greater than or equal to capital N ok. Then choose and fix that small n say n equal to capital N and then which ensures that this is less than epsilon upon 3.

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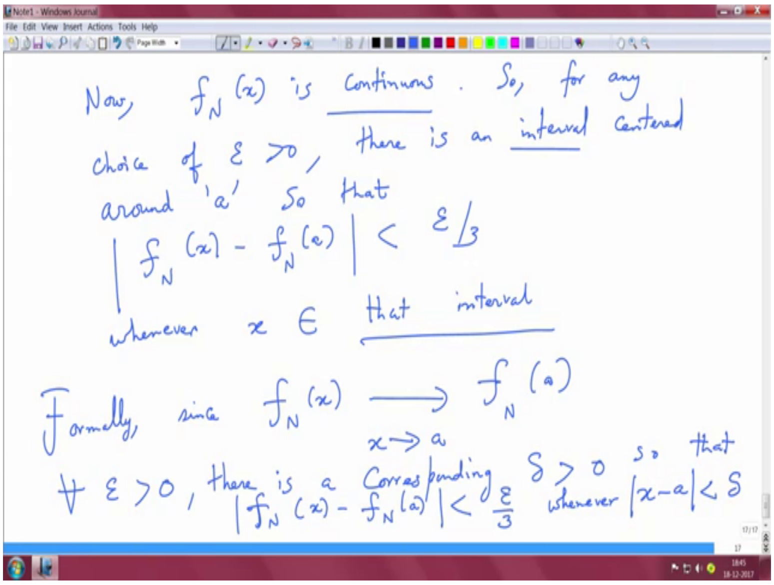

Now, here is a important part if suffix capital N of x is continuous because we said all of these are continuous functions and there exist some small n which is equal to capital N in this that fits this bill and gives you f suffix capital N of x is continuous. Now, this is continuous, so for any choice of epsilon which is greater than 0 there is an interval center around the point a, so that absolute value of f suffix capital N of x minus f suffix capital N of a is less than epsilon upon 3 whenever x belongs to that interval. This interval is not necessarily yes. There exists some interval because it because we want a no continuity here right there exists some interval that is centered around a such that this is holding true that is absolute value of f n of x minus f n of a is within epsilon upon 3.

So, formally since f suffix capital N of x heads to f suffix capital N of a for x tending to a for every epsilon which is greater than 0 there is a corresponding delta which is greater than 0. So, that absolute value of f n of x minus f n of a is strictly less than epsilon upon 3 whenever mod x minus a is less than delta, right.

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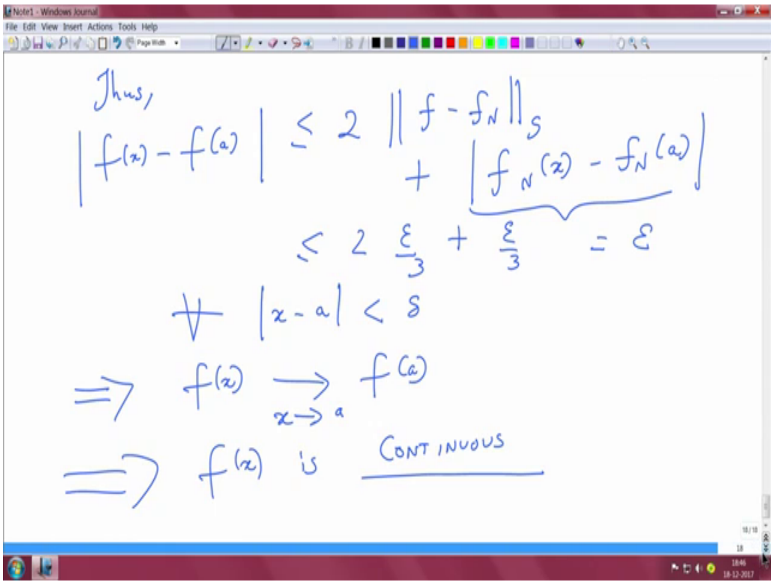

So, thus absolute value of f of x minus f of a is less than or equal to 2 times the superior of f minus f n plus absolute value of f n of x minus f n of a this is less than or equal to 2 times this quantity is epsilon upon 3 and this is also epsilon upon 3 therefore, this is equal to epsilon for all mod x minus a less than delta.

So, therefore, this implies f of x heads to f of a as x tends to a which means f of x is continuous. So, this is a very very important relationship because if you consider a sequence of functions and these functions are continuous and then there is uniform convergence then it implies that the limit function is continuous. So, this also gives us an idea for a quick test towards uniform convergence.

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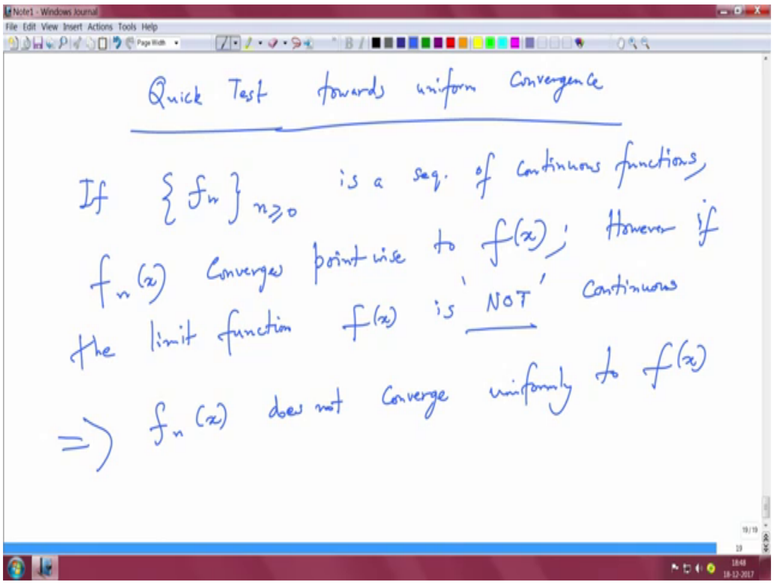

So, if the sequence of functions f n are continuous then f n and f n converges point wise to f of x however, if the limit function f of x is not continuous implies f n x does not converge uniformly to f of x this is a very important quick test. So, I considered a sequence of continuous functions and I assume f n of x small n of x converges point wise to f of x; that means, each individual function converges point wise. However, if the limit function is not limiting function is not continuous then the sequence does not converge uniformly to f of x and this is a very useful result when we have to touch upon subtle aspects in the convergence of the Fourier series.

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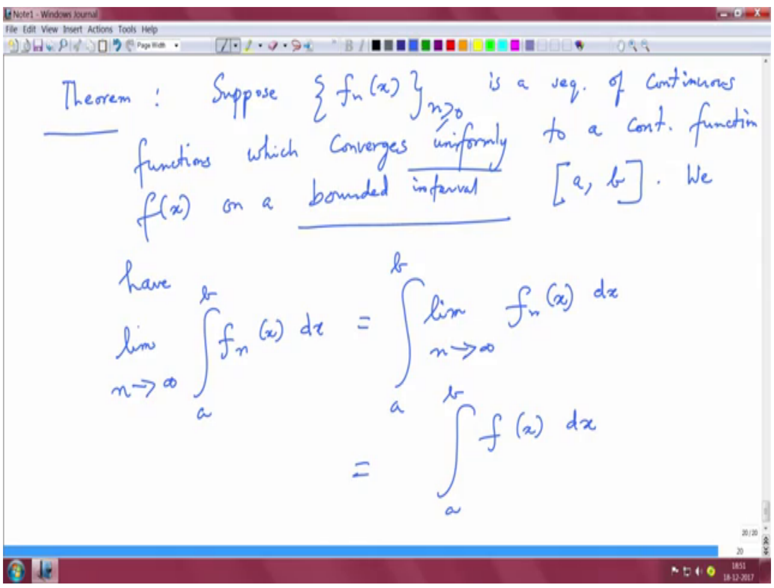

I will conclude with a theorem, another theorem. Suppose f n of x is a sequence of continuous functions which converges uniformly to a continuous function f of x on a bounded interval this is a small twist here it is a bounded interval. Let us suppose it is the closed interval a to b we have limit as n goes to infinity integral a to b, f n of x dx is integral from a to b limit n goes to infinity f n of x dx which is basically the integral from a to b of the limiting function f of x.

Recall in one of the earlier examples when I took the integral of a continuous function and took the limit of that versus taking the limit of the function and then integrating it there was a case when they were not the same right we looked into that example right and that was this tenth function. So, now, and we saw the issues there, right. So, now, we will see under what conditions that we can take the limit of the integral as the same as the integral of the limit right and that happens when there is uniform convergence over a bounded interval. So, we will we will prove this result carefully.

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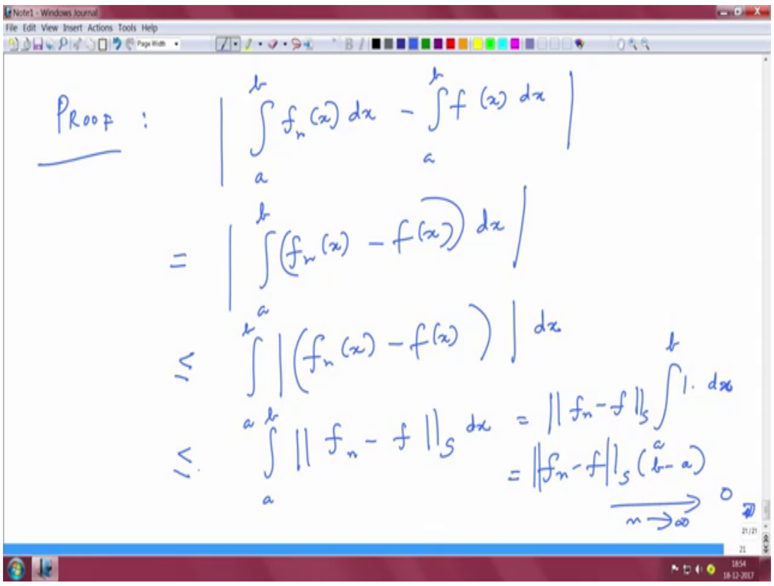

So, consider the norm I mean absolute value of this quantity which is basically the integral of the limit minus the integral of the sequence. So, this is basically absolute value of f n of x minus f of x dx. I just wrote the integral sign before and I just took f n of x minus f of x.

Now, this quantity here is certainly less than or equal to integral a to b absolute value of f n of x minus f of x dx absolute, but I think I have to be careful here, I have to put the dx outside. So, basically I take this deviation I integrate it and then I take the mod this is definitely less than or equal to if I did absolute first and then I integrate it over the interval a to b that is what it means right. And this because there is uniform convergence I can write this as the integral from a to b. The supremum exists and I can pull the supremum outside and this is basically the integral over this interval right because this is just one quantity which is maximum over all x belonging to that set, right. So, or x belonging to R and this is a subset of R.

Now, I can I can get the super supremum outside this is just the integral and this is bounded. So, therefore, it is f n minus f times b minus a and this hits 0 as n goes to infinity and this proves this result right. I mean this is this norm basically heads to 0 as n goes to infinity because of uniform convergence and therefore, if you take the limit of the integral of this take the step is a function in the sequence integrate it look at the limit that is the same as the integral over the limiting function and this is a very important result and this holds for uniform conversions. So, if you look at a sequence of functions which uniformly converge to a limiting function over a bounded interval then limit of the integral of the sequence is basically integral of the limit function ok, and that is basically equivalent is equal to the integral of the limiting function over that interval is a very very important result.

So, with this we have sort of given a basic background into the notions of convergence and the implication of uniform convergence to a limit function which is continuous. And these subtleties we will we will see when we discuss the Fourier series summation of the for; I mean I mean if I look at the Fourier series does the limit converge. And if the limit converges to what does this limit converge to and if there is a discontinuity point then what is the limit and some of these subtle aspects have to be dealt with when you deal with Fourier series and for this some of the notions of the convergence of the functional series is very very important ok.

So, a lot of other details will be dealt in functional analysis or if you graduate course in an analysis all these would be covered. So, I am giving you basically a background into this, into these aspects because you will require this in signal processing.

So, we will end this lecture here.