Mathematical Methods and Techniques in Signal Processing - I
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Lecture- 65
Basic analysis: Pointwise and uniform continuity of functions

So, in signal processing we will often have to encounter with limits sequences of functions, you know series summations and. In fact, each term in the series could be a function so on and so forth right functional series etcetera.

And, there are also these notions of continuous functions discontinuous functions and we need some treatment into this. So, the route to these ideas stem from analysis in mathematics, I will not delve too much into analysis because it is a separate course in itself in the field of mathematics and you could do a rigorous course for 30 hours as part of your NPTEL, but I will just briefly give you some ideas about continuity convergence etcetera as part of the discussions here, which is going to be useful in your understanding of some of the signal processing ideas ok.

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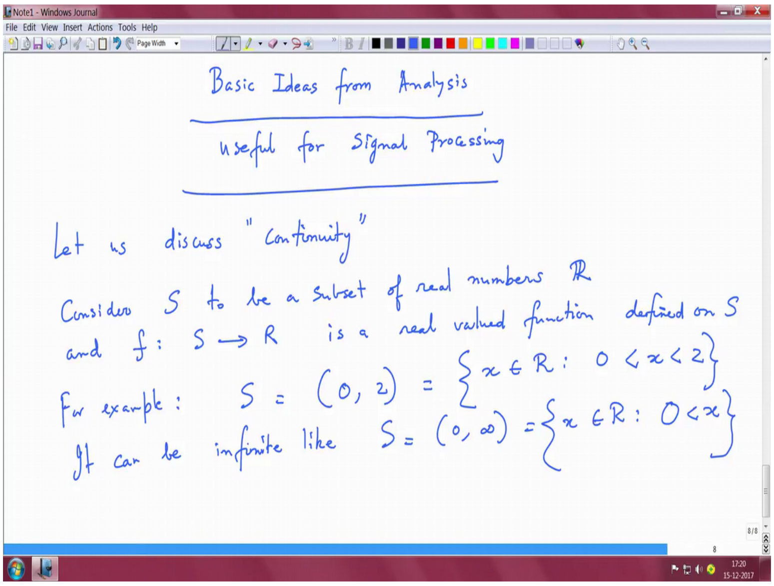

Let us, first talk about 'continuity' and forms of continuity. Now consider S to be a subset of real numbers R and f is a mapping from S to R and this is a real valued function defined on S right.

There is some subset S of real numbers and there is a mapping from this subset to real and this is a real valued function defined on the subset S. So, S for example, if you think of S as say 0 say 2 this is basically all x belonging to the set of real number such that 0 is strictly less than, x less than 2 that is it is open on both sides. It can be infinite like S is 0 infinity, which is basically x belonging to R such that x is strictly positive right, because 0 is not included it is strictly positive.

So, we may encounter many such type of these sets. So, let us start with the notion of what continuity is so, I will start with the definition.

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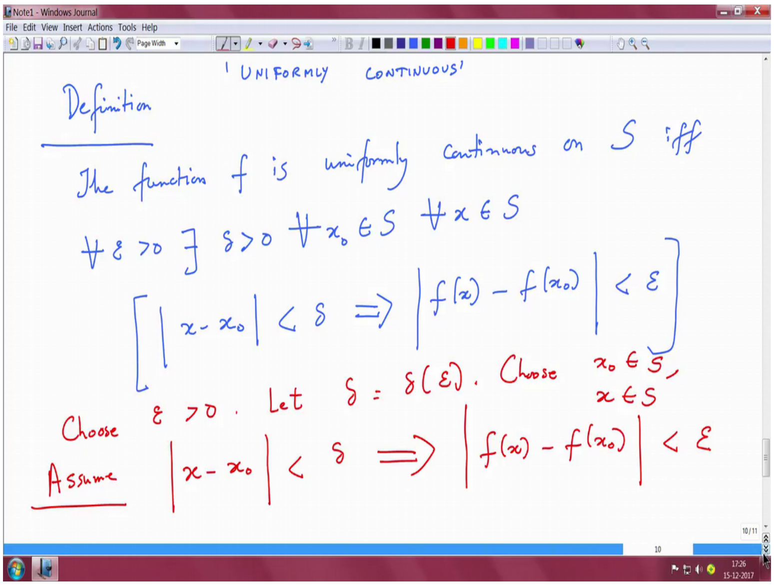

For some function to be continuous on this subset. The function f is said to be continuous honest if and only if and only if for every x 0 belonging to S for every epsilon, which is greater than 0 there exists a delta, which is greater than 0, which depends upon x 0 and epsilon for every x belonging to this subset the following holds that is if x minus x 0 absolute is less than delta, this implies the absolute value of f of x minus f of x 0 is strictly less than epsilon and it is not continuous on S if this is greater than or equal to epsilon.

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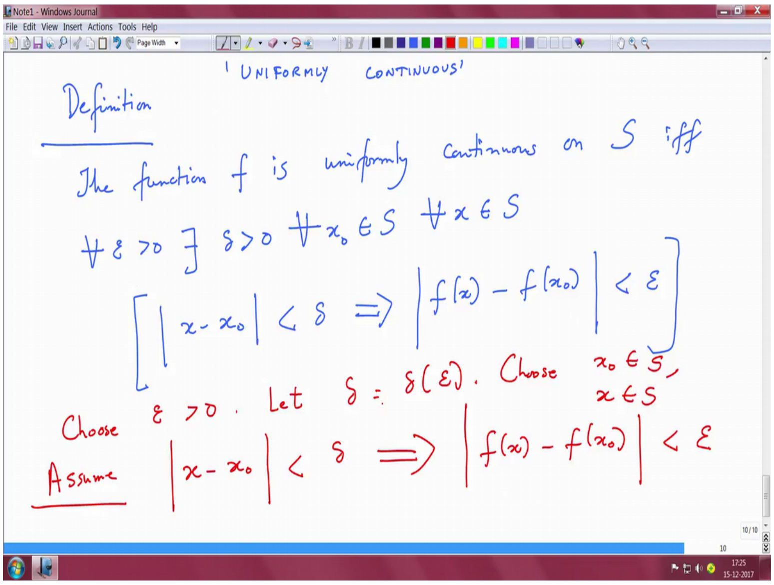

There is a notion of I will go to the next slide there is a notion of uniform uniformly continuous. So, I can call this concept as uniformly continuous. So, the function f is uniformly continuous on S, if and only if for every epsilon greater than 0. There exists delta greater than 0; that means, delta is great is depending only on epsilon right. And then you choose points x 0 belonging to S and x belong to S choose these 2 points, if absolute of x minus x 0 is less than delta this implies that absolute of f of x minus f of x 0 is strictly less than epsilon.

Now, you have to sort of be careful how the definitions are worded very carefully. So, in the case of uniformly continuous you choose first an epsilon greater than 0. So, I will just write it down in a red here. So, we choose epsilon greater than 0 and we let some delta, which is basically delta which is a function of epsilon. Then you choose x 0 belonging to S and x belonging to S you assume, that the absolute value of x minus x 0 is less than delta and this implies that absolute value of f of x minus f of x 0 is less than epsilon this is the English way of writing things this is a little more mathematical way of writing things.

So, I think you have to observe very carefully that in the order I first for every epsilon greater than 0 I choose delta, which is greater than 0 and this delta depends upon epsilon and not really on the point right. So, just going back one slide where we define continuity property on this subset. So, here the difference is as follows. So, here we choose x 0 belonging to this subset, then we choose epsilon which is greater than 0 and the let delta which is basically a function of both the point x 0 and epsilon observe this very careful point, that it has to depend on this x 0 on this point, delta depends on this point. And then you choose some x belonging to S and assume that x minus x 0 is less than delta and if you assume this with all these conditions this implies an absolute value of f of x minus f of x 0 is strictly less than epsilon. So, at least we got an idea of what is continuous and what is uniformly continuous?

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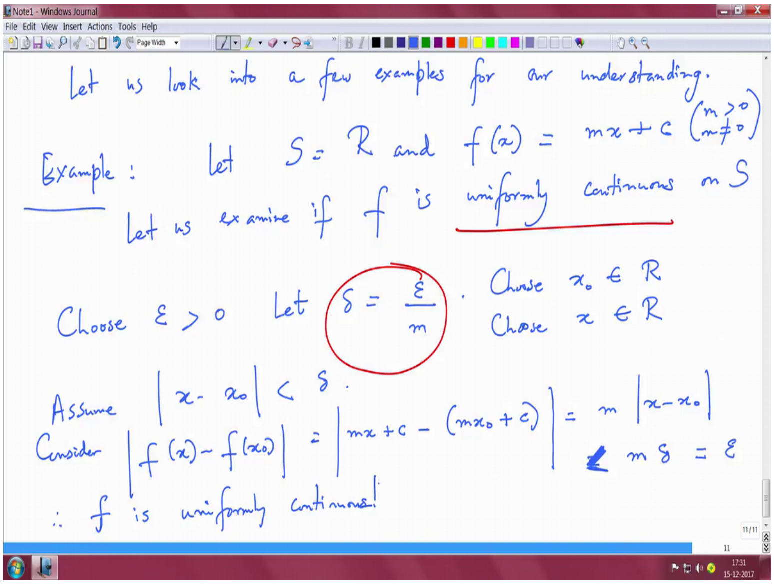

Let us look at some examples towards this understanding. Let this upset be the real line itself choose the mapping to be mx plus c. Let us examine if f is uniformly continuous on S right. So, what do we know what do we need to do here. So, first we choose some epsilon which is greater than 0, because we have to examine uniform continuity.

So, we first choose some epsilon, which is greater than 0 let delta be epsilon upon m I will tell you why we choose this m I mean at the moment just trust me and if things become very clear choose x 0 belonging to R choose x belonging to R assume x minus x 0 is less than delta.

Now, consider mod f of x minus f of x 0 this is mod mx plus c minus m x 0 plus c which is so, I will assume that m is just to make things a little easier assume m is greater than 0 and m 0 equal to 0. So, that this is taken care of now this is m times mod x minus x 0 and this quantity is if this is basically m times delta and this is strictly this is quantity strictly less than delta right. So, I can say this is quantity strictly less than m times delta and delta is epsilon upon m.

So, therefore, this quantity modulus of fx minus fx 0 is strictly less than epsilon right and that is reason why I chose delta to be epsilon upon m. So, now, you can see the choice of delta equals epsilon upon m why it has been carefully chosen right and this is satisfied therefore, f this function, which is a mapping from S to R is uniformly continuous.

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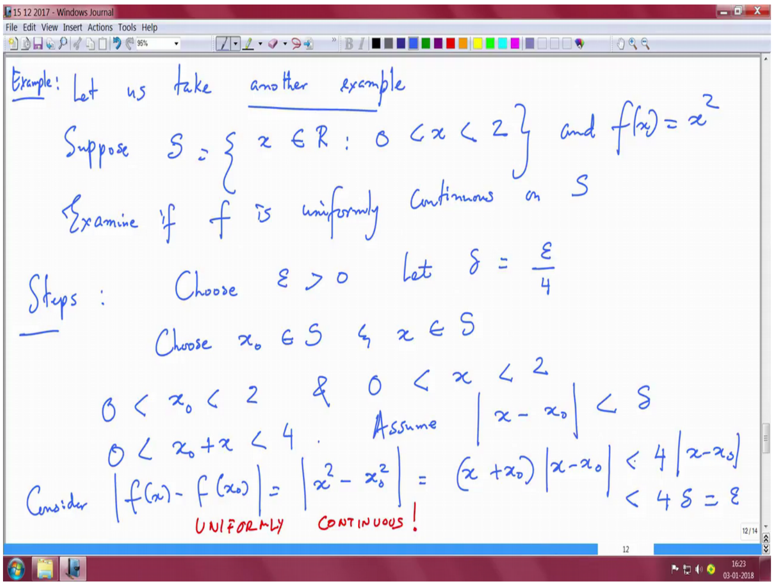

Let us, take another example suppose S is given by x belonging to the set of real numbers such that 0 is less than x less than say some 2 and f of x equals x square. Let us examine if f is uniformly continuous on S. So, it is straight forward let us just work through these steps here right.

So, what we have to do is we have to choose some epsilon, which is greater than 0 let delta be epsilon upon 4 I will tell you why this thing appears like this, then we choose x 0 belonging to S and x belonging to S ok. Now 0 is less than x 0 less than 2 and 0 is also less than x less than 2 therefore, if I add these 2 intervals 0 is less than x 0 plus x this is less than 4 ok.

Now, let us assume the absolute value of x minus x 0 is less than delta, now consider mod f of x minus f of x 0 in this case is x square minus x 0 square, you can factorize this as x plus x 0 into mod x minus x 0 because x plus x 0 is basically definitely greater than

0 right. We know this quantity therefore, I had I did not put the absolute here and this is basically 4 times I would say this is basically 4 times mod x minus x 0 and this is basically less than 4 times delta. And therefore, this is if I choose delta to be epsilon upon 4 this basically satisfies the condition that absolute value of f of x minus f of x 0 is strictly less than epsilon ok.

So, this is again uniformly continuous well there is a small catch here, because the range is from 0 less than x less than 2 precisely x plus x 0 will be strictly less than 4. So, this equality here would have to be an inequality, which is strictly less than and this is something to note.

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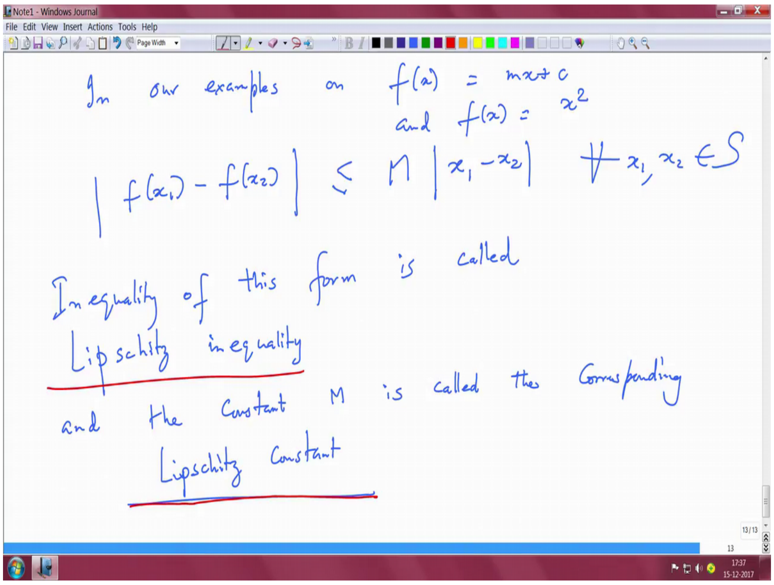

In our examples on if x is mx plus c and if x is x square that is be considered linear function we considered a quadratic function we landed up with a situation where mod f of x 1 minus f of x 2 is less than or equal to some m times mod x 1 minus x 2 right we landed up with a situation like this, for all x 1 x 2 belonging to S right. Inequality of this form is called Lipschitz inequality and the constant m is called the corresponding Lipschitz constant ok.

So, this is a an important note I think we understood what this idea of continuity is what the idea of uniform continuity is I gave you examples of uniform continuity? I will give you some homework exercises that you would want to work.

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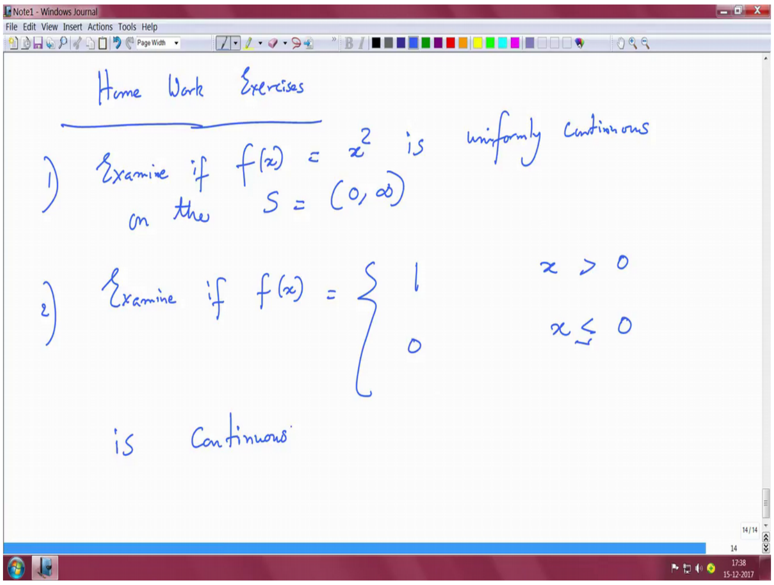

First examine, if f of x equals x square is uniformly continuous on the interval S, which is basically 0 to infinity and it is basically an open interval. So, this is one exercise that I would want you to try and the second would be examine if f of x given by one if x is positive 0, if x is less than or equal to 0 is continuous.

Of course, you know the answer to the second problem, because there is a jump here right I mean so, like a step function there is a jump here and you know it is a point of discontinuity, but using the definitions, that we learned just now on continuous functions and uniform uniformly continuous functions, I would want you to examine if this is continuous or not ok.

So, I think with this we recover the essentiality of continuous property, then the next we will talk about convergence and then we will discuss some of the certain issues related to convergence of sequences and then convergence of functions which is essential for your signal processing understanding.

Thank you.