Mathematical Methods and Techniques in Signal Processing - I

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Lecture- 64
Time frequency localization

So, towards the end of wavelet us, we have to understand the following that there is a resolution that is happening at different scales right and there is a resolution in time. And what about in the frequency is a resolution the same in frequency right, I mean imagine I have a short burst of a pulse right. And I can measure the burst or I can measure some event in the a of the signal in and the variance of the signal in time and some statistical properties of the signal in time, and can I have the same sort of resolution that I can get in the frequency domain. So, this is one of the questions that one would naturally think about.

So, fortunately for us somebody did investigation on this time frequency uncertainty principle. This follows a kin to the position momentum uncertainty Heisenberg's uncertainty principle that we are aware in quantum mechanics right. And the same sort of idea is exists actually in when we look at time frequency uncertainty in signal processing. And there is a more deeper theory using operators in mathematics and will not get there ok.

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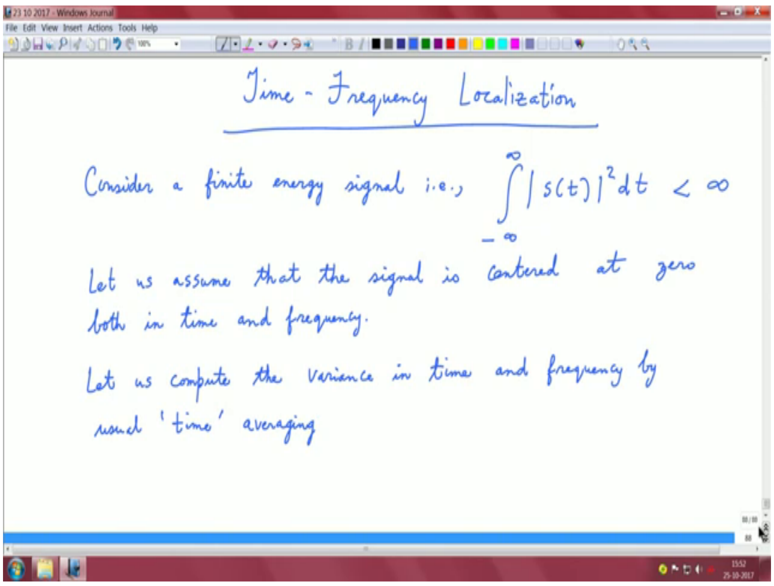

So, let us begin consider a finite energy signal right. If it is a finite energy signal which means integral minus infinity to plus infinity, I mean you can put any range for this, I mean this is very general, I mean you really will not ideally reach minus infinity and plus infinity, but this is just for pedagogical purposes and if you see some range or you just have to put the appropriate range here in your definite integral. So, this energy is strictly less than infinity.

Now, let us assume that the signal is centered around 0 I would say precisely at 0 both in time and frequency, that is it is 0 mean if it is not 0 mean you can always subtract the bias and get it to 0 mean ok. Now, when we think about signals we discussed in the very beginning of module 1, when we looked into signal geometry, that we can think about different kinds of averaging statistics 1 is just a normal time average you are given the samples you compute all these statistics, whether it is mean or the variance or second moment third moment so on and so forth right. You can compute all the moments across the signal or you could do the statistical average right statistical average. So, let us look at the time average time average ok.

Now, let us compute the variance in time and frequency by usual time averaging ok.

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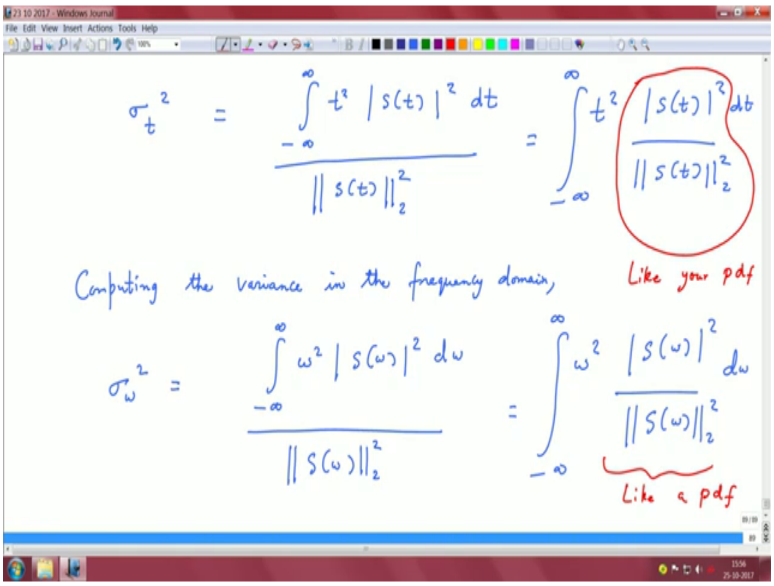

Now since we said we are dealing with 0 mean we can compute the variance in time sigma t square, as integral minus infinity to plus infinity t square mod s of t square dt and there is a normalization factor here, which is the square of the l 2 norm of the signal right. And why should we do this is a question that you might want to ask why should I normalize?

And, if we treat the signal as a random signal there should be some distribution over which we have to weigh it right. So, if you take if you fold this norm inside right. So, if you can you can you can basically rewrite this as essentially integral minus infinity to plus infinity t square mod of s of t square dt divided by the l 2 norm. Now can you appreciate that this is like your density right this is like your like your PDF probability density function, because when you integrate this is basically this integrates to 1 right. So, therefore, you can compute a moment a second moment like this ok.

Now, we will do this in frequency computing the variance in the frequency domain, we get sigma omega square is integral minus infinity to plus infinity omega square modulus of s of omega square d omega divided by the norm l 2 norm over the spectrum right. Again you can interpret this as a density like what we did earlier right there is no big deal. So, this is also like you can interpret this as ok.

So, now, I think we are sort of set with our problem. Now, you can also think about in the notion of quantum mechanics, you can think about this as position this is momentum and you can think about these operate or these quantities and you are also looking at the variance here right or some other 2 set of quantities.

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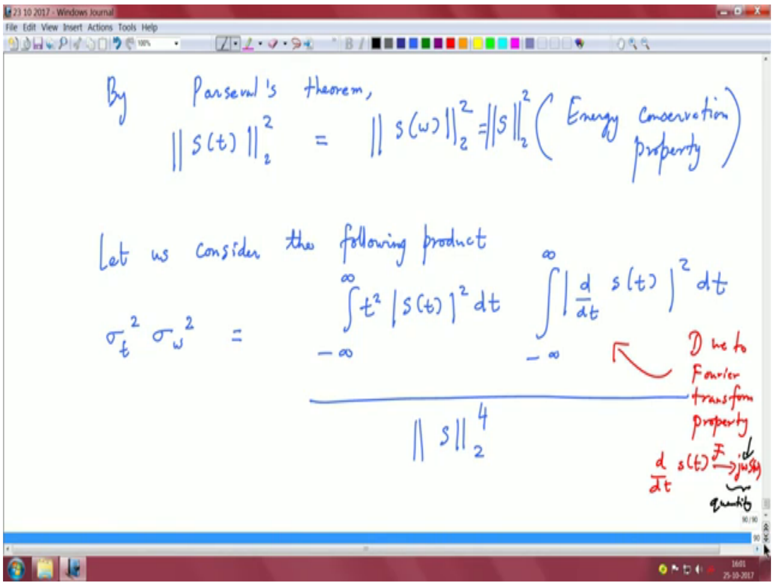

Now, by Parseval's theorem from your undergraduate norm s of t squared l 2 norm of s of t is the same as s of omega square right. Because, if you apply the Fourier transform you are not going to alter the energy in the signal, otherwise it would be the transformation is useless if you lose the energy right, because of energy conservation property. So, notation wise this is easy for me I can just write it like this, some norm of s square I will get rid of t and omega I am saying norm of s square because it is the same whether it is time or frequency ok.

Now, let us consider the following product, which is the variance in time and the variance in frequency I consider this product. This is essentially integral minus infinity plus infinity t square mod s of t square dt times. Now, we can interpret the omega times modulus of s of omega as this quantity, because from duality if you take d by dt of s of t in the frequency domain, it is you are multiplying by g omega right and the modulus will pull the magnitude of j to be one right and you have omega square s of omega square modulus of s of omega square right.

So, this is an important step divided by power 4, because you have a norm s square for this term and another norm s square for this term. So, you have a power 4 in the denominator and in this quantity it is basically due to Fourier transform property right take the Fourier, it is j omega right. So, you have this additional jmo this j omega quantity which justifies what we have.

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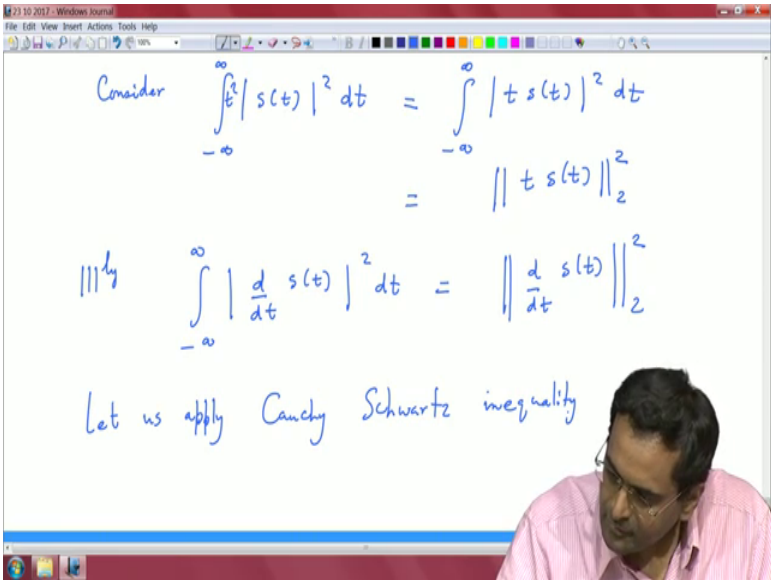

So, now, consider integral minus infinity plus infinity modulus of (Refer Time: 14:00) I will say t square times s of t square dt consider this this can be written I can fold the t inside the modulus there is no problem with it. So, I can write this in this form modulus t times s of t square dt. And you can interpret this quantity as the l 2 norm square of t time's s of t ok. See how we are kind of building the logic right from starting from the variance to interpreting this into the norm.

And, similarly integral minus infinity to plus infinity modulus of d by dt s of t square dt can be interpreted as the l 2 norm of the derivative of the signal right, l 2 norm square of the derivative of the signal. Now, we have a product of 2 norms. So, your intuition should tell that there is Cauchy Schwartz somewhere kicking in ok. So, let us apply Cauchy Schwartz inequality.

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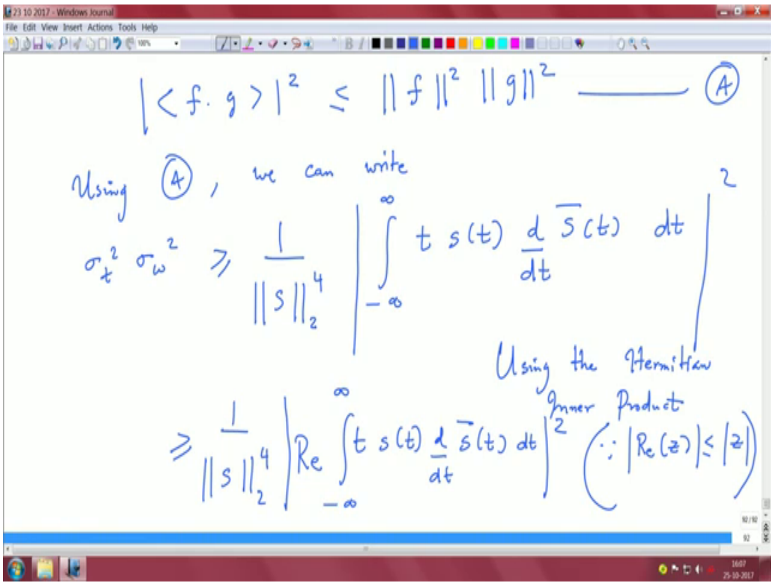

So, the reason states that if you look at the norm of 2 functions, that are possibly complex then the modulus of the in the product square is basically less than or equal to the norm square of the product of the individuals ok.

Now, using this we can say and say this is relationship a using A, we can write sigma t square times sigma omega square as a quantity, which is greater than or equal to because now it is existing in product form exactly. So, therefore, the inequality is this way right one over norm of s l 2 norm of s power 4 times I bring the modulus integral minus infinity to plus infinity t times s of t times there is one careful observation, that you have to make where I am bringing in the conjugate of this signal under the Hermitian inner product form.

Now, this quantity that we have modulus of some integral square so, this is this can be a complex quantity. So, we can simplify this little carefully. So, this is one over the norm of s power 4 times I can write the absolute value of the real part of integral t times s of t times d by dt of s of t bar dt square why? Because, you know for complex numbers modulus of real of z is certainly less than or equal 2 modulus of z right. This is a straight forward result.

So, you take just for a cross check 3 plus 4 j is that vector the length is 5, then the real is 3 and when the complex part is 0 then it coincides with equality. So, you have this relationship.

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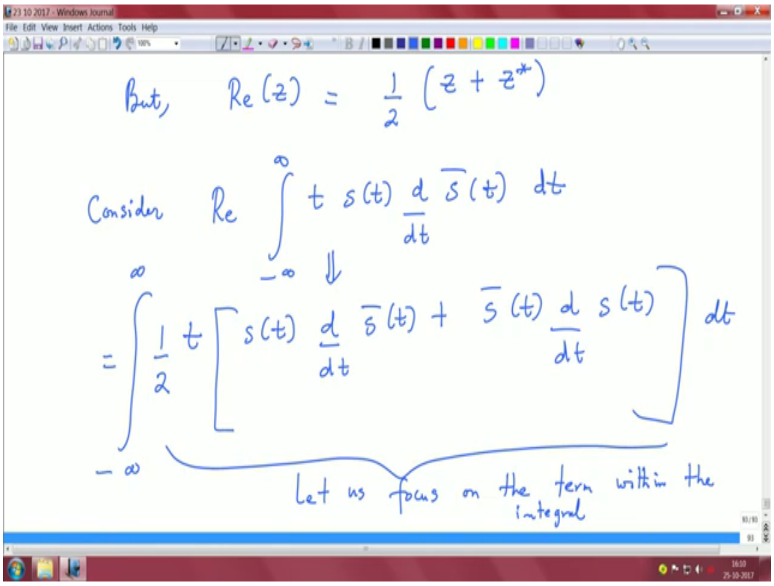

Now, real part of some complex quantity z can be written as one half z plus z conjugate, this is a straight forward result. Now consider the real part of this integral minus infinity to plus infinity t times s of t times derivative of the signal conjugate dt right. I can write this as 1 half I will pull the t factor outside because it is just a scalar it is time variable it is a scalar right.

So, we can write it as s of t times derivative of s bar of t plus s bar of t derivative of s of t right using this room using this step one half of z plus z conjugate I just did that yeah right. So, I just take this as my z here I just conjugate it I will know I will come to the point I am just yeah I need the integration there yes a real part. So, I just have t times I just I am looking at this term here I am just rewriting this thing here and I just have to put if you are with I am just focusing on this term just the argument of the integral.

Now, let us focus on the term within the integral let us focus on the term within the integral. So, this is basically I would say term within the integral is basically half t times d by dt of modulus of s of t square why? Because, you can write modulus of s of t is s of t is power of t now you take the derivative using chain rule and it automatically comes ok.

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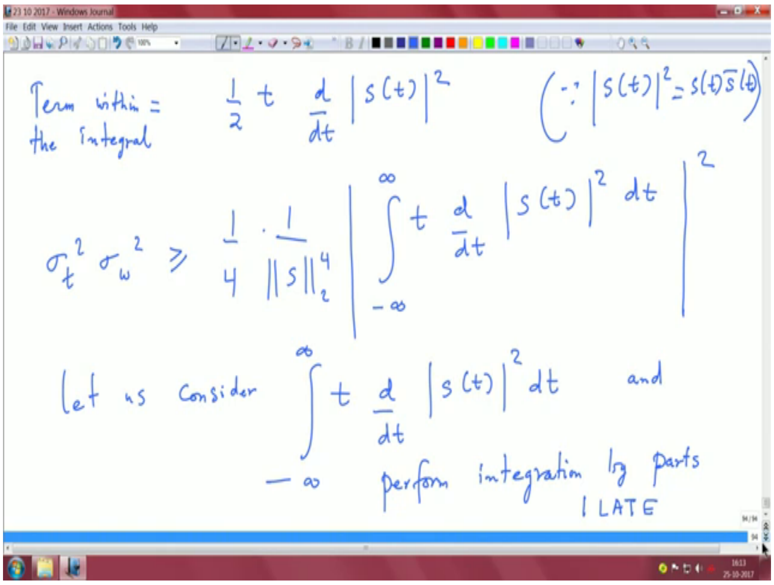

So, therefore, sigma t square times sigma omega square is greater than or equal to now I have to square this quantity right if it is one-fourth, I have a norm s power 4 and then have an absolute value here then this integral minus infinity to plus infinity t times d by dt of modulus of s of t square dt.

Now, still it looks complicated let us consider integral minus infinity to plus infinity t times derivative of modulus of s of t square dt and perform integration by parts right. We have the familiar this is what I recall from my high school I late this is a inverse function logarithmic. So, on and. So, forth right you use that last is exponential. So, you apply this rule integration by parts.

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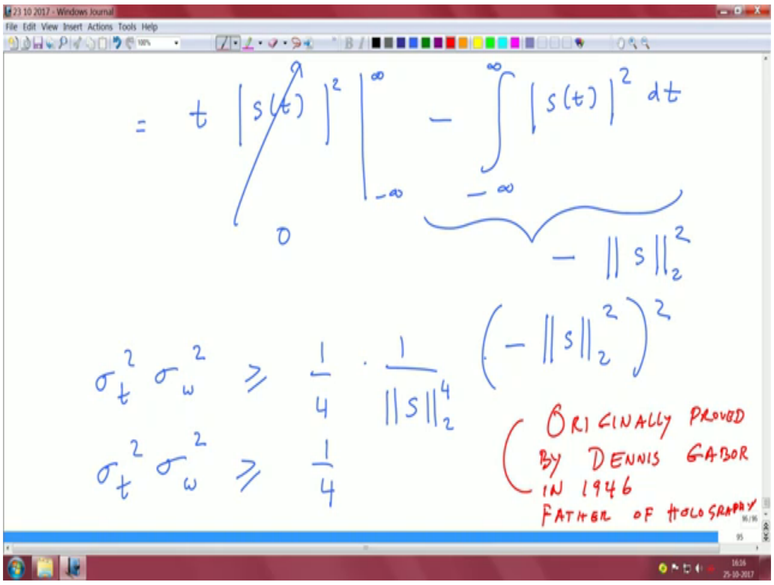

So, what we get is t times s of t square minus infinity to plus infinity minus integral, because I take the derivative I integrate that I get this and then I take a differentiate time, then I get one there right first function integral or second function minus integral of second function derivative of the first function ok.

Now, this quantity goes to 0 and this is basically interpretable as negative norm of s square. Therefore, sigma t square times sigma omega square is greater than or equal to 1 upon 4 times minus norm s square and if you square this quantity because I have that modulus of that integral square is what I have. So, this is basically just 1 by 4, because norm s power 4 and norm s power 4 cancel in the numerator and denominator right. It is a very interesting result, I mean it is a very interesting result that you look at the variance in time and the variance in frequency and that is lower bounded by 1 upon 4 take any signal take any signal any s of t.

It is at least this much that you will live with your uncertainty in your measurement in time and your measurement in frequency, it is a very very profound profound result and this result was proved by no less than the Nobel prize winner Dennis Gabor who is the father of holography. So, let us say the roots this is originally proved by Dennis Gabor in 1946 and of course, Dennis Gabor was great in many areas including in signal processing the signal processing result, but of course, he is a Father of Holography.

So, now we have a very important principle uncertainty inequality or uncertainty principle in signal processing, that we cannot simultaneously localize in time and frequency no matter what? So, a follow up question arises when does this equation when is the inequality is satisfied with equality or when is the equality satisfied, when can we get this actual limit of 1 by 4.

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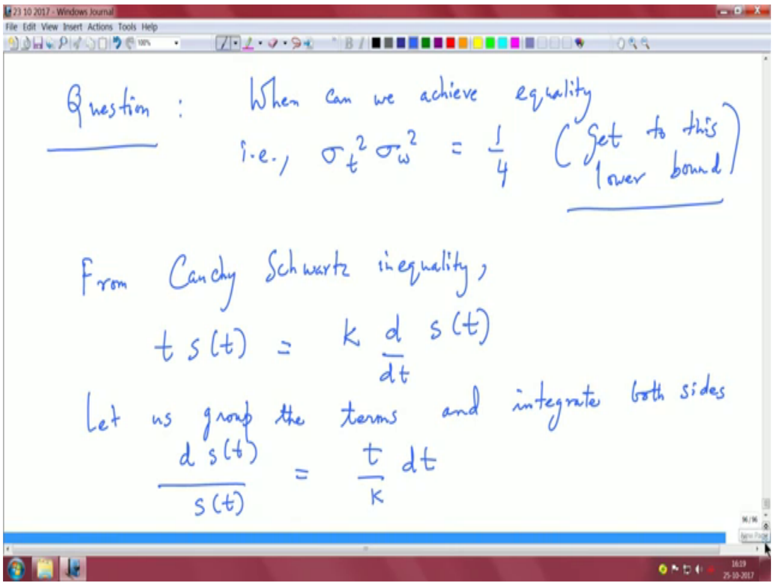

So, for this we asked the next question when can we achieve equality that is sigma t square, sigma w square, sigma omega square, equals one-fourth we need to I need to get to this lower bound. Now fortunately we have a solution from Cauchy Schwartz again that we are comfortable, we can say it happens when t times s of t is some constant k times derivative of s of t and I assume at the moment I just will remove the conjugation here and I will just assume s of t assuming real signals.

Now the k is this factor right I mean when this k times the derivative of s of t, then this is equal the inequality becomes a equality right. Now, we have a differential equation here. So, let us group the terms and integrate both sides. So, how we do this we say derivative of s of t by s of t is t upon k times dt and then I need to do an integration on both the sides ok.

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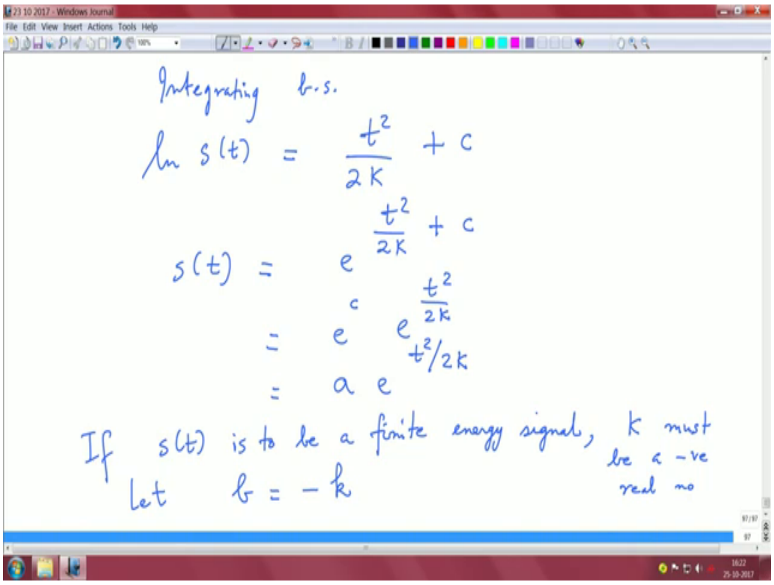

So, if I integrate I get log s of t is t square upon 2 times k plus some constant of integration.

Now, s of t is of the form e power t square upon 2 k plus some constant, which is if you write it carefully it is e power times c times e power t square upon 2 k let me call this as some constant a times e power t square upon 2 k well we have done the integration, but let us see if things are making physical sense for us right.

So, if you recall in the very beginning I said let us consider finite energy signals. Now I have a solution s of t is of this one a times e power t square by 2 k and if you if you let k to be a positive constant the energy in s of t would be infinite potentially it could be infinite right. So, therefore, if s of t is to be a finite energy signal, then k must be a negative real number ok.

So, therefore, let b equals minus k right let b is some negative k. So, if I did that then s of t is some a times e power minus t square by 2 b.

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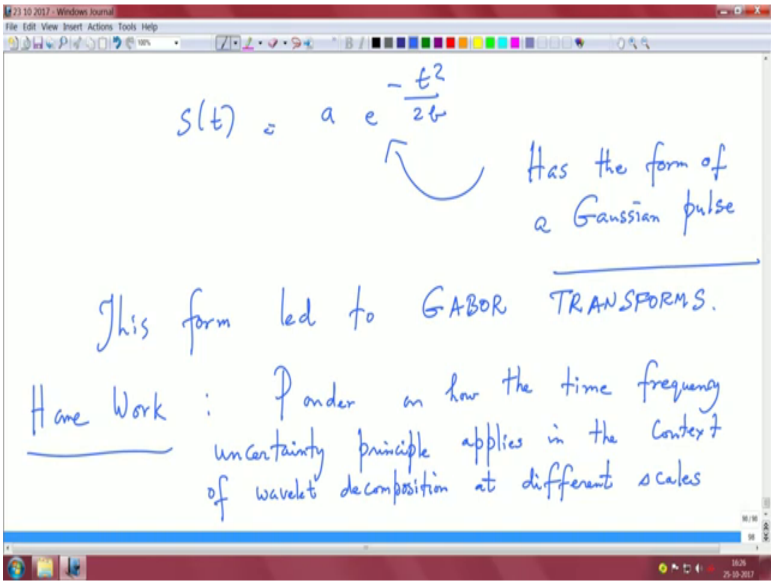

Now, this is easy for us to think about because this is like has the form of a Gaussian pulse, you are not making it a distribution I mean if you if you want if you can choose a to be one upon root of 2 pi b, then it becomes like a distribution t degrades to one otherwise this is just a Gaussian pulse.

Now, this basically led to Gabor transformations. So, this is form this form led to Gabor transforms I mean I am not going to deal with Gabor Gabor transforms, I am not going to deal with Gabor's Gabor's transforms, if we were to go people to go into more details of time frequency analysis we would start possibly with this as our first step and then go into the details ok.

So, but and you can also appreciate why you think about the Fourier transform of a Gaussian pulse why this has the same, shape? And Gaussian has very interesting consequences one of this is this uncertainty relationship, the Gaussian PDF has the maximum uncertainty. If you if you look at the differential entropy right and evaluate this for the Gaussian PDF it gives you the maximum uncertainty right.

And that is the reasons why you look people look into Gaussian PDFs you know when they are in their study of signals with noise and particularly noise having Gaussian PDF. And Gaussian PDF again comes from your central limit theorem results.

So, somehow whether it is consequential or it just nature is providing you that way it is just there are a lot of very interesting properties with Gaussian, whether it is distribution or a pulse or a transform or whatever it is and you see this again and again in signal processing and information theory ok.

So, we will so, we will we will stop at this stage I think this is something which you have to really understand. And, before we conclude I would sort of give you a homework problem to ponder on how the time frequency uncertainty principle applies, in the context of wavelet decomposition at different scales.

So, we have studied the wavelet decomposition and reconstruction. So, at every stage of wavelet decomposition we have a certain resolution right, it is a time resolution. Now think about if this transformation can meet this bound or how close or how far away is it from the uncertainty principle, if you were to do localization through wavelet us this is something to think about this may or may not appear on the exam, but this is a hint for you to think through this homework exercise. So, we will stop here and we can go to questions ok.