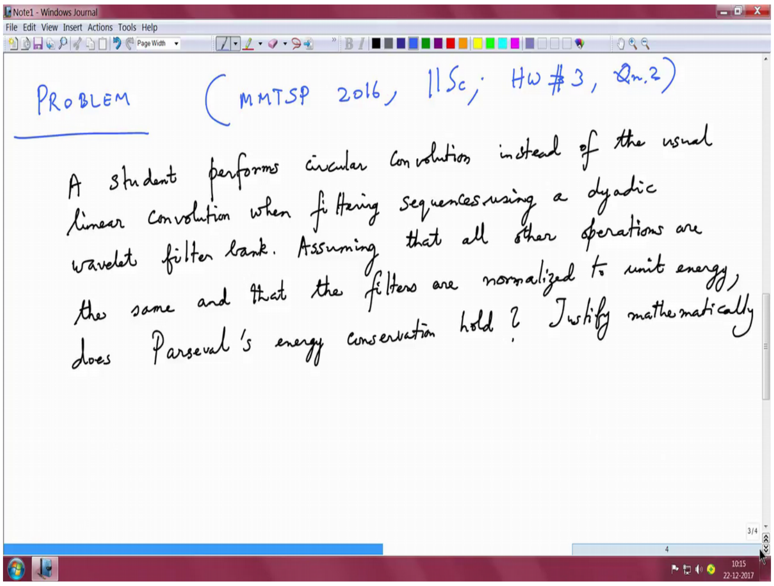
Mathematical Methods and Techniques in Signal Processing- I
Prof. Shayan Srinivasa Garani.
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 63
Problem on circular convolution

So, let us have some interactive problem solving sessions by, my students have taken this course. So, you will see some, illustrations and examples into problem solving, which is useful to understand and digest the concepts learnt during the lectures.

(Refer Slide Time: 00:32)

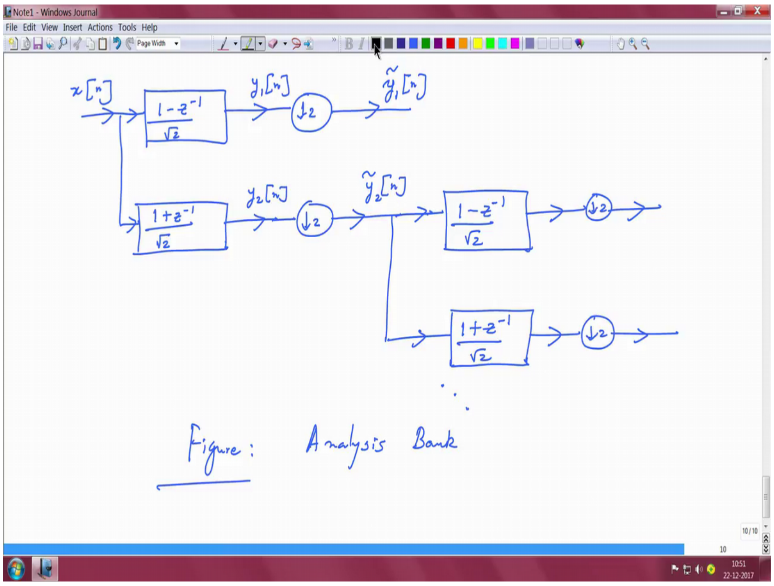


So, this is problem, was posted as part of homework 3, in mathematical methods and techniques for signal processing, in 2016. So, this is a sort of a practical problem which requires theoretical justification. So, here is a situation, a student performs circular convolution instead of the usual linear convolution, when filtering sequences, using a dyadic wavelet filter bank.

Assuming that, all other operations are the same, and the filters are normalized to unit energy, does Parsevals energy conservation hold? Justify mathematically. So, you have learnt linear convolution and circular convolution. Somebody wants to experiment with circular convolution from linear convolution. And there are some advantages of doing circular convolution.

Because, if you do linear earlier convolution, every time, the sequences at the output, go with increasing length, because of the convolution process. So, somebody tried this experiment, and you want to check, if this result is, is correct right.

(Refer Slide Time: 01:45)

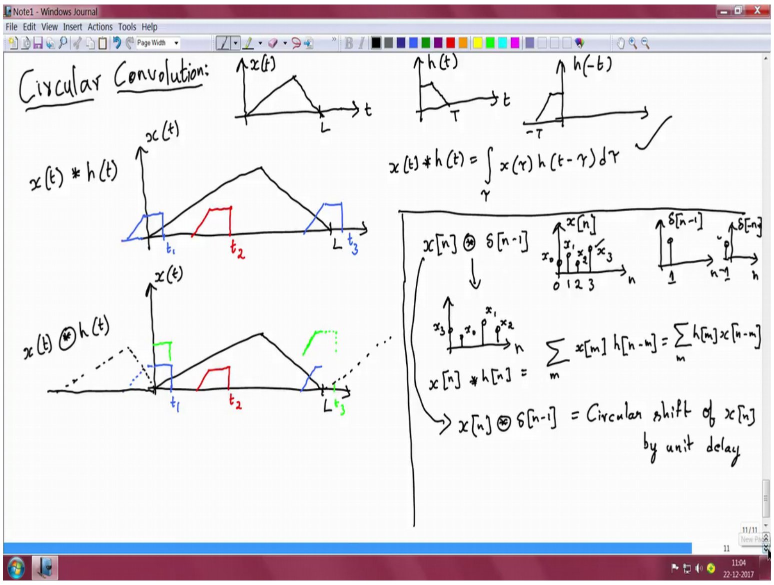


So, we take a signal x of n, we filter it, through a low pass filter, which is 1 plus a delay upon root 2, this is to ensure that the energy is normalized to unity.

So, it goes through a low pass filter here, and a high pass filter, followed by down sampler, down samplers, which is down sampling by 2, and then again, in the low pass band, it goes through a low pass, and a high pass followed by down samplers. And this is the process in the analysis bank, and we would like to, investigate, if Parsevals energy conservation property is satisfied under circular convolution, when you do the filtering operations.

Student: Hello all, I am Chaitanya, a PhD student at IIC. So, I am going to discuss the problem that professor has mentioned. So, here we have an analysis bank, where we are doing a dyadic decomposition. Here the low pass and high pass filters are basically based on the higher, discrete time higher wavelets that we have studied in the class. So, instead that, twist here is, these filters whatever we are seeing, they do not do a linear convolution that we have studied in the class, instead it is a circular convolution. So, before solving into this, I will introduce you to the concepts of circular convolution.

(Refer Slide Time: 03:07)



So, for this Circular Convolution, let us say, we have 2 signals. So, typically so, this is like a transmitted signal x x of t, let us say it has a duration of 2, a length l. And we want to convolve it with another signal. Typically, a filter response, let us say h of t. So, let us say, it is given by this, with some duration, let us say t.

So, now typically, you know the linear convolution operation, how it is done. So, what we do is, for the linear convolution, which is times h of t. So, what we do is, we take, the signal x of t, and we take the reflection of, the other signal which is h of, h of minus t. So, which, we would look something like this. And what we do is, we overlap this particular h of minus t, at different shifts, with respect to h of t, with respect to x of t, and then add it, multiply and add it.

So, essentially for example, with some shift t 1, let us say, we call it as t 1. So, if we want to get the convolution output at t 1, So, what we do is, we overlap this, reverse filter response, in this particular fashion, multiply the 2 signals, and then add that, sum it up, or integrate for the continuous time signal. So, that way we will get, result of convolution, at shift t 1. Similarly for, any arbitrary shift t 2 so, we overlap at that particular arbitrary shift t 2, and then multiply these 2 signals, and then add them. And, for looking at under shift t 3, so, we do the same thing again. So, this is how, we do a linear convolution.

So, essentially we have, we have x of t, the convolution x of t convolved with h of t is nothing but, here in this particular case integral over tau x of x of tau times h of t minus tau d tau. So, this is essentially the regular convolution that we have. So, what happens with circular convolution is, slightly different. So, I will illustrate it with the same example on how we do the circular convolution.

So, let us say, I represent the circular convolution using the convolution symbol with a circular on it, h of t. So, now, so we have this signal x of t. So, when we are performing the circular convolution, what we do is, see this translate at time t 1. So, in this particular case, what we do is, we take the same filter response t 1. So, till the signal x 1 is present within the limits of the signal x 1, we multiply the same signal t 1, but wherever it is going beyond the limits of the x 1 signal, we wrap it back from the end.

So, it goes back like this, to then, and now we multiply these two and add it. So, this is the circular convolution. So, if we are looking at a signal somewhere in middle, it is not going to be different from the linear convolution, but let us say, if I look at under shift t 3 like we have seen here. So, let us say t 3, in order to avoid confusion I will just put the, this shifted version, version here. So, till this particular point it, is the, we actually make it, make the product, but this particular part of the signal is not overlapping with x 1.

So, what we do is, we ensured, take this particular part back to the beginning of the signal, and then multiply these two signals, and do it, and add them up. So, this is essentially the idea of a circular convolution. So, here another thing that we can see is, this is same as, repeating this particular x of t signal, periodically at intervals l, and performing same linear convolution operation.

So, if you look at the shift t 1. So, this blue, response is overlapping with the, replicated original signal x 1. So, this is somewhat similar to, doing a linear convolution operation when the signal is repeated. So, this is essentially the circular convolution. So, more fundamentally, if we look at a discrete time response so, let us look at, what happens when we do x of n, circular convolve with, let us say delta of n minus 1. So, in this particular case, what happens is, we have x of n. So, it might have some impulse responses. So, this is a discrete time signal, and we have delta of n minus 1.

So, what happens here is, when we multiply these 2 signals. So, at time shifts so, here we have time shift 0 1 2 3. So, these are the values of x of n, at different time, different time instances. So, now, when we look at different times so, now, if we reflect this signal back. So, delta of minus n minus 1. So, it is going to be, at minus 1. So, now, if we at time shift. So, now, what we are going to do is, perform the circular convolution operation.

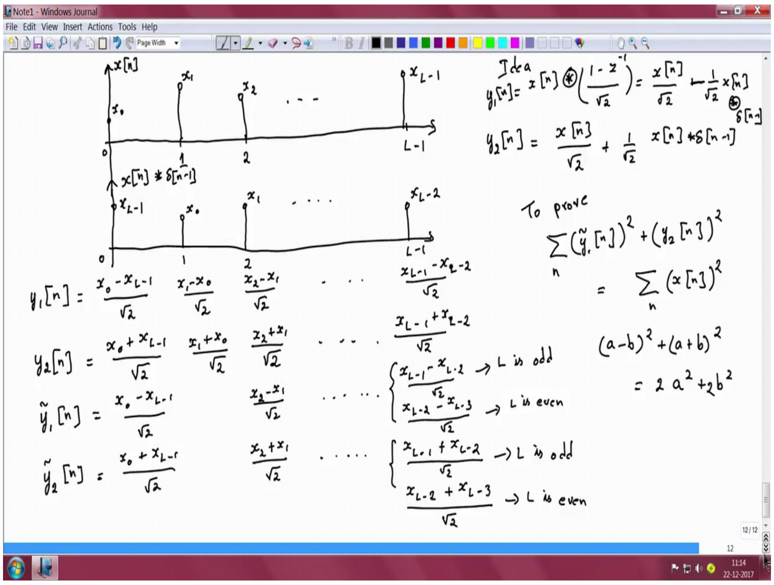
So, we take this particular reflected signal, delta of minus, minus, minus 1, minus n minus 1 multiplied with x of n. So, here with time shift 0, it is not overlapping with of x of n. So, hence we, wrap it back to time instant, this delta, for, this pulse will be wrapped back to this particular instance t equal to 3.

So, what happens is, this is what we get. So, let us say these values are x 0 x 1 x 2 and x 3. So, the resultant signal is going to look like this. So, this, here you have x 3 x 0 x 1 and x 2. So, essentially what we have seen is, when you are delaying the signal, it has wrapped around. So, for a linear convolution operation, when we delay the signal, it is a linear shift in time. Here, what we have seen is, a circular shift in time.

So, a delay corresponds to a circular shift, not the linear shift. So, that is the fundamental idea of circular convolution operation. So, just like how we have derived this one. So, we know that, x of n convolved with h of n is sum over m h of n minus m, or equivalently it is it can be written the other way as well, you take h of m times x of n minus m.

So, a similar result can be proven here. So, what we have seen here, is so, from here, what we have is x of n circularly convolved with a delta function n minus 1 is essentially a circular shift, shift of x of n minus 1. So now, this gives us what a cyclic convolution is. So now, we use this property of circular convolution to, look into the problem.

(Refer Slide Time: 15:13)



So, for this particular problem, we have the input signal x of n. Let us say, it goes from 0 1 2 so, until l minus 1. So, it has l nonzero values, or the duration of a signal is l. So, let us say these values are x 0 to x 1 x 2 so, until x l minus 1. Now, the idea is as follows. So, essentially x of n convolved with, our circularly convolved with, 1 minus z power minus 1 by root 2, is nothing but x of n by root 2 minus 1 by root 2 times x of n circularly convolved with delta of n minus 1.

So, in this case, we have this, as x of n. Now what will be x of n circularly convolved with delta of n minus 1. So, it is going to be circularly shifted signal. So, here we will have, here we will have x of l minus 1, here we will have x naught, x 2, it will and, the final value this x 0 will come back here, at a time l minus 1.

So, this is what we have. So, this one is y 1 of n, and similarly y 2 of n is nothing but x of n by root 2 plus 1 by root 2 times x of n convolved with delta of n minus 1, yeah. So, now, from essentially, what we often can be obtained by subtracting these two signals, x of n and this these two signals, and scaling by root 2, and y, y 2 often can be obtained by adding these two signals, and scaling by root 2. So, that way, the values that we have here, y one of n will be x naught minus x l minus 1 by root 2 at dimensions 1, it is going to be x 1 minus x naught by root 2.

Here, yeah there is a type of. So, this is x 1. So, here we have x 2 minus x 1 by root 2, and the last one is going to be x l minus 1 minus x naught root 2. Similarly we have y 2 of n, is instead of these negative signs, will just have the addition signs. So, here we have x naught plus x l minus 1 by root 2 x 1 plus x naught by root 2, x 2 plus x 1 by root 2, and the last one is going to be plus x naught by root 2. So,.

Student: Sir, x l minus 2 is not there.

Ok, yeah, thank you. So, yeah, there is a type of here, yeah, when we do the circular shift, here we will get x l minus 2. So, these things also will change, x l minus 2, x minus 2. So, now, we have this y 1 and y 2, we are going to down sample by 2. So, essentially we are going to discard, every alternate element. So, if we look at y 1 tilde of n, we will retain these samples x 1 minus x l minus 1 by root 2.

So, this is kept. We have x 2 minus x 1 by root 2, and every alternate things are skipped. And the last one that will be left here, would be either x l minus 1 minus x l minus 2 by root 2, this if, l, l is odd, and all the previous one which is x l minus 2 minus xl minus 3 by root 2, if l is even, and y 2 tilde of n similarly is going to be the same set of samples with just the positive sign.

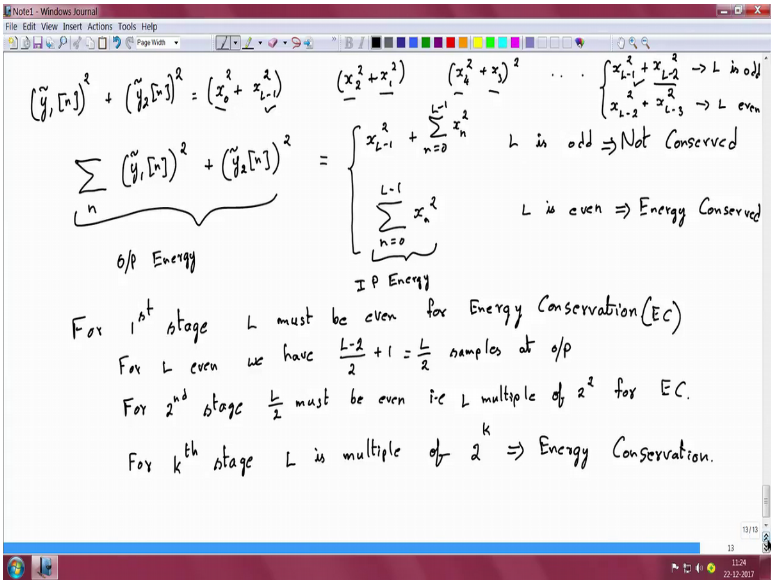
So, now, these are the outputs after the first stage of decomposition, and now we have to compute the, energy of these 2 signals at the output. So, for that essentially what we are trying to prove here is.

Student: (Refer Time: 23:11).

Yeah. So, had a small correction. So, these values are less. So, we have the outputs at the first stage of decomposition, and to prove the Parsevels theorem so, what we need to prove, here is summation of all these y 1 tilde of n square plus y 2 tilde of n whole square summation of n, this should be same as, summation over n x of n whole square. So, this prove, this is essentially, what we are trying to prove. So, we will use, these properties. So, here we see that y 1 tilde of, of n and y 2 tilde of n, are of the form a minus b and a plus b.

So, we know that a minus b whole square plus a plus b whole square is 2 times a square plus 2 times b square. So, I mean, using this we can compute this y one tilde of n square plus y 2 tilde of n square.

(Refer Slide Time: 24:28)



So, the values that we have, we will get right now for y 1 tilde of n whole square plus y 2 tilde of n whole square.

So the values that we will get, from the previous slide so, we, here for the first term, we have x naught square plus x l minus 1 whole square, and because of the factor 2 here, this root 2, the contribution of this root 2 will get cancelled out. So, you will have x naught square plus x l minus 1 square, and the next term is going to be, x 2 square plus x 1 square, and the next term is going to be x 4 square plus x 3 square, and the last term, depending on whether l.

So, here we have 2 cases, whether l is odd or even. So, depending on the case that we have, it will be x l minus 1 whole square plus l minus 2 whole square, if l is odd thought, and for l equal to even, the final value, it is going to be different, it is based on, it has contributions of x l minus 2 and x l minus 3. So, we have x l minus 2 whole square plus x l minus 3 whole square for l even. Now we have these terms.

So, to get the energy at the output of first stage, we just need to add all these terms. So, for which so, adding all these terms.

So, again depending on whether l is odd or even we will have 2 cases. So, if we look at l odd case. So, x 1 appears only once, x 0 appears only once, x 1 appears once, x 2 appears only once, x 3 appears only once, x 4 appears only once, x l minus 2 also appears once, however x l minus 1 appears twice. So, here the output is going to be x l minus 1 whole square plus summation n equal to 0 to l x l whole square, and when l is odd. When l is even, you see that this repetition of x elements 1 does not happier, happen and, each individual term x 0 till x l minus 1, they appear only once. So, essentially the output is going to be 0 x n whole square l is even.

So, from here so, this is, energy at the input. And this is a output energy, of output energy. And for l equal to even so we have energy conserved, and for l equal to odd, energy is not conserved. Now so, here we have done this for the first stage, now what happens at the output of second stage so, essentially we have that, for first stage, l must be even for energy conservation.

Now, let us see how many samples, we have after the first stage. So, if we, for l equal to even. So, here we have x 0 x 2 x 4 so until x l minus 2. So, for l even, we have l minus 2 by 2 samples at output, sorry, it is going to be l minus 2 plus 1 samples, at output equal to l by 2 at output. So, if we want energy conservation. So, basically for second stage, if we want energy conservation, l by 2 must be even, that is l multiple of 2 square, for, let us say, I called it as energy conservation for energy conservation.

So, for similarly if we action to k th stage so, l is multiple of 2 power k, implies energy conservation. So, in summary we have looked into this circular convolution operation, the circular convolution operation is typically done on finite duration signals, as you can see here. So, for, infinite duration signals, there is no concept of wrapping it around to the end or to the front. So, circular convolution is usually done for a, finite duration signals, where the delay here is a circular delay, not the li, linear shift.

And we use the circular convolution, to analyze, the, energy conservation principle for the particular structure, that was given. And we see that, only when, the length of the signal satisfies some certain properties, we see that the energy conservation holds true.