Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 61
Haar wavelet and link to filter banks

So, let us try to link the Haar wavelet decomposition and link these 2 filter banks ok.

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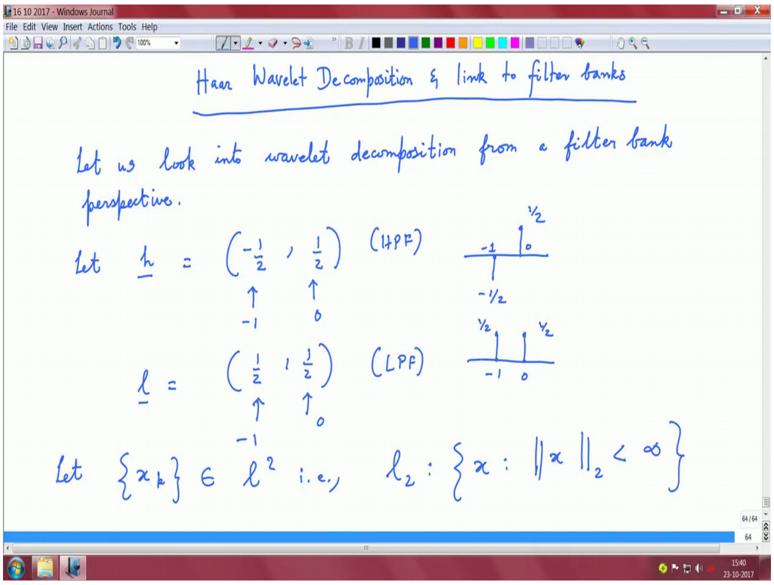

So, let us look into wavelet decomposition from a filter bank perspective. Usually when people teach filter bank theory that is multi rate signal processing, they, they have portion of their syllabus explicitly on systems that is multi rate systems and then there is also a portion on wavelet us and we will see the link between the two. I think you will appreciate it much better. So, let us start with the following. Let h be this vector which is minus half and plus half. So, I say this index is minus 1.

This index is 0. Now we can think about it as 1 half time index 0, time index minus 1, it is minus half and let l be 1 half, 1 half, this is index 0, index minus 1. I think you can very clearly see that this is like your high pass filter, h is like your high pass filter; first order this is like your low pass filter, first Haar right. I mean if you just look at the coefficients you can get sort of an idea just where we are eluding. Let the sequence x k belong to l 2 that is l 2 is this space such that the norm l 2 norm is basically finite.

Now, let us do the following operations.

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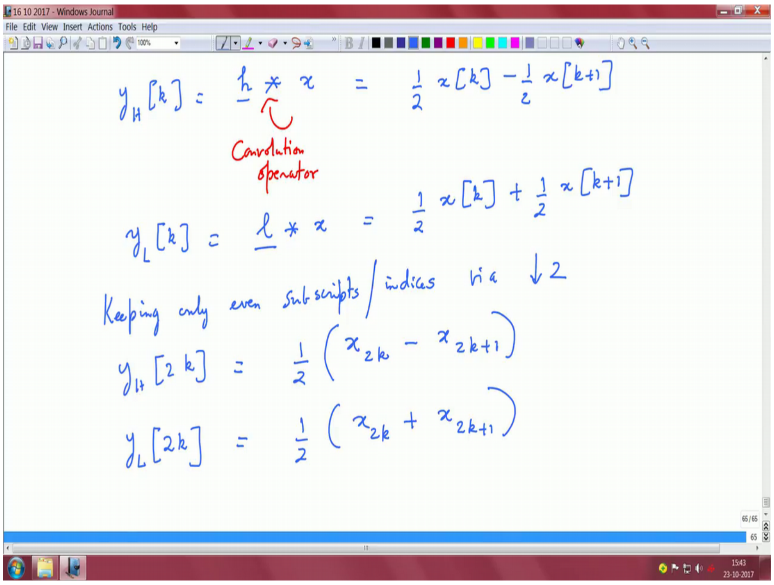

Let us consider y subscript h of k is h involved with the sequence x right and this is the convolution operator. Of course, people who think about it in computer science, they think it is a star, it is a multiplication but for us it is convolution ok. Now this is one half directly you can just convolve it with x, this is half x k minus 1 half, x k plus 1 ok. This is straightforward. Similarly we have y L of k is the filter l convolved with x and it symmetrically follows it is 1 half x k plus 1 half x k plus 1. Now if we keep only the even subscripts or I would say indices, we are down sampling by 2 right, then I would say y h of 2 k is 1 half x 2 k minus x 2 k plus 1.

I think putting it like this is a little bit of an old form of notation. You could put the extra braces, you can say x of 2 k, but you know, I will just put this as a subscript sometimes interchangeably but nevertheless you can just compile this through your head that this is indeed what it is; y L of 2 k is 1 half x 2 k plus x 2 k plus 1. Now this is sort of familiar, if we kind of bring in the scaling into picture and the idea of down sampling and filtering from what we have done through the wavelet decomposition.

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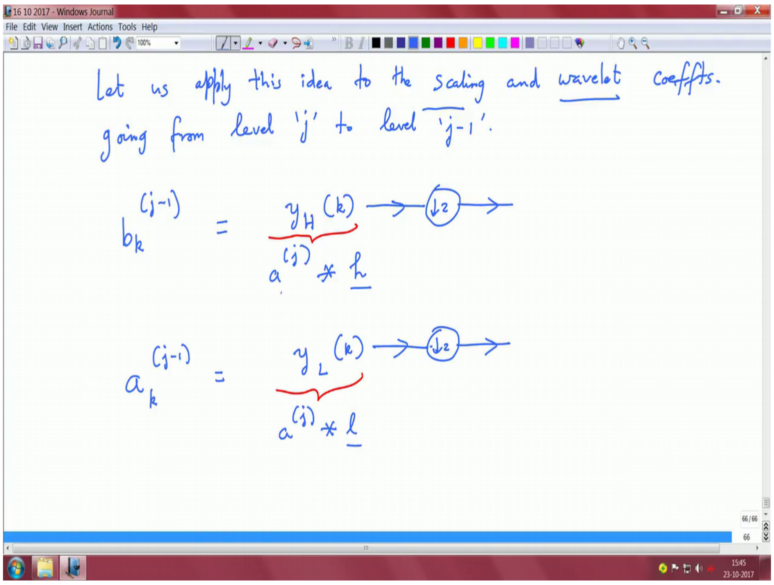

So, let us apply this idea. That is what we just did to this scaling and wavelet coefficients going from level j to level j minus 1. Now, b suffix k j minus 1 can be interpreted as some y h of k; take this and down sample this by 2 alright. This is basically y h k is basically a j convolved with h.

Now you can think of a k j minus 1 as basically y L of k and feed this to a down sampler and you can interpret y L of k as sum a j convolved with a filter l and then down sampling it right. This is sort of straightforward. What we have done is basically, we have used the identity for linking a j, a j and a j minus 1 and a j with b j minus 1. This is what we have done. So, if you think about it in terms of filtering operations, you can say that you have a j here.

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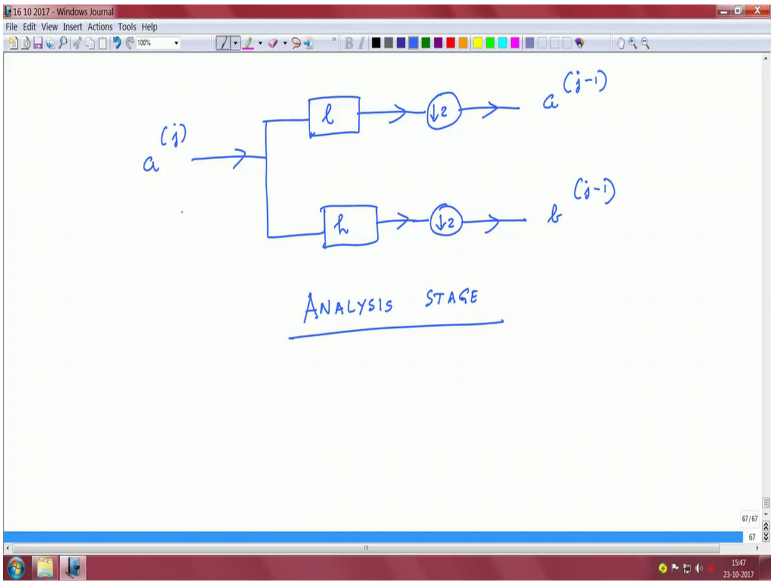

This goes through filters l and h, then there is down sampling by 2 and you have a j minus 1 here and you have b j minus 1 here. This is basically like the analysis, analysis stage of a filter bank. We have done nothing except that we have sort of interpreted the wavelet decomposition through filtering the coefficients from s k l j through appropriate filters ok. Now we can think about it in a similar way for the reconstruction procedure.

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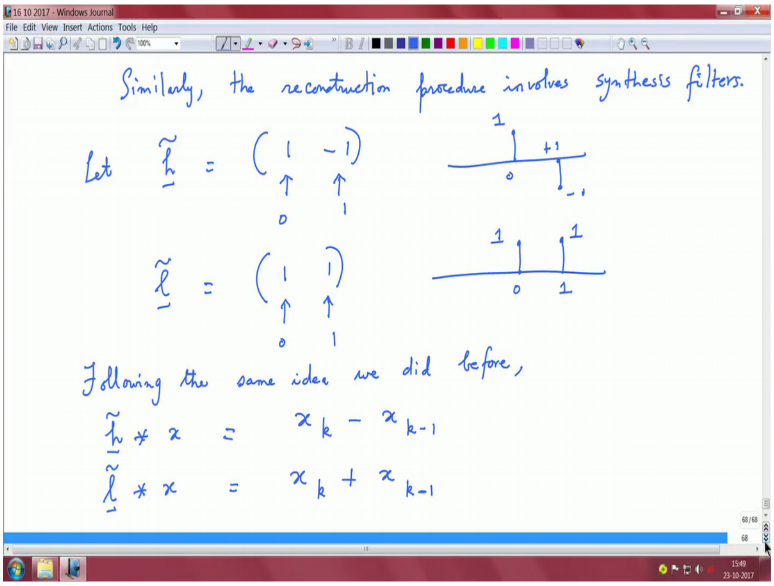

Similarly the reconstruction procedure involves synthesis filters as how this can be done. Let h tilde b 1 and minus 1 right. I will just index this a 0 index this is 1 and you can think about it as this point I have 1, this point I have minus 1 and I have l tilde as 1 and 1. This is 0, this is one; just interpret this as 1, 1, 0 and this time this is 1 here ok.

Now, following the same idea, we did before we consider h tilde convolved with x and this is x k minus x k minus 1 and l tilde convolved with x is x k plus x k minus 1 ok. This is straightforward, just filtering operations.

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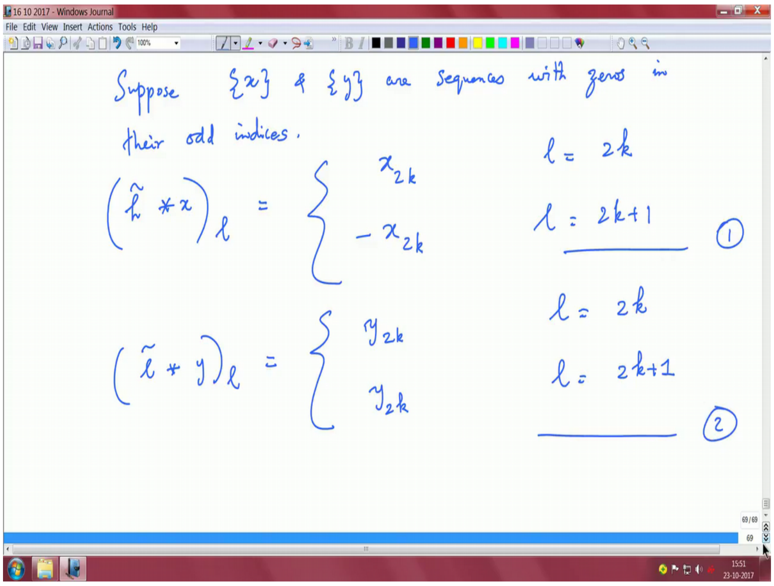

Now suppose x and y are sequences with zeros in their odd indices. Suppose there are they are sequences with zeros in their odd indices. Now h tilde convolved with x is basically x 2 k when l equals 2 k minus x 2 k then l equals 2 k plus 1. We said these are sequences with zeros and they are indices right.

So, they are they are already sort of up sampled. You can interpret the sequences as sort of up sampled because as zeros in their odd indices. So, therefore, if you look at, if you if you have index with k which is even with k minus 1, it is basically odd right. So, therefore, one of them would exist. Similarly you consider, l tilde convolved with y indexed as a, indexed by l and this is basically y 2 k then l equals 2 k and it is same as y 2 k and l is odd which is of the form 2 k plus 1. Now let me call this equation as 1, this equation as 2.

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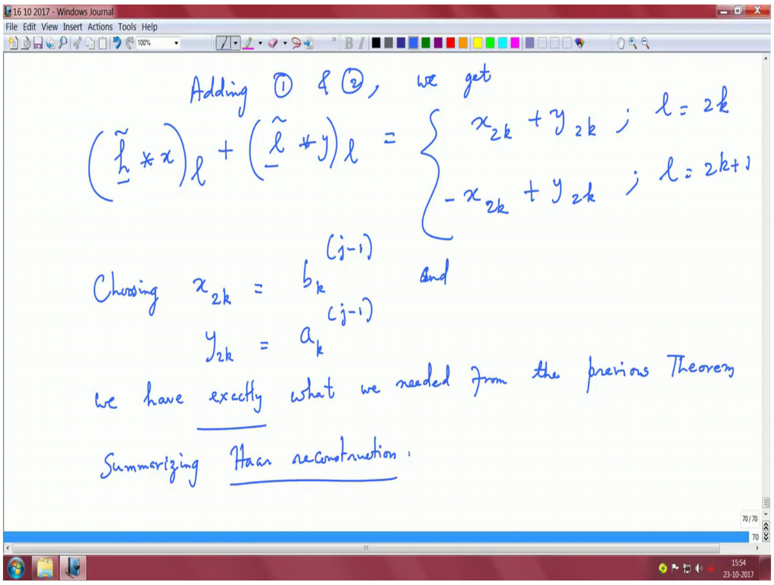

Adding 1 and 2, we get h tilde convolved with x and l tilde convolved with y is x 2 k plus y 2 k, then l is e 1 it is x 2 k negative plus y 2 k when l is odd. Now choosing x 2 k as b j minus 1 and y 2 k as a suffix k j minus 1; we have exactly what we needed from the previous theorem summarizing Haar, summarizing Haar reconstruction right.

So, let me give you the picture.

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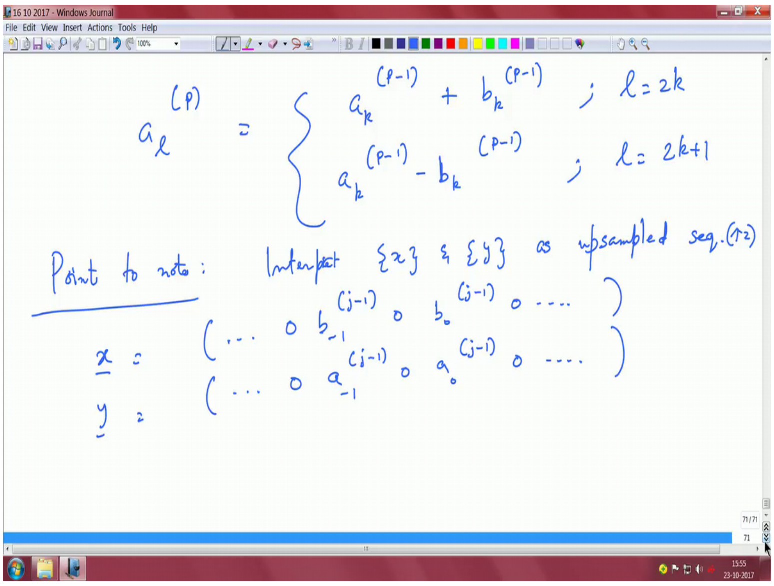

To just recall you had a l of p is basically a k p minus 1 plus b k p minus 1 where l equals 2 k and a k p minus 1 minus b k p minus 1 where l is odd. So, we are sort of set. I think the only point to note as I mentioned earlier, interpret x and y as up sampled sequences that is x is and this is up sampled by 2. So, you have something 0 b minus 1 j minus 1, 0, b 0 j minus 1, 0, dot dot dot and y can be interpreted dot dot dot 0, a minus 1 j minus 1, 0, a 0 j minus 1, 0, dot dot dot. Now I think with this we are sort of set in terms of interpreting this filtering operation.

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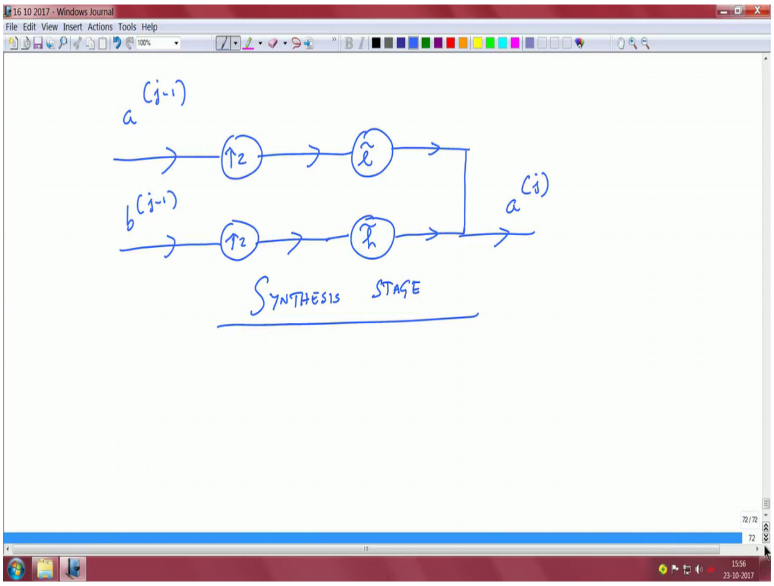

So, we can think of a j minus 1 being sampled and then going through this filter l tilde and we have b j minus 1 being up sampled going through this filter, h tilde and then combining them to get to give you a j right and this is basically the synthesis stage ok.

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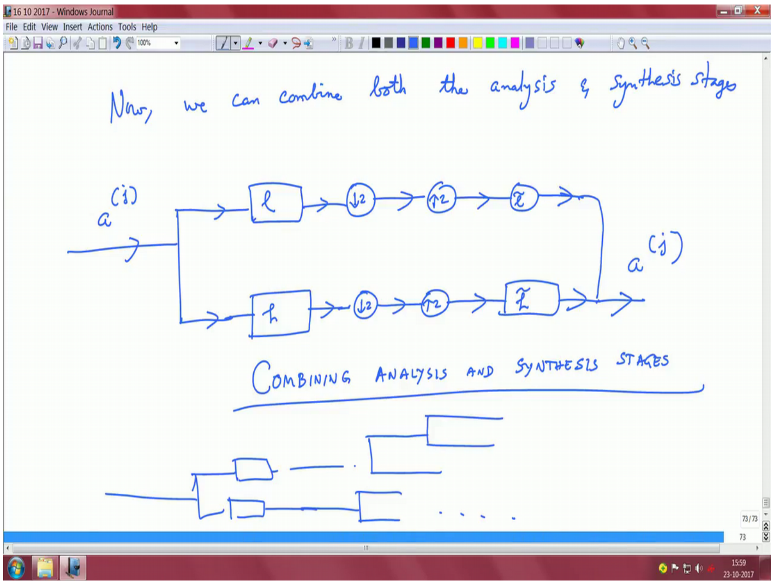

Now we can combine both the analysis and synthesis stages. So, what we get is, we have these a js in scale j that goes through some filter l, followed by down sampling by 2, followed by sampling by 2 with some l tilde in the first branch.

And in the second branch, it is basically filtering by h, then down sampling by 2, then up sampling by 2 and then you have h tilde and then you combine from both the branches and you get a j and this is basically our filter bank basic filter bank with you know the low pass filters and high pass filters ok. And one can extend this at different stages, now if this is stage say 0 j equals 0, then say j equals 1, j equals 2 and therefore you can you can solve at every point, you can think that this you know you can you can say that this is you can expand this out.

So, basically there is one stage here. This again can break up into 1, this possibly can break up into the next so on. You could have a dyadic decomposition or you could have a binary tree decomposition where you can basically decompose on the other side as well, you can fork here ok. Now we can clearly appreciate the role of the radix 2 there and the down sampler and up sampler right down sampling by 2 and up sampling by 2 and the radix 2 in the Haar wavelet and then the interpretation of these filters. So now, given any wavelet decomposition, maybe 3 adic or 5 adic, you can kind of sort of construct the filter bank equivalent representation ok. So, I think this completes filter banks and wavelet us and what you think of the what how good a wavelet is can be thought about and interpreted using the Fourier transform. So, that is the reason where you have this notion of filter that is wavelet coefficients being filtered through appropriate filters and you can see the signature of the filters. We have Fourier analysis ok. So, both of these are sort of coupled.

So, with this we are towards the end of wavelet us, we have completed wavelet discrete wavelet transform basically the decomposition, reconstruction, certain properties and interpretation and linking them to filter banks ok. So, this is by any means not fully comprehensive, I mean one has to go through harmonic analysis for a very rigorous treatment on wavelet us but at least it will give you a very good feel from engineering perspective; how to go about constructing the transformation itself starting from the definition of multi resolution analysis and then building the necessary tools ok.

So, I think this we are going to stop here.