Mathematical Methods and Techniques in Signal Processing - I
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 60
Wavelet Reconstruction

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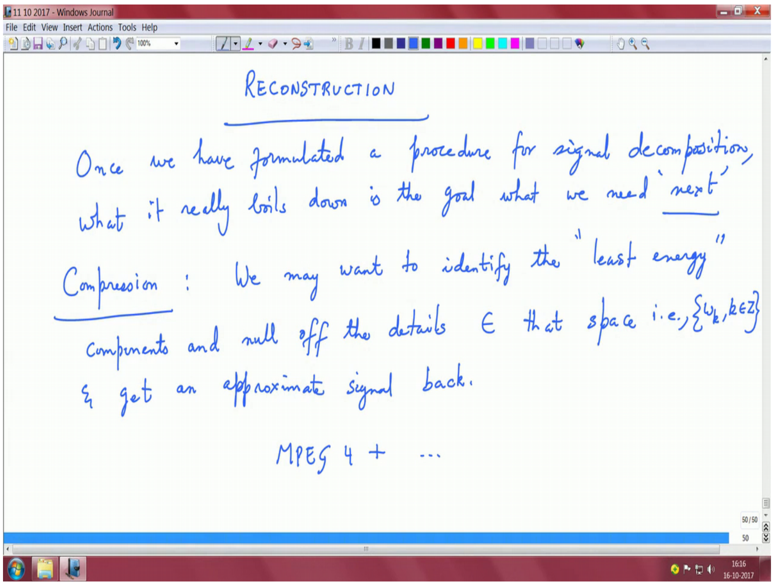

So, now let us get into the reconstruction part, once we have formulated a procedure for signal decomposition, what it really boils down? Is the goal, what we need next? Right, I mean we have to understand what we need with this with this technique. So, I will give you 2 important applications, the first is for compression. We may want to identify the least energy components and null off the details that belong to that space. That is, those W k s k belonging to z, you might figure out which of them are least energy components and null them off.

And get an approximate signal back right. Let us say you are in some scale say w2 and that energy was just probably 1 percent of the total energy or you are in scale w3 that was just 0.1 percent of the total energy. That means, you figure out where the energy is the least null them off because, when you reconstruct it you would not have lost a lot of information. You would have lost some information, the information I mean if it has to be completely lossless it has to be perfect transformation right? I mean you transform it, you retain everything and then you get it back. And if you think that you know I want to compress my data idea, why do I need a lot of these signals in various of spaces?

Then you figure out where the components have the least energy in terms of what your threshold is in terms of energy threshold, null them off and from there you reconstruct things back. You get an approximate signal not what was original, but that way you will be able to represent the approximate energy a signal, post nulling with minimal number of bits or fewer number of bits ok. This is one idea and this is done routinely in MPEG etcetera, MPEG 4 plus standards, then these days they are really very much evolved, but I mean MPEG 4 I think had wavelet transformations.

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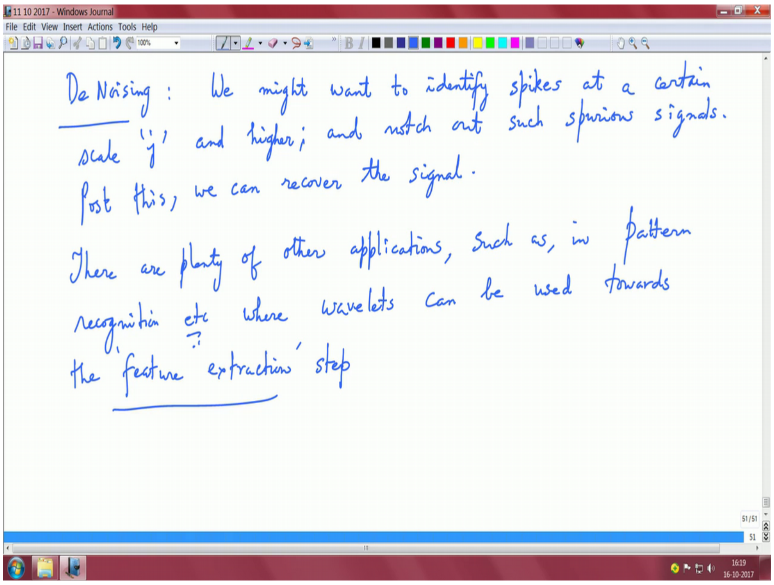

The second application is De Noising, we might want to identify spikes at a certain scale j and higher and notch out such spurious signals. So, basically after cleaning the signal basically after removing these spikes posters, we can recover the signal.

So, and there are plenty of applications from this purpose I mean we want to have a clean signal, we want to remove the noise first, get a clean signal to start with and from which you can do many things. And I would also suggest that there are plenty of other applications such as in pattern recognition etcetera where wavelets can be used towards the feature extraction step. So, this is another possible application you do a wavelet decomposition let us say got the coefficients in for f naught, you got the coefficients for w naught, w1, w2, w3 so, on right and.

You can use those coefficients and in some form basically towards pattern recognition applications. So, if you have to recognize certain shapes etcetera you first you can do a decomposition and the decomposition gives you certain features and you might want to use those features in some way. So, wavelet can be in some cases a good tool for feature extraction ok.

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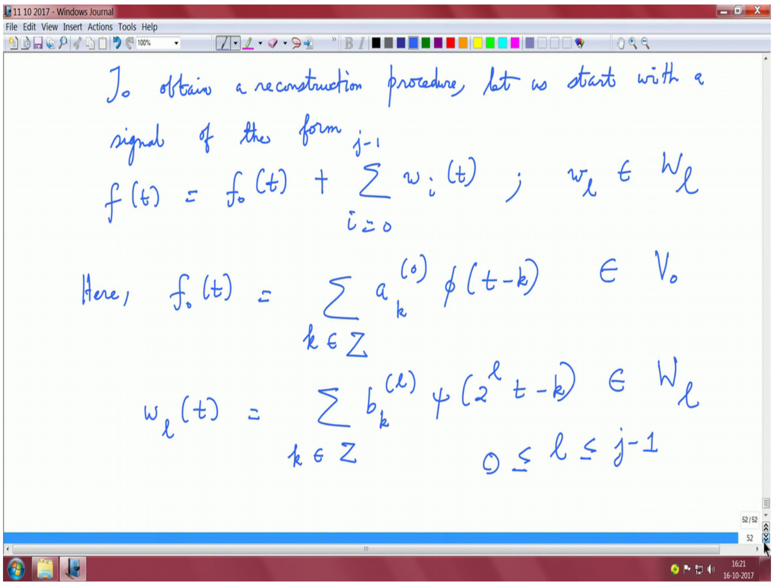

Now, let us formally begin the process of reconstruction. To obtain a reconstruction procedure, let us start with a signal of the form f of t is f naught of t plus summation i equals 0 to j minus 1 w i of t, where some w l belongs to the space capital W l.

Now, here f naught of t is basically summation k belonging to the set of integers b k use a different subscript here. So, I will call this is a k a k 0 phi of t minus k and of course, this belongs to the space v 0. Now, w l of t is summa k belonging to the set of integers this I will use b k l chi of 2 power l t minus k this belongs to the space W l for 0 less than or equal to l less than or equal j minus 1 ok. We have these basics to start with now, what is our goal?

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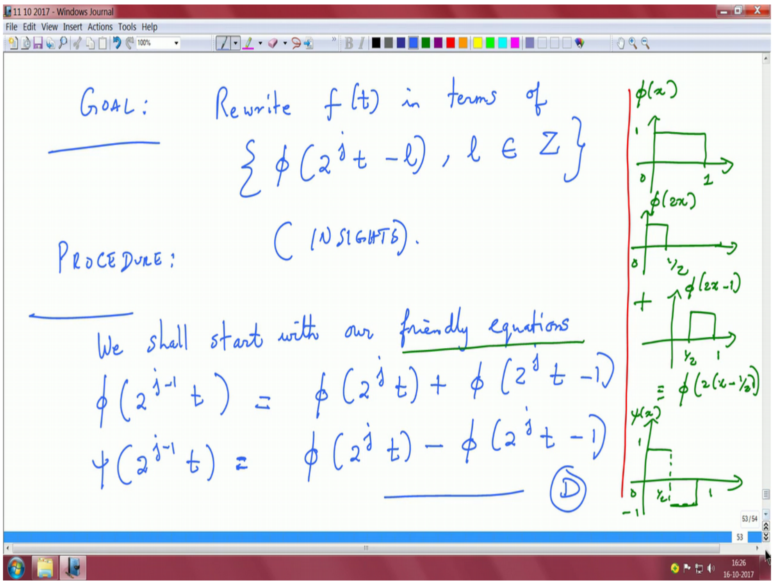

Rewrite f of t in terms of phi of 2 power j t minus l, l belonging to the set of integers right, this is our goal. So, we have all our f naught and w l's from which we have to get this in this form right, this is our reconstruction goal.

Now, I will start with the procedure or some important steps. So, we should start with our I call them friendly equations and I call them friendly equations because we have seen them over and over again. This is phi of 2 power j minus 1 t which is basically phi of 2 power j times t plus phi of 2 power j times t minus 1. And then chi of 2 power j minus 1 t is phi of 2 power j t minus phi of 2 power j t minus 1. And I just have to recall this carefully basically this if you think about this, draw this aside here.

This picture should be very clear in your in your mind so, if this is phi of x 1. This can be decomposed using phi of 2 x plus it is translate phi of 2 x minus 1 right, which is phi of 2 times x minus half. That is you have this is the same as phi of 2 times x minus half because, if you have sketched phi of x what you are doing is just translating this by half a time step here right.

So, this picture should be very clear in your mind and you are chi of x is basically this wavelet function right. And we will take the (Refer Time: 13:38) and basically using this you can that is the reason why you have a negative sign. If you if you just get this picture in your mind rest will be very easy to go through.

So, I call them our friendly equations, this maybe I will just call this equation number as D, just for consistency.

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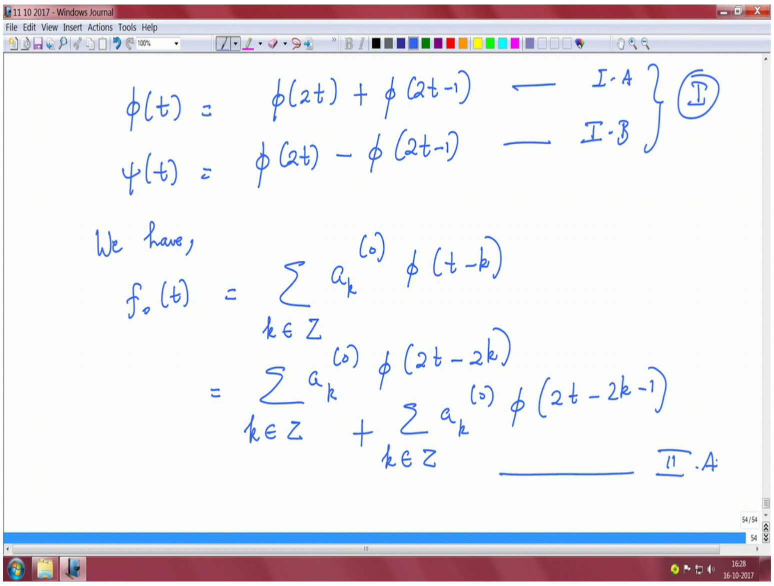

So, phi of t is basically phi of 2 t plus phi of 2 t minus 1, I call it 1A. chi of t is basically phi of 2 t minus phi of 2 t minus 1, I call it 1B, together this set is 1.

Now, we have by definition f naught of t is basically in the linear span of phi of t minus k right, which is basically some a k 0 phi of t minus k and this I write it slightly in a different way. This is k belonging to Z a k 0 phi of 2 t minus 2 k plus summation k belonging to Z a k 0 phi of 2 t minus 2 k minus 1. Just am using 1A here basically and then what I am doing is going through the odd and the odd and the even parts. So, any shift k can be an odd shift and even shift.

So, using that I basically decompose this in this form, ok. Now, I could call this as say equation 2A.

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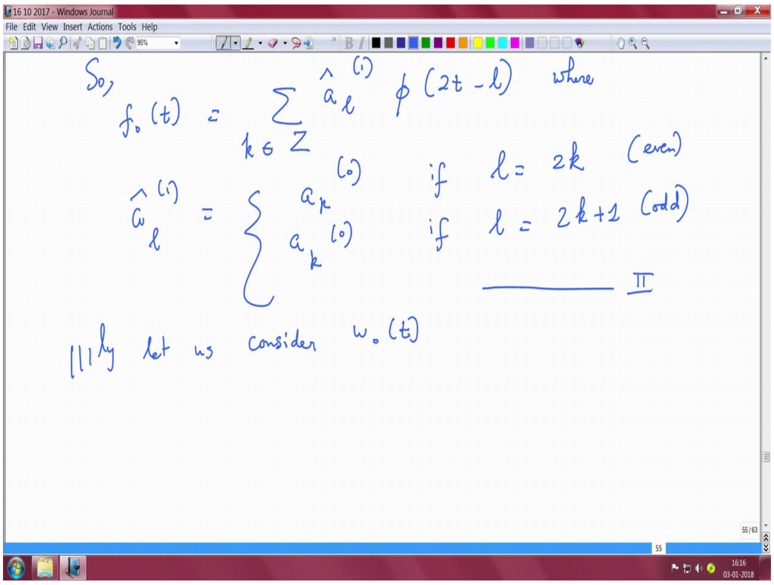

Now, f naught of t is basically summation k belonging to Z a l I just put a hat here for reconstruction. phi of 2 t minus l, where this a l hat belonging to scale 1 is a k 0, if l equals 2 k it is also a k 0 if l is 2 k plus 1. This is the even case; this is the odd case ok. So, similarly let us consider w naught of t, let us denote this equation as 2B.

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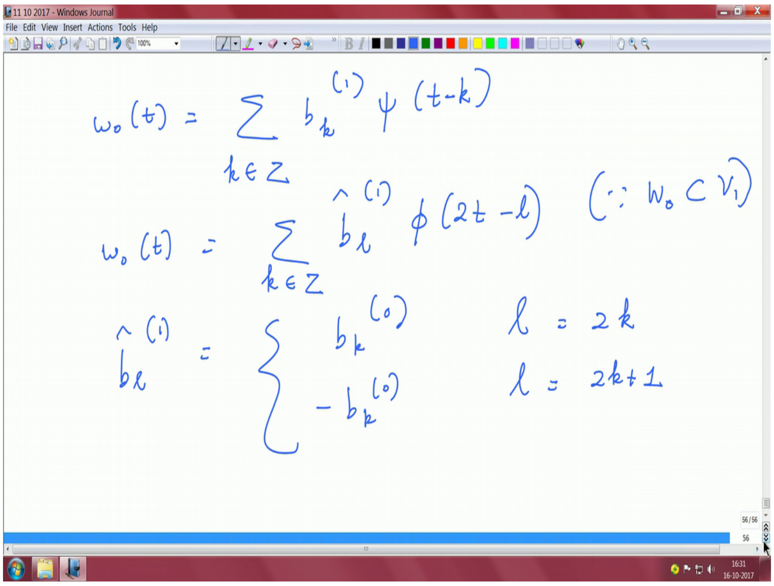

Now, w naught of t equals summation k belonging to Z b l of 1 chi I could put I have k here some b k of 1 chi of t minus k.

Because, it is the wavelet using the wavelet basis I can expand this. But, I can write w naught of t as summation k belonging to the set of integers some b l hat of 1 phi of 2 t minus l. Because, w naught belongs to V1 right, w naught belongs to V1.

So, basically I have done 2 forms of expansion, one in this w naught space itself and other is in the space in use is using V1 as the basis. So, now, b l hat of 1 is basically b k of 0 l equals 2 k and minus b k of 0 l equals 2 k plus 1 right. I think this is this is what we be land up with right because if it is e one this is b k 0 if it is odd it is minus.

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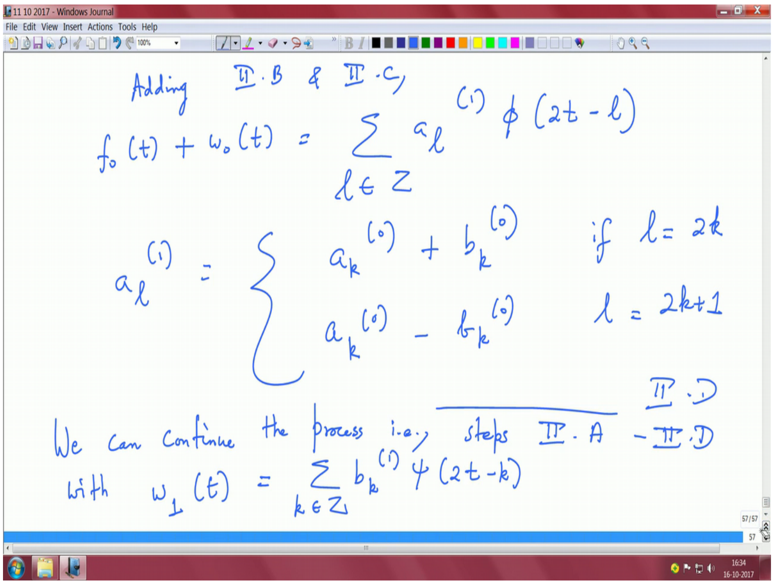

Now, if you sum so, let us designate this equation as 2C.

Now, adding 2B and 2C we get f naught of t plus w naught of t is summation l belonging to z a l of 1 phi of 2 t minus l. Where, a l of 1 equals a k 0 plus b k 0, if l equals 2 k. And it is a k 0 minus b k 0 for l equals 2 k plus 1, this is basically equation 2D.

Now, we can continue the process that is steps 2A to 2D, starting with w 1of t which is summation some k belonging to sum of integers b k 1 chi of 2 t minus k right. Like w 0 that we had earlier here hiring this has to be a 0 here, I think this is 0 here because scale 0 here instead of 1 chi t minus k. And then this is basically b k 1 chi because, it is scale 1 ok, consistent with this.

So, now we do this process 2A through 2D recursively, w1, w2 so, on right so, what do we how do we go about this?

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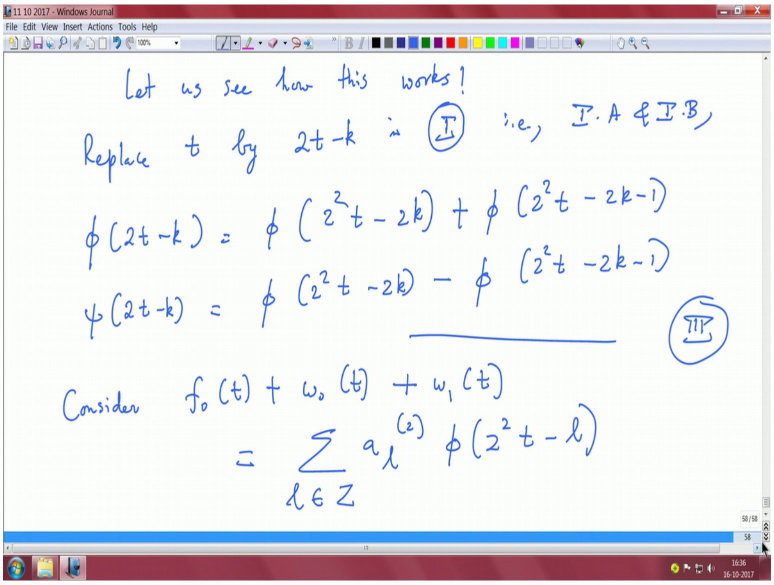

Now, let us see how this works. Now, you replace t by 2 t minus k in equation 1, that is 1A and 1B that we started with. What we get is phi of 2 t minus k is phi of 2 square t minus 2 k plus phi of 2 square t minus 2 k minus 1. And then we get chi of 2 t minus k such as phi of 2 square t minus 2 k minus phi of 2 square t minus 2 k minus 1. Let us call this equation 3, right just we are going to 1 scale down.

Now, let us again consider f naught t plus w naught t plus w1 t, this should exist in scale j equals 2. So, therefore, we can write this within the linear span of the shift orthonormal basis in scale 2, using these scaling functions appropriately. Now, if you take this function and therefore, you can now compare with the previous level like what we had. We can conclude.

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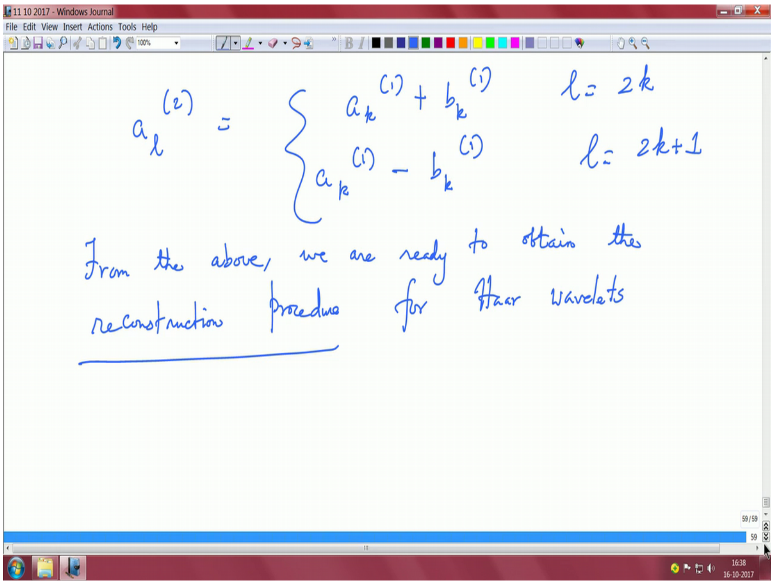

That a l in scale 2 is basically a k in scale 1 plus b k in scale 1 for l equals 2 k and a k in scale 1 minus b k in scale 1 for l equals 2 k plus 1 ok. Like what we had for the previous step from 2A to 2D, we just do this by adding f naught w naught and w1 and then decomposing that and then linking it to a previous scale.

Now, from the above we are ready to obtain the reconstruction procedure and I mean this is for Haar for Haar wavelets. We ready to obtain the reconstruction procedure and this can be summarized more formally and I will state this as a theorem.

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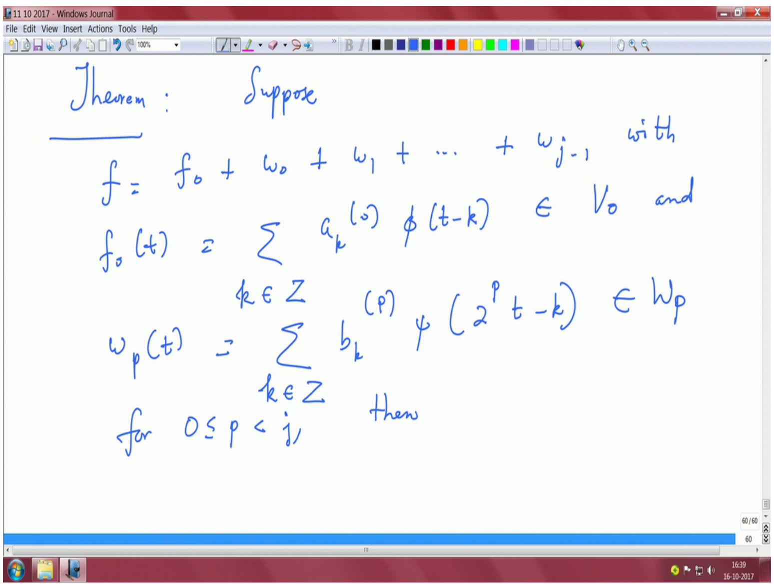

Suppose f equals f naught plus w naught plus w1 dot dot dot till w j minus 1 with f naught of t equals summa k belonging to set of integers a k 0 phi of t minus k belonging to V0. And w P of t which is summa k belonging to the set of integers b k of P chi 2 power P t minus k belonging to the space W p for 0 less than or equal to P less than j.

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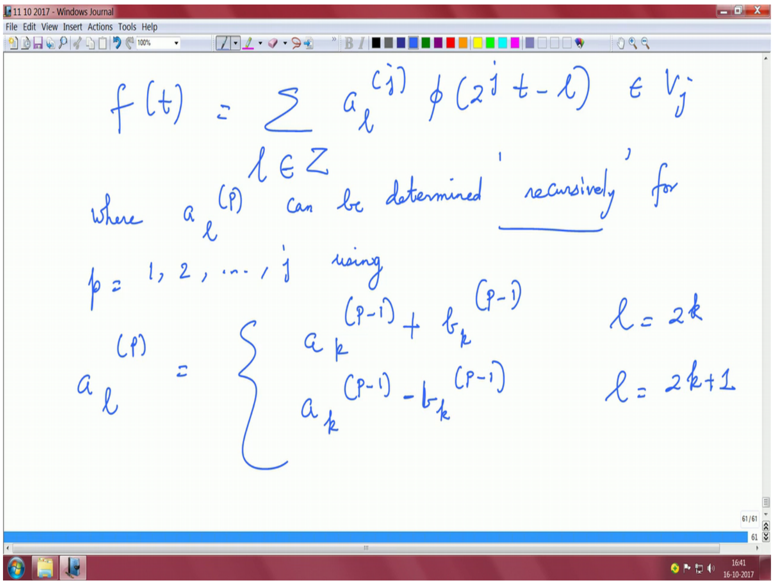

Then, if this is true then f of t can be written as l belonging to the set of integers a l of j in scale j phi of 2 power j t minus l belonging to the space V j. Where, a l of p can be determined recursively, this is very important because we are linking one scale to another scale.

For P equals 1, 2 so, on till j using the recursion a l of P equals a k P minus 1 plus b s of k P minus 1 for l equals 2 k and as of k p minus 1 minus b s of k P minus 1 for l odd, which is 2 k plus 1 form. So, I think this basically this is the statement of the theorem and the procedure that we derived is basically the proof of the theorem ok.

So, I could have stated this theorem to start with and then intimidatory with a 2 statement of the theorem here and continued with the proof, but I just gave you the feel for how the reconstruction procedure works and then stated the theorem towards the very end so that you can just connect the pieces together. I will give you a homework exercise here.

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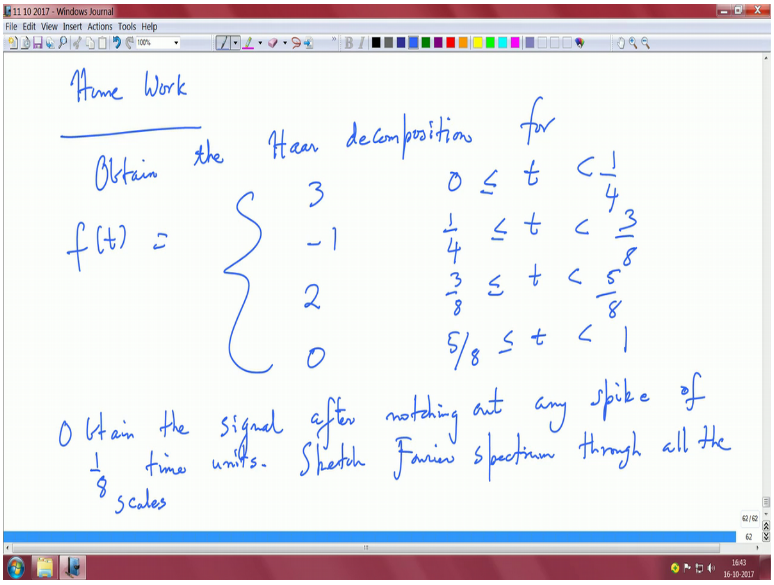

Obtain the Haar decomposition for f of t equals 3 0 less than or equal to t less than one-fourth minus 1 between one-fourth and 3 8 to between 3 8 to 5 8 and 0 between 5 8 and 1. Obtain the signal after notching out any spike of one-eighth time units sketch, the Fourier spectrum through all these skills ok.

So, you have a signal here and perhaps for the signal, any spike of one-eighth time step is a noise. So, null out this and then look at the frequency domain or a spectrum of the signal at each scale and then analytically obtain what the reconstructed signal is and in the frequency domain also interpret what your results are going to be ok.

So, this completes the wavelet Haar wavelet decomposition as well as reconstruction. Now, we will in the next lecture interpret the wavelet decomposition and reconstruction within the framework of filter banks and then you will see how filter bank theory is linked to wavelets, ok.

We will stop here.