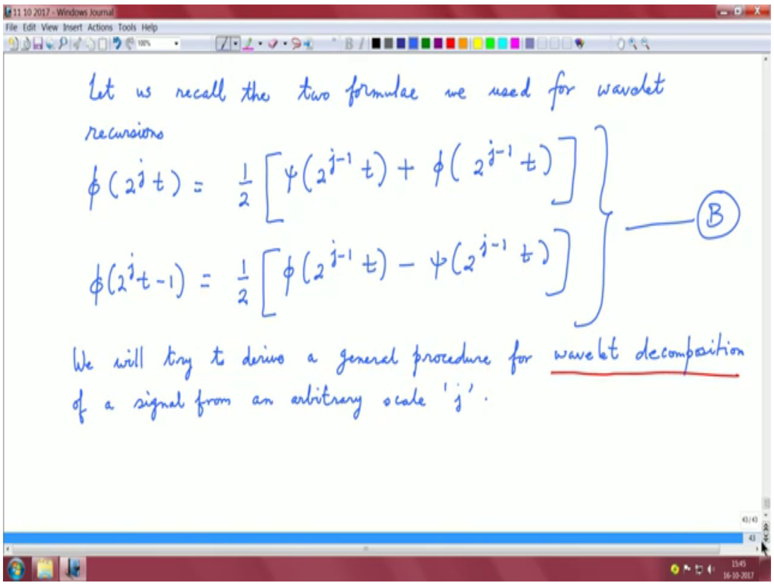
Mathematical Methods and Techniques in Signal Processing- I
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 59
Haar decomposition – 2

(Refer Slide Time: 00:34)

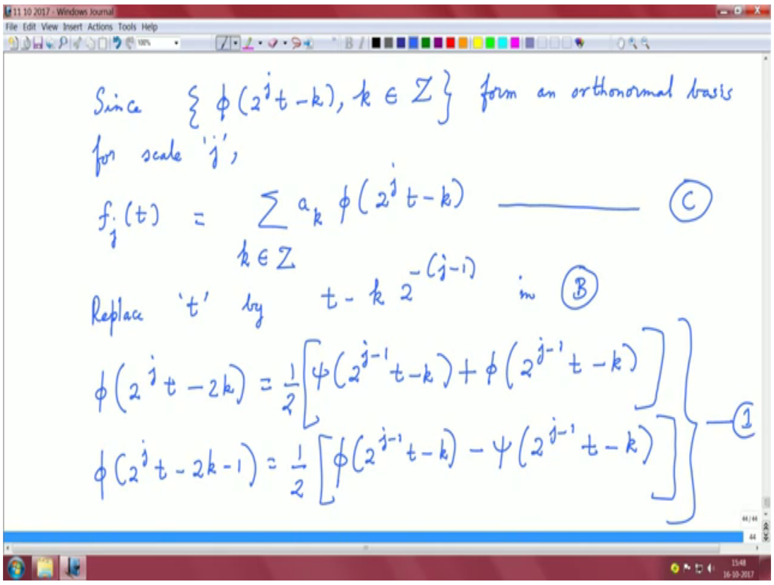


So let us get started with a general form for wavelet recursion. And we will also see through the inverse transformation, and recovery after you have done the forward wavelet decomposition ok. So, let us start let us recall the two formulae we used for wavelet recursions.

So, the first one was phi of 2 power j times t is one half of psi 2 power j minus 1 times t plus phi of 2 power j minus 1 times t ok, this is straightforward. And then we look at a translate of this 2 power j times t minus 1 equals one half of phi of 2 power j minus 1 times t minus psi 2 power j minus one times t ok. Let us designate this set of equations as B. I hope you have not used B before so some letter B ok.

Now, we will try to derive a general procedure for wavelet decomposition of a signal from an arbitrary scale j ok. We start with some arbitrary scale j and we will derive a general procedure for the wavelet decomposition.

(Refer Slide Time: 03:24)

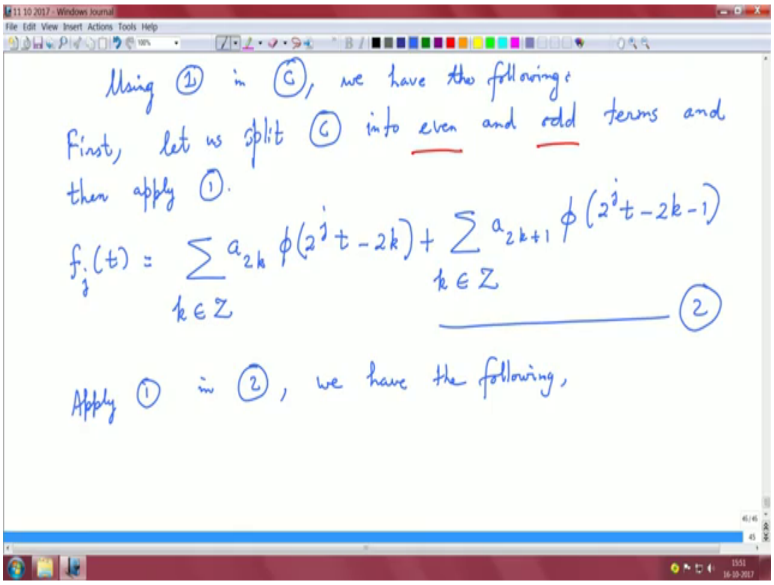


Now since we set of these boxcar signals in scale j the form orthonormal basis that is phi of 2 power j times t minus k k belonging to set of integers form an orthonormal basis for scale j. We can write some function f j of t as summation k belonging to the set of integers a k phi of 2 power j, t minus k that is we are expressing this as a linear combination using the basis for the scale j right this is straightforward. Let me call this equation as phi.

Now replace t by t minus k times 2 power minus j minus 1 in equation B right what will happen? So, basically instead of phi of 2 power jt I have phi of 2 power j times t minus 2 k equals one half psi 2 power j minus 1 t minus k plus phi of 2 power j minus 1 t minus k just a change of the variable.

Similarly for the translate by one time step is 2 power j t minus 2 k minus 1 this is one half of phi 2 power j minus 1 times t minus k, minus psi 2 power j minus 1 t minus k ok. Let us call this say some equation 1.

(Refer Slide Time: 06:26)

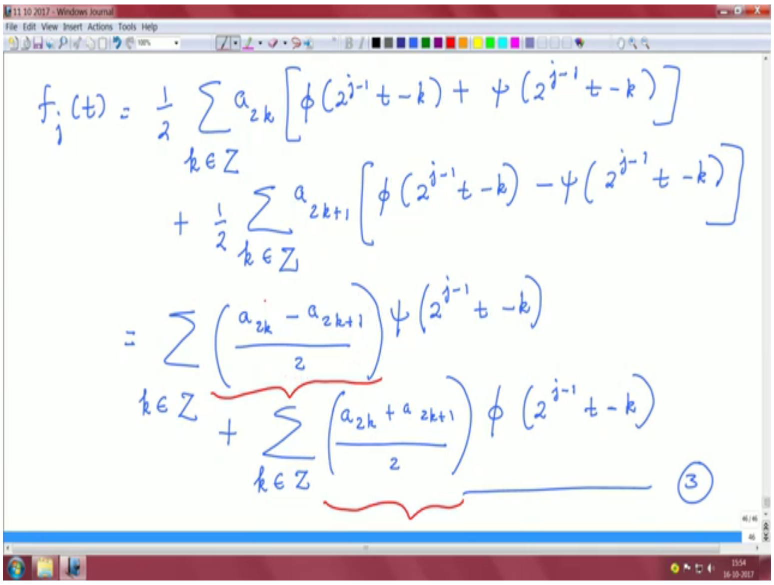


Now using equation 1 in C, we have the following. First let us split equation C into even and odd terms and then apply equation 1. This is a normal divide and conquer approach which people adopt you might have seen this in many algorithms in computer science there is no different.

So, what happens is as follows? So, f j of t equals summation j belonging to the set of integers a suffix 2 k times phi of 2 power j times t minus 2 k plus summa k belonging to the set of integers a 2 k plus 1 phi of 2 power j times t minus 2 k minus 1 ok. This is no big deal for us straightforward straight forward algebra. Now let me call this equation two of the apply 1 into.

So, what is one just if you want to recall one is this equation which connects the phi function at a scale j time translated by 2 k and 2 k minus odd and even translates linking with the scaling function and the wavelet in a lower scale j minus 1 right. So, that is what we do here, so we plug in 1 in 2 alright. So, applying 1 in 2, we have the following.

(Refer Slide Time: 09:24)



So, the equation is quite lengthy so I am just going to use a different page. So, f j of t is one half summa keep belonging to the set of integers a suffix 2 k. So, you have to put the brace outside phi of 2 power j minus 1 t minus k plus psi 2 power j minus 1 times t; minus k plus one half summation k belonging to the set of integers; a 2 k plus 1 phi of 2 power j minus 1 times t minus k minus psi of 2 power j minus 1 t minus k ok.

Now we can simplify this as follows this is basically summa k belonging to the set of integers what I do is I group the terms. So, I have a 2 k minus a 2 k plus 1 by 2 times psi 2 power j minus 1 t minus k, plus summa k belonging to set of integers; a 2 k plus a 2 k plus 1 divided by 2 times phi of 2 power j minus 1 t minus k right, I call this as equation 3. Usually it is a long lengthy equation, but I think what is important is we have decomposed it carefully by grouping them into odd and even parts.

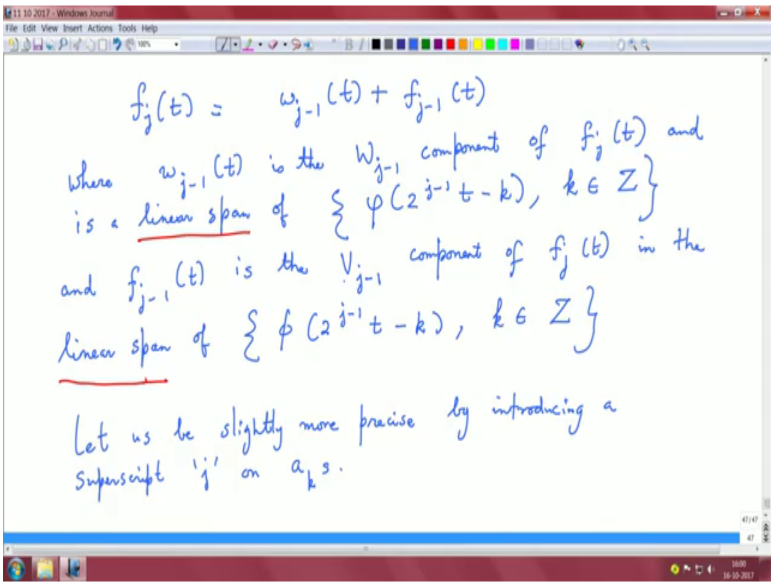
But I think what you should pay attention is just not what this equation is, but observe this term carefully and observe this to observe both the terms in this summation carefully and this psi we know it is a wavelet function right. And look at the coefficient of this wavelet basis it is basically doing some difference operation here a 2 k minus a 2 k plus 1 by 2 right.

It is basically taking some difference operation and it is basically this is basically the scale scaling I mean scaling by a coefficient is the coefficient itself is basically taking some difference of some set of coefficients from a different obtained from a different scale right, and you can imagine this is like our detail right. If you were to detect some sharp edge in your image maybe you take the difference between the image at that point and the previous point to the left or to the right only when you take the gradient you can see that change right. So, this is like a sort of difference operation sort of like a difference operation, will be will be making it more explicitly clear how these are linked to the filters etcetera right.

For example if you think about the high pass filter right if it 1 minus z power minus 1 is a high pass filter 1 minus z 1 minus the unit delay is a high pass filter. And if you look at its equivalent right it is basically is busy if you take xk and filter it through that you get xk minus xk minus 1 it is basically differentiator, a high pass filter is basically a differentiator.

And why it is a differentiator? Because it basically looks at successive differences in some way simplest high pass filter first order, and that is that is why you can observe some change in the signal the same thing is what so that that kind of interpretation comes through when you when you look through this detail. And this is basically some sort of averaging happening over the scaling function. So, this is sort of a loose sketch of this, but we will make it very precise then we when we further get down and interpret the wavelet transformation into filter banks ok.

(Refer Slide Time: 15:15)

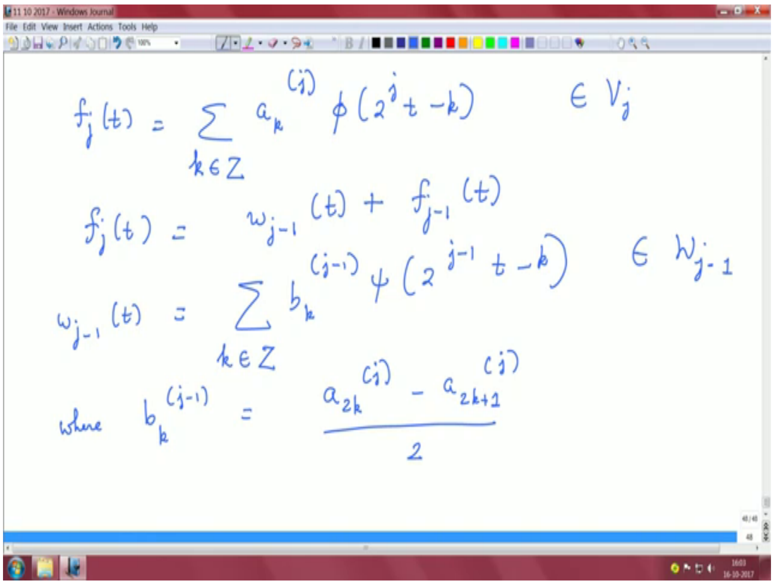


Now, f j of t is basically w j minus 1 of t plus f j minus 1 of t where w j minus 1 of t is the w j minus 1 component of f j of t, and is a linear span of psi of 2 power j minus 1 t minus k k belonging to the set of integers. And you can also interpret f j minus 1 of t this is basically the v j minus 1 component of f j of t in the linear span of phi of 2 power j minus 1 t minus k belonging to a set of integers this linear span is important.

Now, let us be slightly more precise by introducing a superscript j on all of these a k s right. So we generally took a scale j we decomposed it into two components, one in wj minus 1, the other in vj minus 1 space right. But the ak s are collected, ak s are the coefficients for the decomposition and they are collected.

Now if you want to program this numerically alright you would have to you know how to make it in a way that you could compute right. So, therefore, this ak s have they need to have a superscript or a subscript indicating which scale they belong to in the process of decomposition. So, we will make it a little more precise by introducing this superscript j on these a k and try to link all of them.

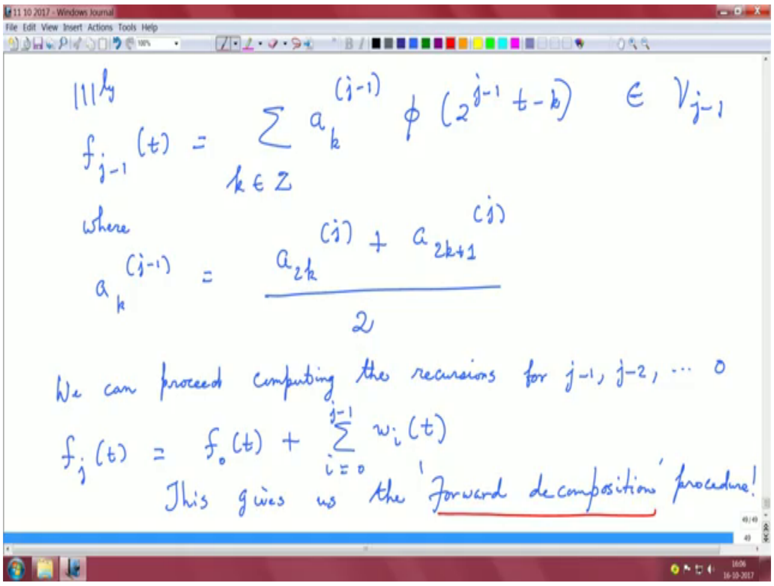
(Refer Slide Time: 18:41)



Now f j of t is summa k belonging to the set of integers a k of j phi of 2 power j t minus k and this belongs to the space v j right. What I did is I just put the subscript sorry the superscript j here just to make it a little more precise. And now f j we know just if you recall this is basically w j minus 1 of t and f j minus 1 of t and w j minus 1 of t equals summa over key belonging to the set of integers sum b k of j minus 1 psi of 2 power j minus 1 t minus k right.

I am just trying to expand this because this is a signal which is belonging to the space w j, where bk of j minus 1 is a 2 k of j minus a 2 k plus 1 of j upon 2 right. We saw in that equation a 2 k minus a 2 k plus 1 upon 2 of psi of 2 power j minus 1 t minus k right; we did this in equation 3 right. I am just rewriting that a 2 k minus a 2 k plus 1 with the superscript j and linking it with the coefficient in scale j minus 1 because this belongs to this is j minus 1 right, j minus 1 clear.

(Refer Slide Time: 21:09)



Now what we will do is we will do the similarly we can write f j minus 1 of t has summa summation k belonging to set of integers. So, I will just use this as a itself sum a k, but I will use j minus 1 here to be little more precise phi of 2 power j minus 1 t minus k that is all shift orthonormal translates of the basis function this belongs to the space v j minus 1, where a k j minus 1 is a 2 k of j plus a 2 k plus 1 at scale j upon 2 right.

What I did was nothing express except expressing equation 3 that I had earlier by making it a little more precise by bringing this superscript indicative of the scale. So, that when you decompose you know you know you know with scale that function resides in order signal resides in and appropriately you have the coefficients which are appropriate to that particular scale ok.

So, now we have a procedure for decomposition we can proceed computing the recursion puts a recursions for j minus 1 j minus 2; dot dot dot and to 0 to obtain f j of t equals f not of t plus summa I equals 0 to j minus 1 w I of t right. And this gives us the forward decomposition procedure ok.

So, given any signal you just start with an appropriate scale and from there you start decomposing the signal and you know you can you get to whatever level of detail that you want to ok. So, we will stop here and then we will begin with the reconstruction step following this.