Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 58
Haar decomposition – 1

Now let us look into the idea of approximating using step functions.

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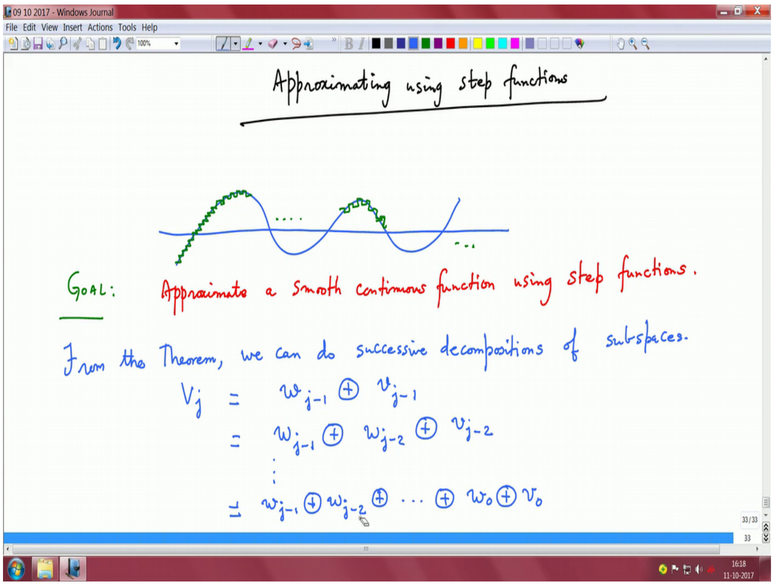

So, I have some continuous signal and then I want to approximate this using some sequence of step functions like this. So, you can you can fill those diagram throughout ok. Now our goal would be to approximate a smooth continuous function using step functions of course, you may even philosophically ponder about step function is a discontinuous functions and how would I want to approximate a smooth continuous function with discontinuous functions right. I mean this is a natural question one would get and the answers to all of these have been thought about by folks working in analysis, I mean, this is a whole topic of study, there I mean what is the measure of such a discontinuity, etcetera, etcetera because you cannot have the measure is really finite.

Then it would be very hard I mean if you should have finite number of such discontinuities, right, I mean, you cannot have many discontinuities I mean which is possibly you know uncountably infinite discontinuities and you want to approximate the function that would be hard there are the certain issues, but from an engineering perspective we will see how we would want to approximate a given function using wavelets or such step functions now from the theorem that I stated we can do successive decompositions of subspaces.

So, what do I mean by this? So, we start with our equation V j is w j minus 1 direct sum with V j minus 1, right. So, I keep w j minus 1 the way it is i; now decompose V j minus 1 that can be written as w j minus 2 direct sum with V j minus 2 and now, you get a trend if you continue this in a sequence you have w j minus 1 direct sum w j minus 2 direct sum dot dot dot finally, you will land up with w 0 and V 0.

So, what this means is each function f in V j can be decomposed uniquely as the sum of functions which belong into space w j minus 1 w j minus 2 so on and so forth, right. So, that is the meaning of this decomposition.

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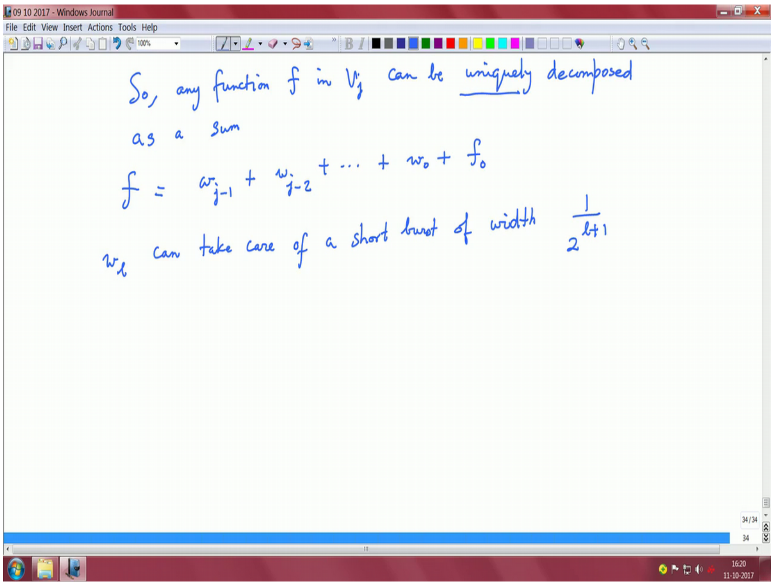

So, any function f in V j can be uniquely decomposed as a sum where f is now these are signals I just use a sum w like this w j minus 1, this is a this is not a space, but a function right plus f 0. Now w l can take care of a short burst of width 1 upon 2 power l plus 1; I mean this is the resolution. So, I initially mentioned if you were to do denoising or sort of remove these spikes or you know clean up the signal, then you have to identify to which scale you would want to do the wavelet decomposition then I would figure out those Fourier spikes and then just null them off, right.

So, basically what you have is basically a control then you recompose the original signal into the into various components each component residing an appropriate scale with a resolution, right and you can go to that appropriate space and if you figure out that there is a spike, there you can just null that spike out and this is an important application. Now a more deeper result would be to ponder about in the space L 2 R can be decomposed into an infinite orthogonal direct sum space right if the space itself in L 2 can be decomposed and what is this, this general result.

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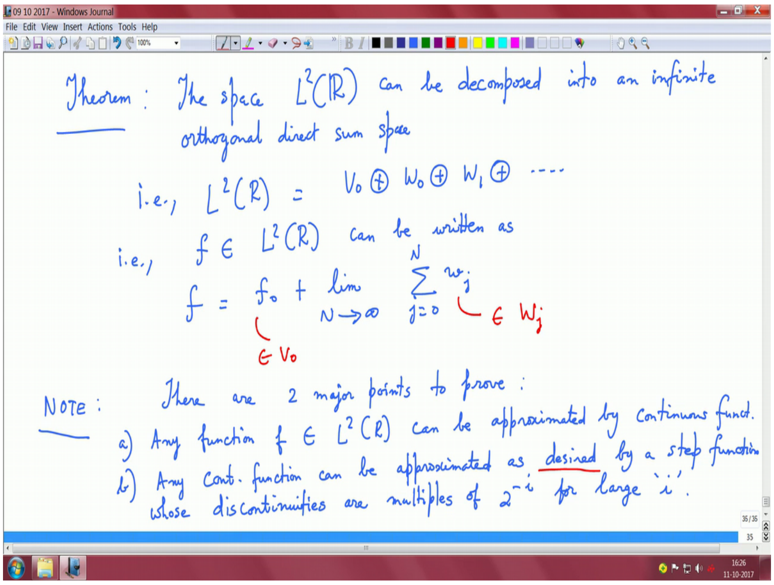

So, let me state this as a theorem and I will give you some ideas towards it a formal proof of this theorem will require a background in analysis in maths.

And I will leave this as a sort of a study exercise, but it is not intended as part of this course the space L 2 are can be decomposed into an infinite orthogonal direct sum space. So, what it means is L 2 R can be written as V sum V naught W naught with W 1 dot dot dot dot till infinity which means some function f belonging to L 2 R can be written as f is some f naught plus limit n going to infinity summation j equals 0 to capital N w j where f 0 belongs to V 0 and small w j belongs to space w j.

So, I will tell you; what is the difficulty in the proof of this result. So, there are 2 major points one needs to really establish the first point is any function f belonging to L 2 can be approximated by continuous functions. There is a first statement that needs to be proved then any continuous function can be approximated as desired by a step function whose discontinuities are multiples of 2 power minus i 4 large I. So, these are the 2 subtle aspects, because this summation is heading to towards infinity right first is you have to establish that any function in L 2 can be decomposed or can be approximated by a continuous function and that continuous.

Function can be approximated as desired this is very important; that means, you have some metric for the error tolerance that you have to account for in terms of some norm by a step function whose discontinuities are multiples of 2 power minus i for larger i. So, I will leave this as sort of just to ponder upon I mean I do not expect it because it will require analysis. But I will provide you supplementary notes to the proof of this theorem just for your understanding, but it is not intended to be part of this particular if you get a detailed course in harmonic analysis probably will have to consider these aspects here ok.

So now, with this with these ideas we have to get into Haar decomposition.

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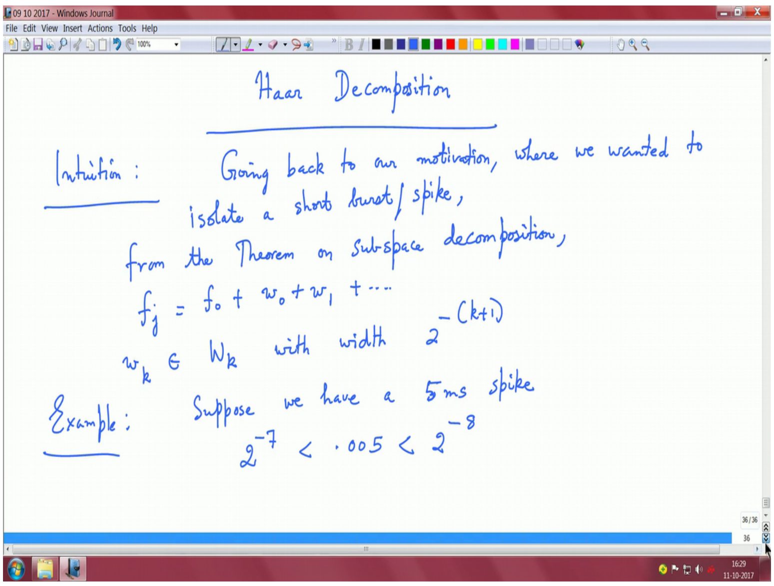

So, what is the intuition? So, going back to our motivation where we wanted to isolate a short burst or spike from the theorem on subspace decomposition right we saw that f j can be written as f naught plus some w naught plus W 1 plus dot dot dot right and a component w k belonged to space some W capital W k with width to power minus k plus 1. So, for example suppose we have a five millisecond spike.

Then we know that 2 power minus 7 is less than 0.005 is less than 2 power minus 8 and you can choose your scale appropriately to a null out this spike and the resolution in the smallest scale, right, I mean that you that you would consider in your decomposition would be roughly; what your sampling has to be these are important things, I think with this motivation in mind, we will start with the wavelet decomposition procedure ok.

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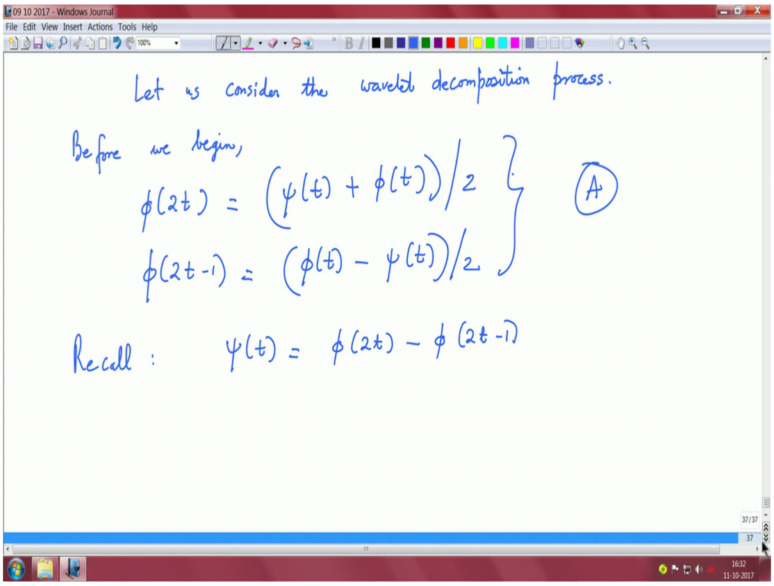

So, let us consider the wavelet decomposition process. So, this is best understood to an example first and then we will formalize. So, before we begin, we would like to recall some relation relations that would be useful in our decomposition phi of 2 t is basically psi of t plus phi of t upon 2; this is one important relation that is I want to realize a boxcar of width half and that I can realize using adding the wavelet and the scaling function right in a resolution which is coarser than that.

Now phi of 2 t minus 1 is phi of t minus psi of t upon 2 call these 2 relationships a write equation a and you must also recall that psi of t is phi of 2 t minus phi of 2 t minus 1 right you just you translate it and you flip it you get the wavelet it is intuitive. So, now, with this let us start to understanding how the decomposition process works through an example.

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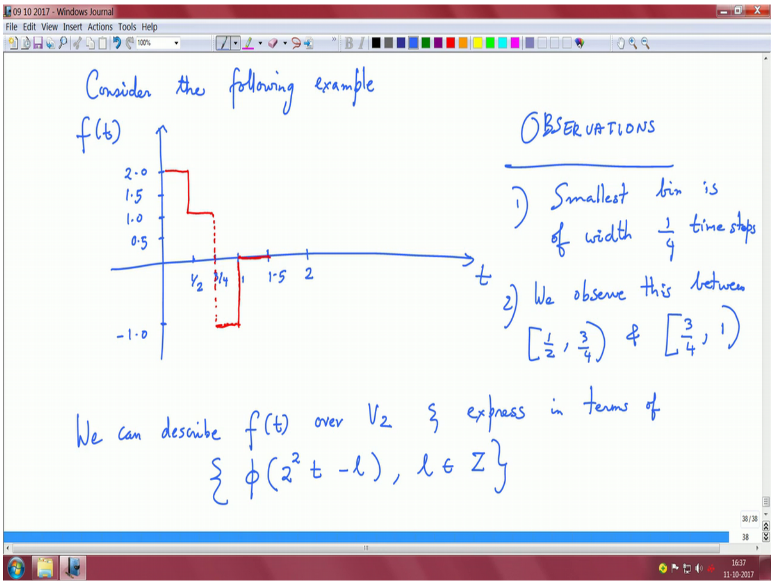

It is 0.5, 1, 1.5, 2, 2.2, 1.5, to v, here I have half one this is 3 quarters and there is 1.5, this is 2 t here n some f of t. So, until half this is basically 2, then there is a sharp dip it comes to one then for the next quarter time step it maintains at 1 and then there is again a discontinuity here this is 1 minus 1, it stays for the next one fourth time duration it and basically returns 0 and then stays at 0.

Now we may find such pulses a good example is if you look into a waveform which is pulse width modulated it appears in your power systems etcetera you will see such staircase type of functions of various bits and so on and so forth, but our goal is to basically decompose this signal now.

If you look at what the smallest bin is the first step call as observations smallest bin is of width 1 by 4 time steps time units and we see this between 1 half and 3 quarters and between three quarters and 1; this is a semi closed interval, just to indicate that this is a point of discontinuity ok. So, now, it is obvious that we can describe f of t over V 2 the space, we can consider is V 2 f of t belongs to V 2, right, we can describe f of t over V 2 which means that we can express f of t in terms of these phi functions in over scale j equals 2 this is 2 square t minus l l and belonging to the set of integers; that means, I can expand f of t using this as my basis.

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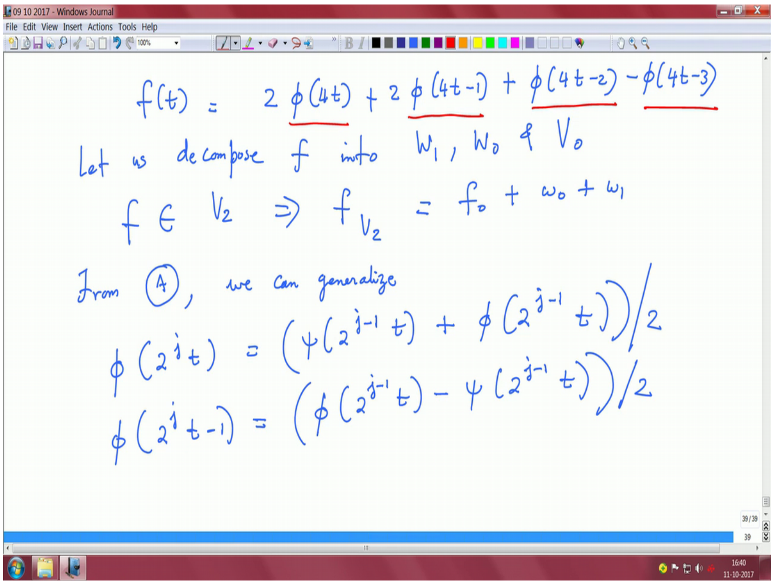

Now, f of t can be written as 2 times phi of 4 t plus 2 times phi of 4 t minus 1 plus phi of 4 t minus 2 minus phi 4 t minus 3, directly if you observe the waveform, you can you can express this using this relationship. Now let us do the wavelet decomposition. So, let us do a wavelet decomposition of f in too spaces W 1 w naught and V 0; that means, you will have a component in W 1 a component in w naught and in V naught. So, f belonging to V 2 implies f in this space V 2 can be written as f naught plus some component W naught.

And some component W 1; now from our equations in a we can generalize it for arbitrary scale phi of 2 power j t is psi of 2 power j minus 1 t plus phi of 2 power j minus 1 t upon 2 and phi of 2 power j t minus 1 just a translate of this n this can be written as phi of 2 power j minus 1 t minus psi 2 power j minus 1 t this upon 2 right this is a generalization you could visualize it for V 0 and 1.

So, we can now extend it to arbitrary scale this straight, this is a straight forward generalization now what we do is the plug j equals to because we have these terms right we have to get you have to get these terms phi of t phi of 4 t minus 1 phi of 4 t minus 2 4 t minus three we have to get all these things to get to this step we have to use these formulae and then decompose things further ok. So, let us see how this works out.

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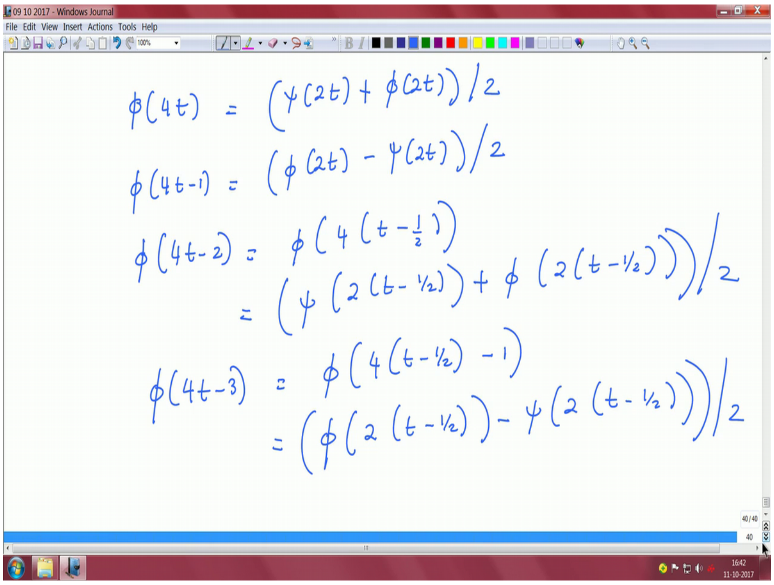

Now phi of 4 t is basically psi 2 t plus phi 2 t upon 2 and phi of 4 t minus 1 is phi of 2 t minus psi of 2 t upon 2. Now how would I do phi of 4 t minus 2 this is phi of 4 into t minus half.

So, now, this is basically psi of 2 into t minus half plus phi of 2 into t minus half this upon 2 because once you realize what phi of t is you can sketch what phi of t minus half is and. So, on and then you can just get the rest and similarly you have 4 phi of 4 t minus three is basically phi of 4 into t minus half minus 1 and this you can write it as phi now use the phi of 4 t minus 1 formula. And then you can write this 2 times t minus half minus psi of 2 into t minus half this upon 2 now we have expanded all these forms and then we will try to group the terms.

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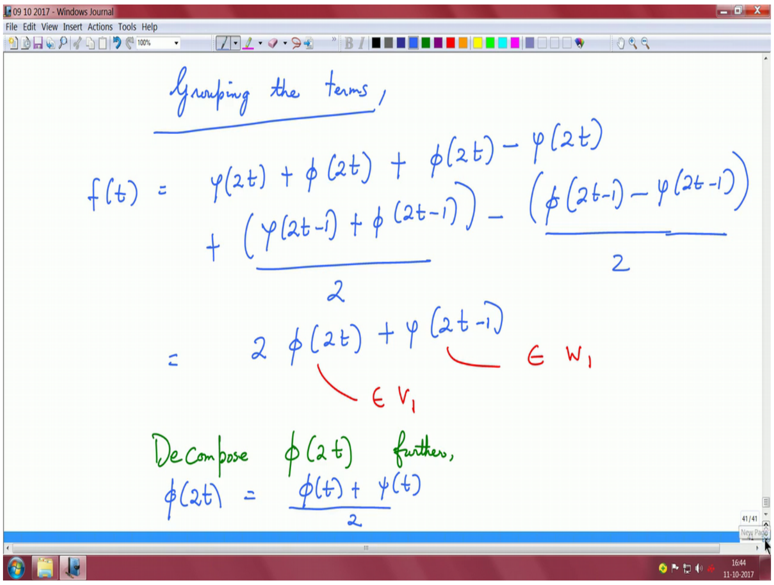

So, grouping the terms f of t is basically psi 2 t plus phi 2 t plus phi 2 t minus psi 2 t plus psi 2 t minus 1 plus phi 2 t minus 1 upon 2 minus phi 2 t minus 1 minus psi 2 t minus 1 upon 2 which is basically 2 times phi of 2 t plus psi 2 t minus 1. Now, this belongs to W 1 and this belongs to V 1 decompose phi of 2 t further. So, what we get is phi of 2 t is basically phi of t plus psi of t upon 2.

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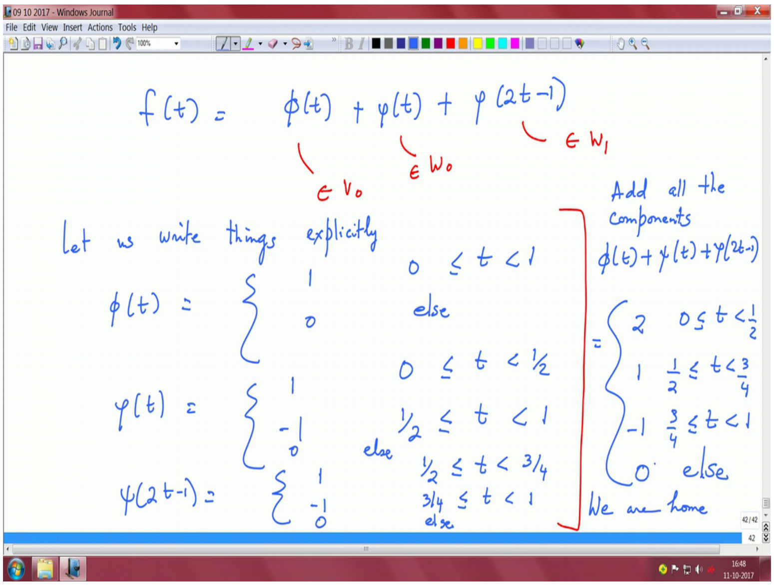

Now we plug everything together, right and then we land up with f of t is phi of t plus psi of t plus psi 2 t minus 1 and if you see this spaces this belongs to V 0 this belongs to W 0 and this belongs to W 1 let us write things explicitly write, we write things explicitly.

So, now phi of t; how does this function look this is one for 0 less than or equal to t less than one 0 else now this psi of t is one 0 less than or equal to t less than half minus 1 between half and one and 2 t minus 1 this is one from one half less than or equal to t less than three quarters and minus 1 from three quarters less than or equal to t less than one and I have to put a 0 else here just to be consistent ok. Now you add all of these right you take phi of t plus psi of t plus psi of 2 t minus 1.

So, what you get is function; now phi of t plus psi of t plus psi 2 t minus 1, if you add all these using these ranges, this is basically 2 from 0 to half, it is 1 between 1 half and 3 quarters minus 1 between three quarters and one and it is 0 otherwise. So, we are we are basically home we started off with this signal f of t we decomposed it in the V 2 space using the basis in V 2 and then we looked at each of the components in V 2 and then before we further decomposed to a component in V 0 and W 0 and W 0 and W 1 ok. So, this is the idea. So, V 2 is recomposed to V one and W 1 and then V one is further split into V 0 and W 0 and we did a 2 level decomposition.

So, in your practical example you may need some d level decomposition arbitrary level because you know what application you want you know to what level of resolution you want right you could go to that resolution and that is where the scale is very important and in that resolution you look at the scaling function you look at the wavelet and wavelet is basically the high frequency component here if you can sort we imagine that in that way. And then you figure out what to null and what to retain depending upon what application you have in mind ok. So, this clearly is a good examples if you want to null out some signal in W 0 you would just basically remove this component, right. If you want it and allowed something in W 1 you would just allow some component here.

Therefore, you have full control over the decomposition part and the part to retain and null whichever components you need depending upon your applications ok. So, now, we have just got an example here, but we will formally generalize this into an algorithm for wavelet decomposition. And we will derive the forward wavelet decomposition and we will also look at the reconstruction part and then we will try to link all of these 2 filter banks.

We will stop our lecture here.