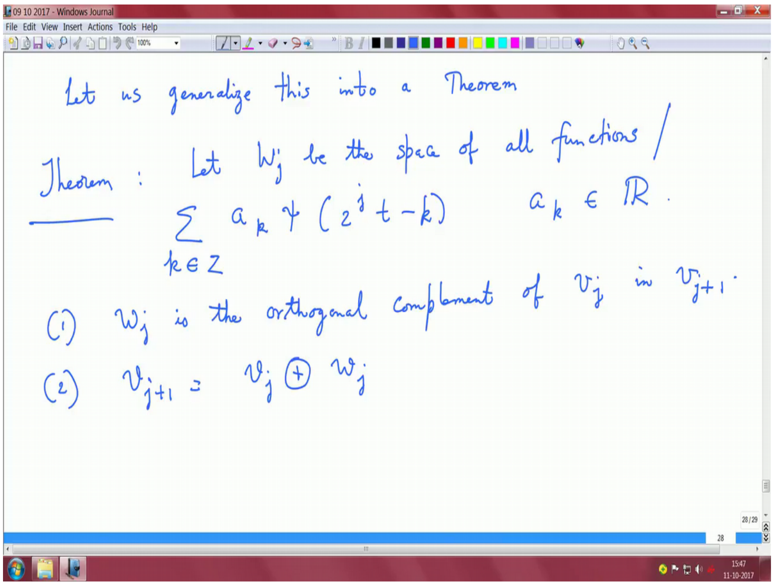
Mathematical Methods and Techniques in Signal Processing - I
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 57
Structure of subspaces in MRA

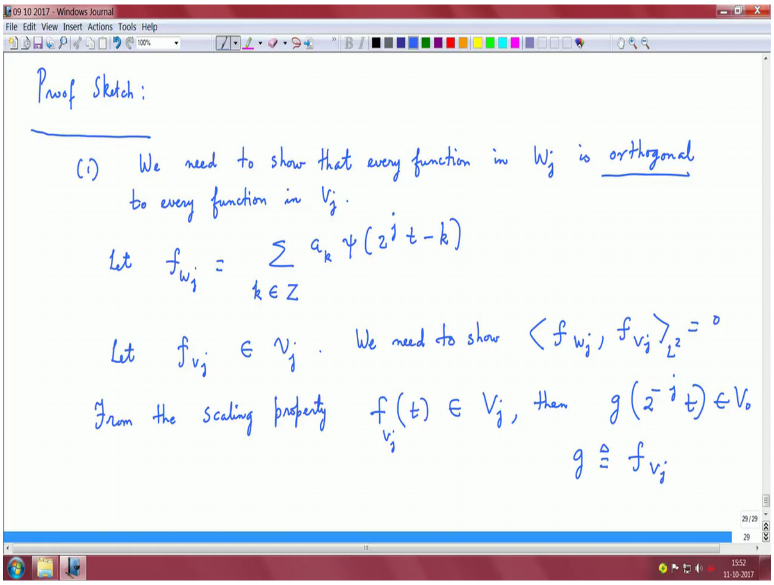
Let us get started with some properties into wavelets and these properties will essentially help us for a wavelet decomposition forwarding decomposition using wavelet discrete wavelet transforms ok. So, we start with this theorem.

(Refer Slide Time: 00:36)



So, let w j be the space of all functions such that sigma k belonging to set of integers a k chi to power j t minus k, this is the function is described by this form and a ks are basically real numbers. So, we need to show that w j is the orthogonal complement of v j in v j plus 1 and then v j plus 1 is basically written as v j direct sum with w j.

(Refer Slide Time: 01:14)



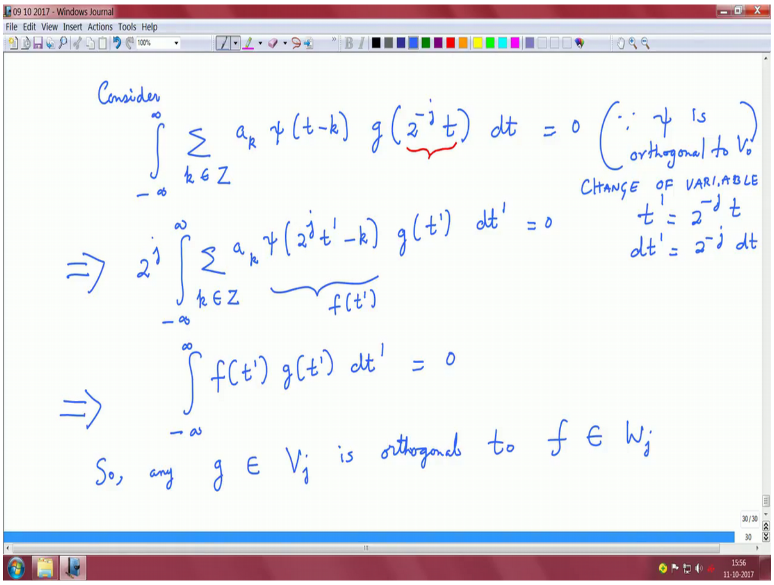
So, I will go with a proof sketch for one of the parts and for the other part I will leave this as an exercise and we can discuss the solution to the exercise. So, for the part one; we need to show that every function in the w j space is orthogonal to every function in the v j space.

So, let us start with some typical function in the w j space and let that function be f w j which is summation k belonging to some set of integers a k chi 2 power j t minus k this is basically a linear combination of the shift are orthonormal wavelets in this scale right in the jth scale.

Now, let some function f v j belong to v j space, right, then what we need is we need to show that f w j and f v j their inner product is 0 in the L 2 space, right, I mean this; their inner product is basically 0 now from the scaling property. So, if f v j belongs to v j, then basically let me just denote this in simple form as some f and some g here.

So, f of v j of t if this belongs to v j, then if I just call this simple function as say g like f v j is g, then I would say g of 2 power minus j times t belongs to v 0 and called g is basically synonymous to my f just f j I am getting rid of this v j. So, that it does not confuse as much; I could call them g and f, but I just put these subscripts v j and w j to indicate what spaces will it belongs to ok. So, by the scaling property this belongs to v 0.

(Refer Slide Time: 05:51)



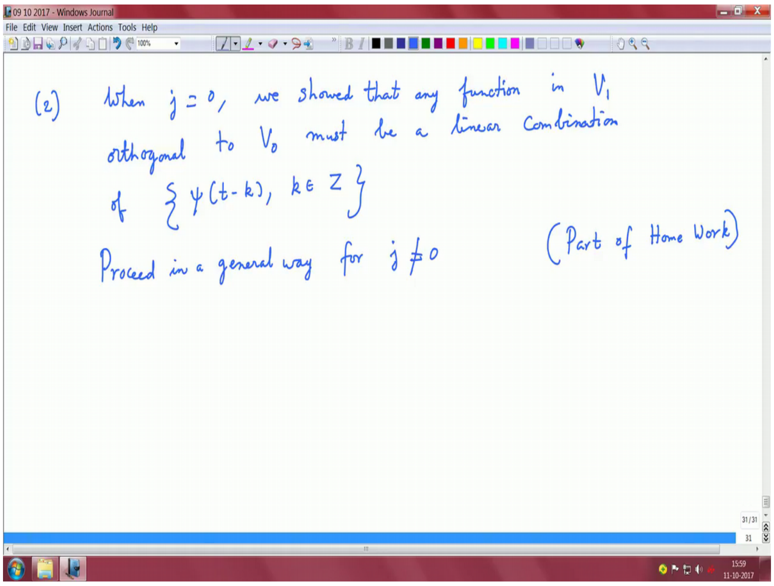
Now, consider there is integral minus infinity to plus infinity sigma k belongs to z a k chi of t minus k now we will just translate everything in the scale for j equal to 1 right. So, now, this times g of 2 power minus j times t; let us consider this inner product and this is 0 because chi is orthogonal to v 0, right, it is orthogonal to the functions in v 0, now this implies I want to simplify this further. So, what I would do is I would define a variable t dash is to power minus j times t.

So, just focus on this I do a change of variable here; so, when I say t dash is 2 power minus j t when I let this change of variable d t dash is 2 power minus j times d t right. Now, I can rewrite this integral slightly differently this is going to be 2 power j times integral minus infinity to plus infinity summa k belonging to z a k chi 2 power j t dash minus k times g of t dash d t dash.

This is basically 0 d t is 2 power j times d t dash. So, therefore, this 2 power j is pulled outside here ok; now what we do is; so, this would imply integral minus infinity to plus infinity I can pull the 2 power j outside. So, let me call this function as possibly f of t dash, right, I call this as f of t dash just for simplified notation this is f of t dash times g of t dash d t dash and this is 0.

So, this implies any g belonging to v j is orthogonal to some f belonging to the space w j ok, this is a straightforward result, there is nothing out there, basically, we use the definition of the inner product figure out what functions reside in what space and then we just expand using the basis in that space and then through the property.

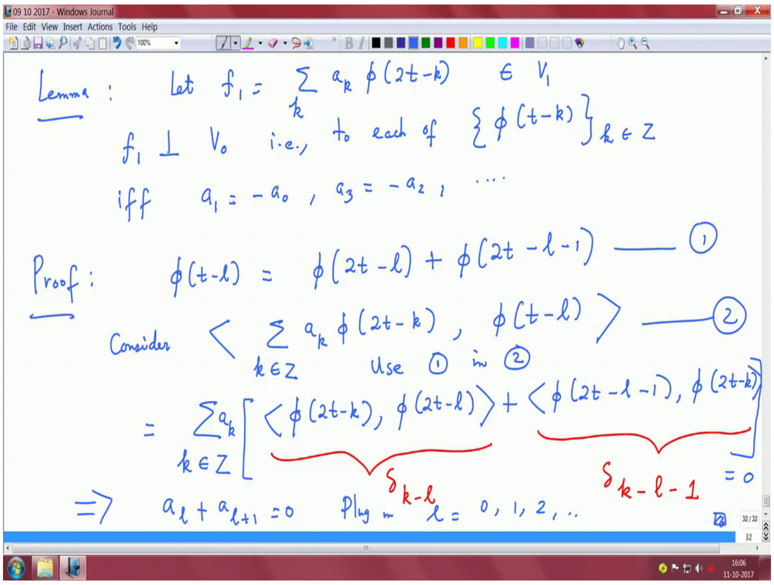
(Refer Slide Time: 10:45)



So, this is pretty straightforward, now for the second part; I will give you the idea and I believe this deliberately as a homework. Now, when j equals 0, we showed that any function in v 1 orthogonal to v 0 must be a linear combination of basically these shift orthonormal wavelets, you saw that example and from an example basically, you have to formalize this result and then proceed in a general way for j naught equal to 0 and fill in the missing gaps ok.

So, now, with this we will have we will we will establish another lemma and probably that lemma might be useful for you in this proof let me sketch that statement.

(Refer Slide Time: 13:02)



Let f 1 be summation over k a k phi of to t minus k and this belongs to v 1, right, this is some function in v 1 this function f 1 is orthogonal to v 0 that is to each of the shift orthonormal scaling functions if and only f this property holds a 1 is minus a naught a three equals minus a 2 so on. So, let us see the proof of this lemma.

Now, phi of t minus l, this is basically the scaling function in v 0 which is time translated, this can be written using the scales in the v one space as phi of 2 t minus l plus phi of 2 t minus l minus 1. Now, let us consider the inner product of summation k belonging to z a k phi of 2 t minus k with phi of t minus l. So, what we need is if f 1 is orthogonal to v 2.

Then this property must hold so; that means we should consider the inner product set that to 0 and then examine what we can; now let us look at the inner product of 1 with phi of t minus l. So, this can be written as sigma k belonging to z because not we can exchange the summation I mean you we have studied this property of inner products.

So, you can exchange the summation and then you can take the inner product of phi of 2 t minus k with phi of 2 t minus l because I expand it let us say if I say this is equation work say this is equation 2, I say use 1 in 2, right, then I could do this way plus the inner product of phi of 2 t minus l minus 1 with phi of 2 t minus k.

When these terms really look ugly, but you know it is very easy to simplify them now just ponder at this inner product here both of these are boxcar functions L 2 space, but with different time translates and they exists when k and l coincide.

So, basically, this can be written as delta k minus l when k equals l it exists otherwise it vanishes similarly you look at these inner product this is basically existing when k coincides with l plus 1 right and this can be written as delta k minus l minus 1 now these are this is a summation on the outside you have some a ks and then you have deltas and they will exist only when k equals l here.

So, therefore, you land up with a trivial statement that a l plus a l plus 1 is 0 this implies a l plus a l plus 1 equals 0 n. So, it will again l equals 0, 1, 2, dot dot dot and I think I should write this be equated to 0 because this is where we start with a condition that f one is orthogonal to each of three of t minus k now there is at least one existential result here you can figure out what is one nature of this function f 1, if it has to be orthogonal to v 0.

So, hopefully, you will use this property in proving the result which I gave you as a homework for part b.