Mathematical Methods and Techniques in Signal Processing – I
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Lecture – 56
The Haar wavelet

Now, that we have seen how scaling functions work and how you can form an orthonormal basis at different scales using the scaling function, let me introduce what the Haar wavelet is and what is it is significance.

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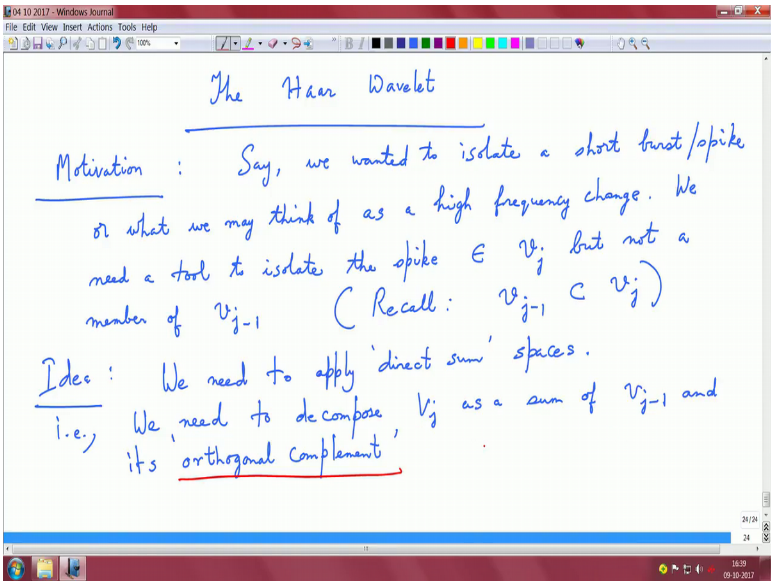

So, before we begin let us start with the motivation to the problem say we wanted to isolate a short burst or spike or what we may think of as a high frequency change. We need a tool to isolate this spike belonging to some space V j, but not a member of the space V j minus 1 why because we have to recall that space V j minus 1 is contained in V j right because when j increases we are at better resolution.

Now, with this in mind we start with the idea. So, what is the idea we need to apply the notion of direct some spaces that is we need to decompose V j as a sum of V j minus 1 and it is orthogonal complement and we have seen all these ideas in our lectures on signal geometry that is take the space V j decompose this space into some V j minus 1 and it is orthogonal complement right and what we mean is V j minus 1 and this other space there orthogonal compliments of each other right and the original space V j is basically decomposed into these two spaces.

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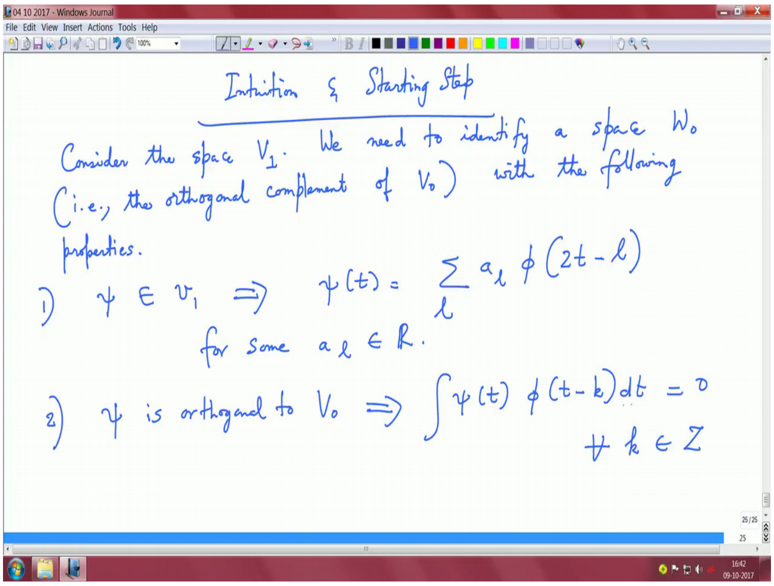

Now, let us start with our intuition and begin this intuition and starting step consider the space V 1 we need to identify a space which is say W 0 that is the orthogonal complement of V 0 with the following properties. Now, if some chi belongs to V 1 this implies chi of t can be written as some linear combination of the basis in V 1 and the basis in V 1 using our scaling functions we can write them as al times phi of 2 t minus l right for some a l belonging to some real numbers.

Now, chi is orthogonal to V naught because that is how we started right we want to write V 1 in terms of decomposition of two subspaces. So, what would this mean this implies integral of chi of t with phi of t minus k this has to be 0 for every k belonging to set of integers,. So, we start with the space V 1 we need to identify a space W 0 which is an orthogonal complement of the space V 0 satisfying the following properties, right and chi belongs to V 1 you can say chi can be expanded using the orthonormal basis in V 1 and we said chi is orthogonal to V 0 therefore, and we know the basis for V 0 would be these functions phi of t minus k.

So, therefore, this function chi with phi of t minus k this product inner product has to be 0, for all k belonging to set of integers.

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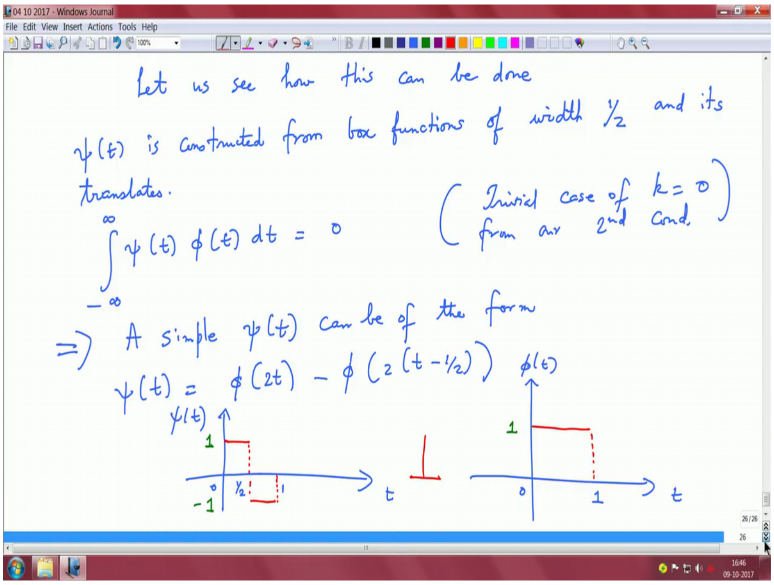

Let us see how this can be done. So, we can construct chi from the set of box functions of width one half and its translates, right. We can start with the box function of width one half and it is translates. So, if we need this condition minus infinity to plus infinity some chi of t it is phi of t dt equals 0, if this condition has to be satisfied. Let us look at the trivial case of k equals 0 from our second condition right it is a phi of t minus k I said phi of t, where k equals 0 ok. If this has to be satisfied a simple chi of t can be of the form chi of t is phi of 2 t minus phi of 2 times t minus half.

Let us get the picture here right if you just sketch this let us say this is 1 and inside this is minus 1 here this definitely is orthogonal to phi of t which is which is this you take the inner product of this with this as you get one and then this is minus 1 it cancels. So, this is basically I would write this as orthogonal to this. Now, what this means chi of t is orthogonal to phi of t therefore, chi of t belongs to the space V 1 and chi of t belongs to V 0 right, chi of t belongs to V 0.

Because this space and chi of t belongs V to W I mean in W 0 I would say not these one I think this is a this is a typo here. So, this should be V 1 compliment orthogonal complement or it belongs to the space sorry I think I think I have to made a small change mistake here. So, chi of t belongs to V 0 because V 0 can be written as V 1 plus W 0 right V 0. So, we start with the space V 1 V 1 this was written as some V 0 direct sum with W

0. So, basically it does not belong to V 0, right. Chi of t belongs to V 1 and chi of t belongs to W 0 ok.

So, it belongs to W 0 space because it is orthogonal complement of V 0 and it belongs to the original space even because even has discontinuities at plus minus half plus minus 1 and so on ok. So, thus W 0 has all functions of the form sigma a k chi of t minus k k belonging to the set of integers and all these a k's are real numbers. Why? If you sort of look at the time translates.

So, if you have one function which is like this 0 half and 1 and another function which is shift this function by one more time step right. So, this is 1, 1 and a half right. This is 1, this is minus 1, this is 1, this is minus 1, these two functions are basically orthogonal to each other and you can expand this space W 0 using all functions of this form in general where this shift orthonormality holds starting with this function chi t and we call chi t as the wavelet function.

So, now we saw that they expand a function using the appropriate basis functions in an appropriate space in appropriate sub space you can expand the function using the basis for that sub space and any particular sub space can be decomposed into two sub spaces which are orthogonal complements of each other, ok. That is the sort of the intuition where we have and then we are now ready to discuss a general form and I will state this as a theorem and then we can go ahead in the go ahead with the proof in the next class.

So, let us generalize this into a theorem the statement of the theorem as is as follows. Let W j be the space of all functions such that sigma k belonging to set of integers a k that you can expand the function in this form that is you are looking at the time trans scaled and the time translates of the wavelet function a k belonging to real numbers the first property is W j if this is.

So, if W j is space of all functions such that this holds, right; then W j is the orthogonal complement of V j in V j plus 1 and you can write V j plus 1 as the direct sum of two spaces which is V j and W j. So, this is the statement of the theorem; that means, we have to prove both these properties ok.

So, if you are clear with the inclusion we had in the last step, how we were able to decompose V 1 into V 0 direct sum with W 0, we can we can just immediately proceed towards proving this theorem.

So, there are a few details we will do this in the next class.