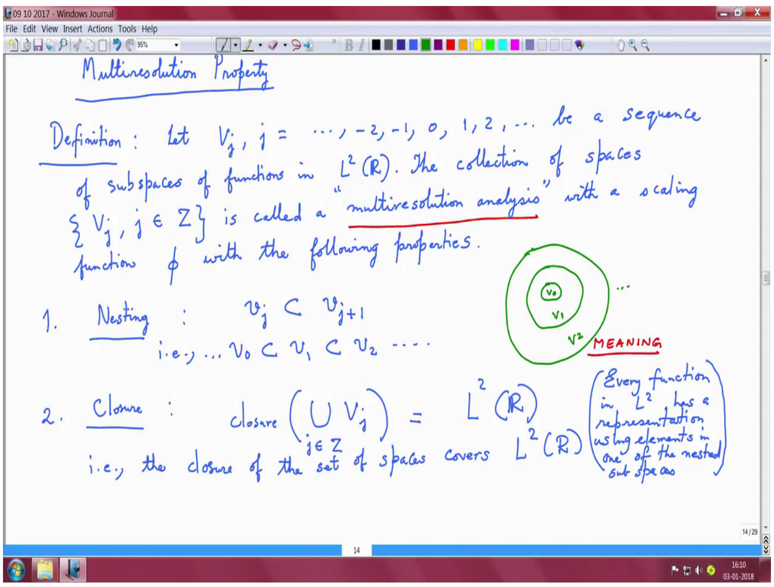
Mathematical Methods and Techniques in Signal Processing - I
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 55
Multiresolution analysis and properties

So, in the last lecture, we discussed sort of a background into wavelets, right; what are wavelets? Why do we need wavelets? What are the shortcomings of the Discrete Fourier transform and so on, right and we said that one of the important things that motivates wavelets is multiresolution; that means, we want to expand a function at different scales; at different resolutions, right and that are applications from denoising to efficient signal representation, etcetera, etcetera, right.

So, now, we will get into the details of the of wavelet analysis. The first and foremost to get towards this is understanding; what multi resolution property is all about ok. So, let us start with the definition first.

(Refer Slide Time: 01:19)



Let V suffix j for j equals dot dot dot minus 2, minus 1, 0, 1, 2, dot dot dot be a sequence of subspaces of functions in L 2 R, right, they are basically square integrable functions.

The collection of spaces which is all V j j belongs to the set of integers is called multi resolution analysis with a scaling function phi with the following properties. So, let V j j equals dot dot dot minus 2, minus 1, 0, 1, 2, 3, so on, be a sequence of subspaces of functions in L 2 R that is all square integrable functions and the collection of spaces V j j belongs to the set of integers is called a multi resolution analysis with the scaling function phi if the following properties hold good ok.

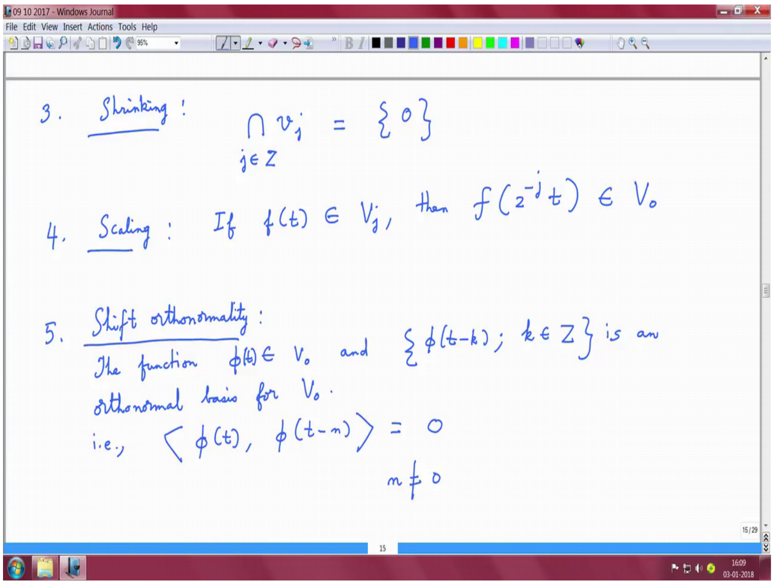
Let us see; what the properties are. The first property is the nesting property, what it means is the space V j is contained in V j plus 1 that is V 0 is contained in V 1, is contained in V 2, so on. So, this is a containment property, this is called nesting and we will see very carefully through illustrations and examples what this containment property is pretty soon ok.

So, I think you can get an idea that one subspace is contained in the other subspace. So, this picture should be there in your mind. The second is closure.

The closure is list of the union of all these spaces is essentially I will put two here the original space of all square into integrable functions that is the closure of the set of spaces covers in L 2 R which means that every function in L 2 has a representation using elements in one of the nested subspaces, right which also means every function in L 2 has a representation using elements in one of the nested subspaces. This is sort of the meaning or interpretation of this of this property.

I will be useful to picture how this these spaces are, right this is our V 0 space, then this is our V 1 space, then this is our V 2 space and so on, I mean if you kind of imagine that the signals are confined in those spaces, I mean what is in V naught is contained in V 1l what is in V naught and V 1 is contained in V 2 and so on. So, this idea has to be clear.

(Refer Slide Time: 08:03)



The next property is shrinking ; that means, if you look at intersection of all these spaces this should be trivially the 0 element, you could call it a vector or you could call it a function or whatever, 0 function or 0 0 element and you could guess this it should be true because it is a linear linear vector space, then is this scaling property is important if f of t belongs to the space V j n, then f of 2 power minus j times t belongs to V 0; I used deliberately the radix two, here you could use essentially any a right you could use any a a power minus g belongs to t; I mean I just used this because we will deal with Haar wavelets in particular and we see a triadic representation there.

So, this two was used right, otherwise you do not have to have to you could have a 3, if it is a triadic decomposition, you could use 5, if it is a 5 adic decomposition or in general for any p adic, you can have a p power minus j ok. So, this is just a small note; then there is something called shift orthonormality what.

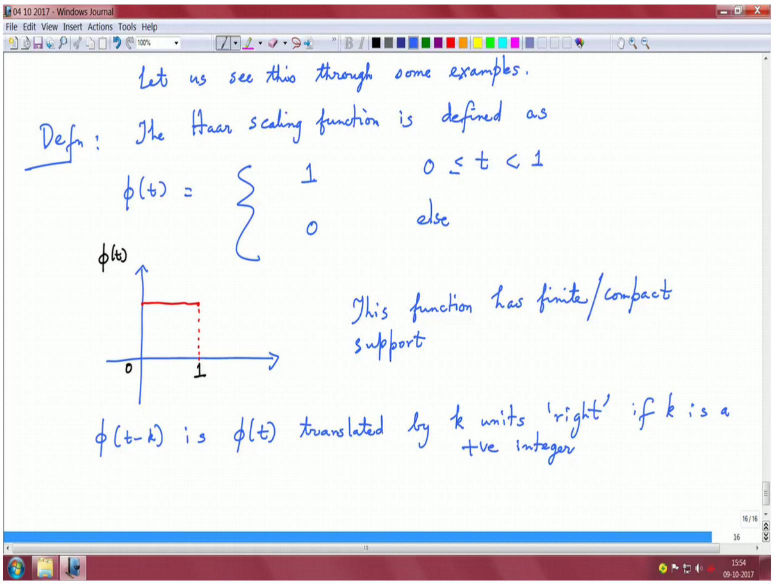
It means is follow is this follows the function phi belonging to V naught and when I say phi, it is phi t here and the set phi of t minus k all its time translates; k belongs to the set of integers is an orthonormal basis for the space V 0; what it means is if you take the inner product of the function phi of t with phi of t minus n for some integer n this must be 0, of course, the normality would ensure that if you take the integral norm for these functions it has to be 1, I mean if you take the integral of phi square t d t or phi square of t minus n d t that has to be 1; of course, when you look at this equation the product of phi of t with phi of t minus n this is equal to 0, this would imply; obviously, for n not equal to 0 because when n equals 0 it is coinciding right, I mean this is never 0. So, so if all these properties are satisfied, then it forms a multi resolution analysis.

So, let us just recall one is a nesting property that is you have a sequence of subspaces that are nested then there is closure; that means, every function in L 2 has a representation using elements in one of the nested subspaces, right, this is akin to saying that you know, I mean I can I mean for example, if you look think about any square integrable function and I have a Fourier basis for that; right, I mean I can think about having a Fourier basis for that nobody, it satisfies certain conditions.

Now similarly, here it is saying any every function in L 2 has a representation using elements in one of the nested subspaces, but the distinction; this is certain other properties like nested shrinkage scaling shift orthonormality, etcetera which is not there in perhaps certain other transforms nesting is one of them then shrinking. So, this is a very trivial thing if you take intersection of all these sub spaces it should be trivially the 0 0 function.

Then scaling if f of t belongs to V j, then you take a scale of that which with to power minus j. So, f of 2 power minus g t belongs to V 0 and shift orthonormality the function V of t and all its integral translates, they form an orthonormal basis for this space. Now it will become very clear if we see through some examples, let us see this through some examples.

(Refer Slide Time: 14:13)



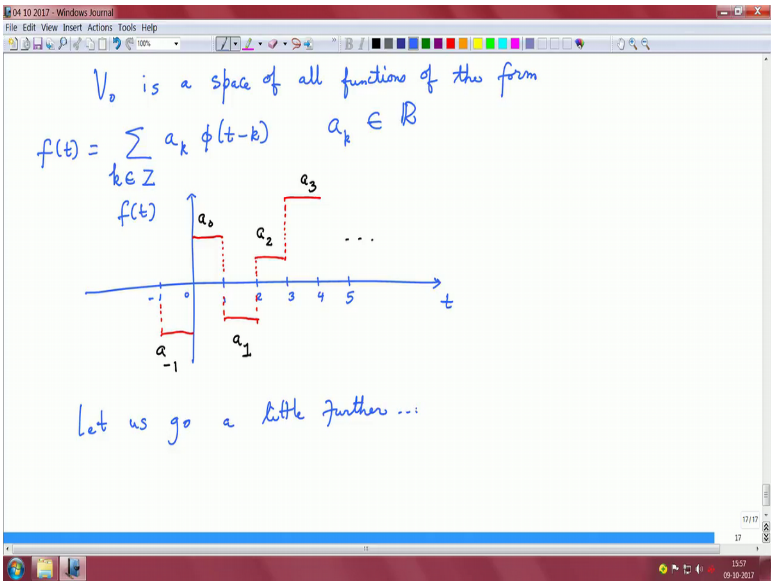
So, we start with a definition. So, we will start with the simplest wavelet which is the Haar wavelet and then we will progressively advance the Haar scaling function is defined as follows; now phi of t is one between 0 and t less than 1.

And it is 0 0 else. So, if we just plot this because how it would look it is like our simple step else that we are used to right now this function has finite or I would say compact support and you can realize translates of this function by time translating phi of t through integral time steps right and this is very important this compact support is very important because unlike the sine cosine bases that form essentially the basis for the Fourier expansion the bases here have compact support you look at sine cosine.

It basically extends from minus infinity to plus infinity and this is one of the fundamental questions that we asked in the very beginning for signal representation do we need functions which are not finitely supported to expand a function which is finitely supported, it is a common sensical question to ask; I mean I have a pulse may be from 0 to say 5 seconds some t seconds against some arbitrary pulse and why would I need sines and cosines in an infinite dimensional basis I mean whose dimension is basically infinite to expand this signal and each of the basis function denzils are not compactly supported can I represent functions using a compact please support an basis set ok.

So, if you ask this question this is where we are sort of eluding ourselves to just another detail phi of t minus k is phi of t translated by k units to the right if k is a positive integer I mean you can translated to the right or to the left.

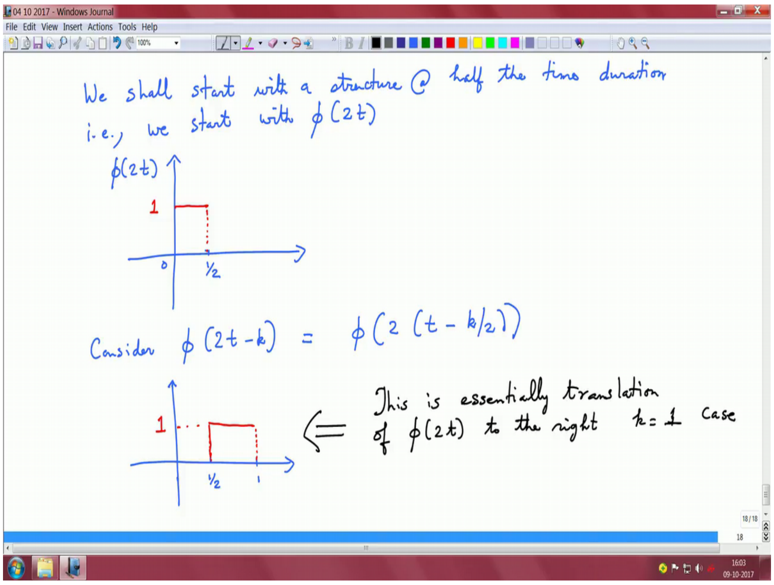
(Refer Slide Time: 18:56)



Now, once we get this picture of compact support and translations, now we are ready to define this space V 0; V 0 is a space of all functions of the form sigma a k phi of t minus k where k belongs to the set of integers and a ks, they are basically real numbers, now how can we picture this function, I think one of the ways we can picture these is as follows. I have time step minus 1 0 times, step 1 time, step 2 times step 3 time, step 4, 5, so on. Now, say a minus 1, this is say a 0, this is a 1, this is say a 2, this is say a 3, right, you observe these discontinuity points right so on and so forth.

So, this is like our a minus 1, this is a naught, this is a 1, this is a 2, this is a 3 dot dot, now of course, you can clearly observe the shift ortho orthonormality right, I mean if you take phi of t with phi of t minus 1 for this simple scaling function it is basically 0 because the support I mean is the overlap is essentially 0 between phi of t and phi of t minus 1, right. So, you now can appreciate the idea why you want to have time translates starting from a simple box care function.

(Refer Slide Time: 02:25)



Now, let us go a little further. So, we shall start with the structure at half the time duration that is we start with phi of 2 t. So, let us catch how this looks. Now phi of 2 t is 1 between 0.

And half and it is 0, otherwise right you plug in when t equals half, it is 1 t, you know plug this into the function. So, you can you can really see this; what is happening? So, therefore, if I have numbers that are greater than a fraction greater than 1 here as the coefficient of t and basically shrinking in time and if it is phi of t by 2, t by 4, t upon 8 and so on, I am basically stretching the function.

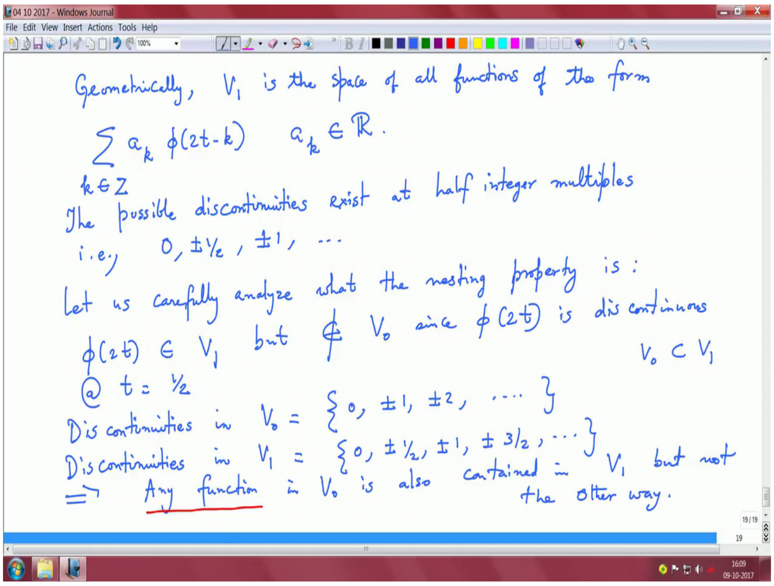
So, I can stretch or I can compress depending upon how I modulate the coefficient in front of t, right and again I can think about time translates of these. Now, let us see, this a little bit further, now consider phi of 70 t minus k. Now, this can be written as phi of 2 times t minus k upon 2, right.

So, what does this what does this mean now I have a phi of t which is basically I mean if I took take this function is basically free of 2 t which is k translated, right or if you write it in this form you start with original function V of t and basically you shrink it by a factor of 2 if you basically take a translation k; k by 2 k upon 2 steps to the right and followed by a shrinking this is what you are going to get. So, you should be able to interpret both of these starting from this function phi of t with translation and shrinking or directly sketching of phi of 2 t and then basically looking at a time translation right.

So, if you sketch this how does it look; let us suppose we choose k equals 1, this is how we will have observe the discontinuity point here, this is one indication, this is one this is essentially translation of phi of 2 t to the right for the k equals one case now we get this picture right.

So, you have a space 0, you have a space V 1, so on and so forth. Now V 0 is basically all leave V 0 the space of all functions which can be realized using the basis phi of t minus k right. So, and now V 1 would be basically using the basis phi of 2 t minus k right that that picture should become pretty clear for us.

(Refer Slide Time: 28:26)



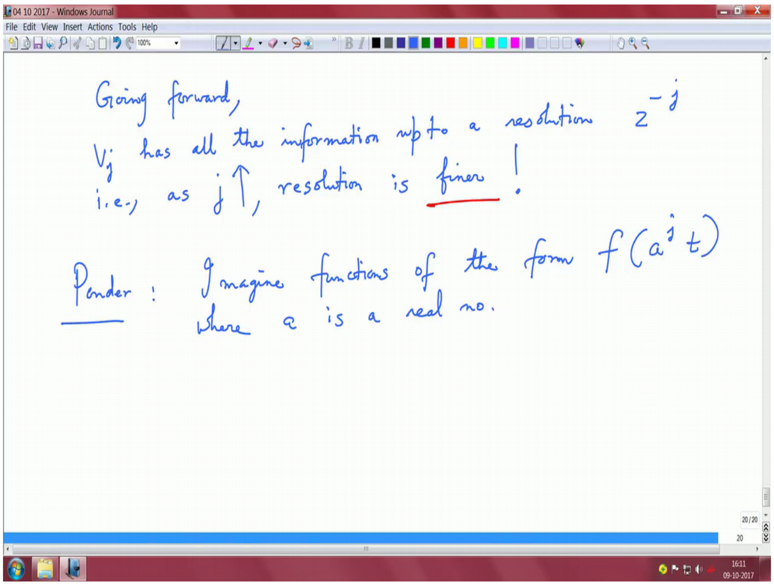
Now, geometrically V 1 is the space of all functions of the form summation k belonging to set of integers a k phi of 2 t minus k where a k is some real number.

Now you may wonder where the discontinuities are and the a discontinuities exists at half integer multiples right the possible this continuities exists at half integer multiples that is 0 plus minus half plus minus 1 dot dot dot. So, let us carefully analyze and see what this nesting property is ok; now a phi of 2 t belongs to V 1 , but does not belong to V 0, right, what is V 0? V 0 contains some signal which can be expanded using phi of t minus k where k is some integer set of all integers right negative and positive integers, but this function phi of 2 t which is basically a boxcar function which is 1 from 0 to half and 0, else can not be realized using V 0 ok. So, phi of 2 t belongs to V 1, but does not belong to V 0 since phi of 2 t is discontinuous at t equals half.

I mean if you think about it in the form of sets for these discontinuities I think it is much easier to picture this way discontinuity in V 0; they exists at 0 plus minus 1 plus minus 2 dot dot dot , but if you think about the discontinuity is in this phase V 1 that is at 0 plus minus half plus minus 1 plus minus 3 upon 2 and so on and so forth which means any function in V naught is also contained in V 1 , but not the other way, this is an important sort of observation that you might want to think about, right.

So, we can conveniently say that V 0 is contained in V 1, but not the other way round which is what we wrote in the very beginning as one of the properties that was that it was the first property or nesting of these subspaces V 0 is contained in V 1; this is what this is what it is now?

(Refer Slide Time: 34:21)

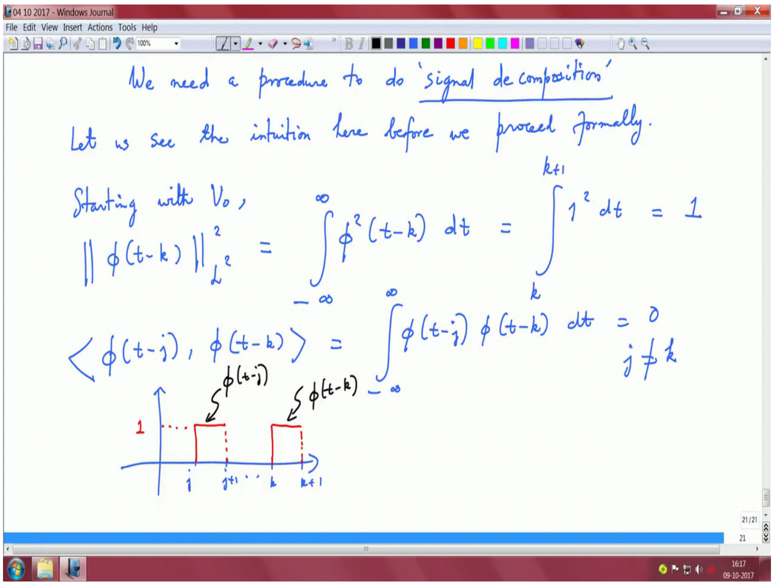


If we go forward V j has all the information up to a resolution 2 power minus j, right and as you increase j your resolution is becoming finer that is as j increases the resolution is finer and it is straightforward and quite intuitive imagine for you to imagine functions of the form f of some a power j times t where a is a real number.

Now, where is useful, suppose, I have a short burst of sometime, let us say, you know root 2 seconds or 1 upon root 2 or some very small time, right, I can figure out to which space it what it belongs to and I can go to that level of detail that I would want to in the appropriate space right rather than having the same bandwidth across all the filters or if you think about in the language here in terms of saying that you we give the same resolution for all the functions.

We say we give sort of a different resolution for different scales that is the whole idea and you have to get that picture very clear in your mind all phi of t minus case all phi of 2 t minus case all phi of 4 t minus case all phi of eight t minus case, so on and so forth, right, if you imagine this space, you can think about yeah I can go to whatever resolution that I want in whichever space and I can expand a function in that space and I have this nesting property one subspace is contained in the in the other.

(Refer Slide Time: 37:47)

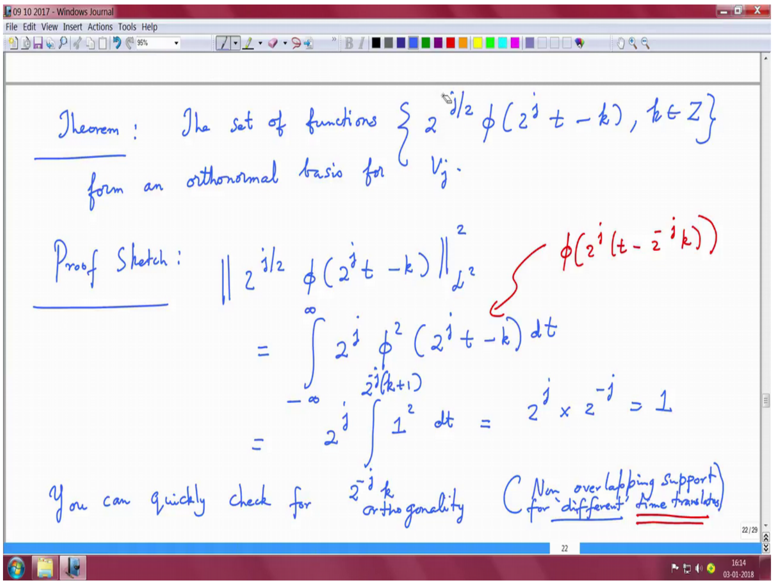


Now, let us go a little further we need a procedure to do signal decomposition. So, let us see the intuition here before we proceed formally I mean; if you think about history of mathematics at any point of time, first you have to have some intuitions some idea to start with that is a key thing and then you start building the theory rigorously formally, right, nobody starts with someone you have some definitions, if you know those definitions, it is great, but if you do not you start with an idea from scratch and then post that you start building things ok.

So, let us see this intuition here. So, let us start the displays V 0 if you look at the integral norm square of the integral norm of phi of t minus k well this can be written as integral minus infinity to plus infinity V square of t minus k d t which is basically phi of t minus k exists from k to k plus 1 and this is 1 1 square is 1 d t and this is basically one and if you take the inner product of phi of t minus j with phi of t minus k this is basically given by this integral which is 0 for j not equal to j equals k of course, it is one. So, how do you visualize this. So, you have some j some j plus 1 dot dot dot some k some k plus 1 and you have this ok.

Now, we have established normality, we have established orthogonality; it forms an orthonormal basis we have understood how these functions look with scaling it could be a stretch or it could be a shrink shrinkage.

(Refer Slide Time: 41:53)



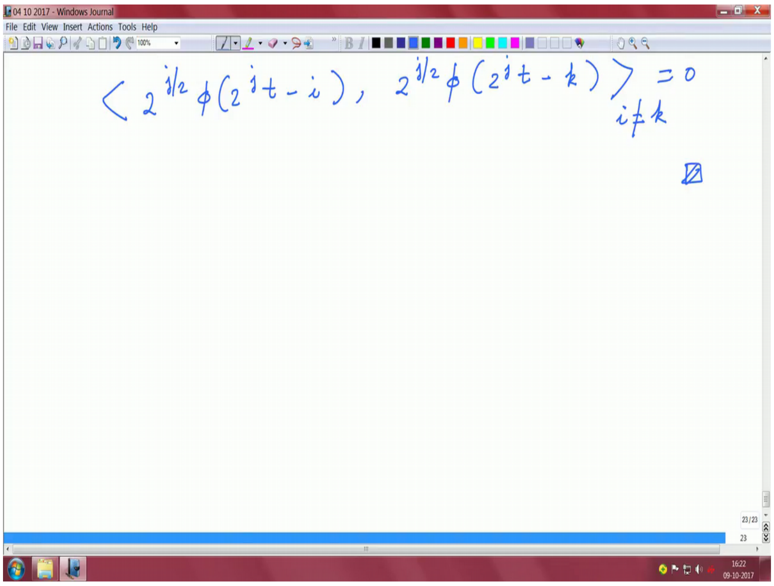
Now, we are ready to basically state a result which is formal and we state this as a theorem the set of functions 2 power minus j upon 2 phi of 2 power j times t minus k for k belonging to a set of integers form and orthonormal basis for V j space V j proof is pretty straightforward, I will just give you a sketch now you start with the norm of 2 power j upon 2 g of 2 power j t minus k the integral norm.

So, basically this is integral minus infinity to plus infinity you have 2 power j this j upon 2 the j upon 2 will be 2 power j then this is phi square of 2 power j times t minus k d t right and this is basically 2 power j pull out and the limits this is existing basically it you pull 2 power j out to get t minus 2 power minus j into k right. So, therefore, this exists from 2 power minus j times k to 2 power j times minus j times k plus one right.

So, there is 2 power minus j; you pull out k to k plus 1 and this is one one square d t and this is basically 2 power j times; this is 2 power minus j which is basically one and then you can quickly check for orthogonality that is to sum k one choose sum k k 2 and like what we did earlier right where we had phi of t minus j and phi of t minus k for k not equal to j you could you could just do the same thing except that now you have to shrinking functions or expanded stretched functions.

And they do not overlap in their support, right; it is very what 4 different time translates is very very important was a small type of here that I just saw right, now this negative there should not be a minus sign here, it is 2 power j upon 2 m because it has to integrate to one.

(Refer Slide Time: 46:12)



So, what it means; if you put in the formal language the inner product of 2 power j upon 2 phi of 2 power j t minus I take the inner product with 2 power j upon 2 phi of 2 power j times t minus some k equals 0 because i is not equal to k and they have disjoint supports and that sort of prove this result, we will stop here.