Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 54
Introduction to wavelets

So, let us get started with an Introduction to wavelet us. We studied filter banks and filter bank theory is intricately connected to the wavelet transform. And basically I will start with the motivation for wavelet us we will study the properties of wavelet us decompositions etcetera. And, then towards the end I will link the wavelet transformation to filter banks and then you will see the connection to multi rate signal processing and wavelet transforms and how one can think about these 2 as a sort of one is an engineering way of arriving at filter banks and there is a mathematical way of defining this object there is a called a wavelet.

And the scaling function and how one can think about basically realizing these operations these transformations as filter banks and then basically linking the filter banks to wavelet us we will see that towards the very end, but before we get into the details we need to start with the motivation to this. So, in the very first module we looked into signal representation basics and we said that the a signal is basically a point in a signal space.

So, as you can think about having basis vectors for representing a vector in a vector space, you can have basis functions for representing a function signal in a signal space right. And we did gram Schmidt orthogonalization process, we were able to construct basis functions for signals etcetera and we got an idea of signal geometry right. Now, we will sort of revisit that here, but with a little bit of twist in the way we think. So, the motivation is as follows.

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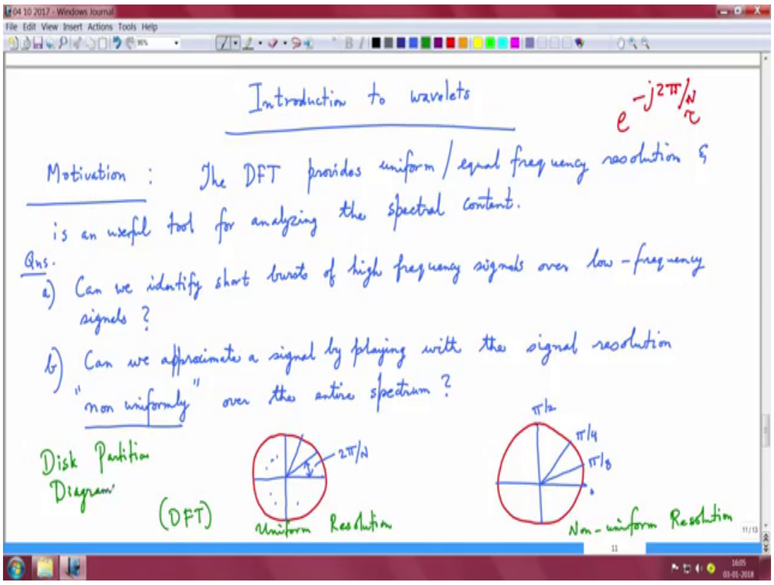

If, you think about the discrete Fourier transform the DFT provides uniform or equal frequency resolution, and is a useful tool for analyzing these spectral content. Some questions come about in practice and one of them is can we identify short bursts of high frequency signals that are riding over low frequency signals, very practical problem I just have some noise pike in my signal and can I identify this short burst of high frequency signal it is very practical.

Second problem can be approximate a signal by playing with the signal resolution "non-uniformly" over the entire spectrum?

Now, if we ask these questions and why would you want to have signal resolution non uniform over the entire spectrum. Like, if you think about the DFT as I mentioned it provides uniform frequency resolution and you have this e power minus j 2 pi by n for an N component DFT right; that means, you are placing the same resolution for all the frequencies.

So, I think before we wrap up with this slide I think it will be good to picture how non uniform resolution versus uniform resolution is in the disk partition diagram. So, let us draw these circles probably use a red one. So, this is assuming this is a 0 to 2 pi.

So, if you imagine the Fourier right you have uniform which; that means, the resolution is basically 2 pi upon N here right if this is whole 2 pi then 2 pi upon N is this cone. So, it has it places uniform emphasis for the frequency components and if N is really large basically you can resolve to that frequency, but the resolution is the same across all the frequencies, but in the case of wavelet us one can think about having resolutions, which are different for different frequencies. For example, I can split this in the band from 0 to pi by 2 to 0 to pi by 4 and pi by 4 to pi by 2. So, this is say 0 this is say pi by 4 this is pi by 2 then this cone is 0 to pi by 4 can be further split to pi upon 8 and so on.

So, therefore, there is a sort of non uniform emphasis for the frequency bands right and this is the distinction that we have to start with when we play with approximating signal with such non uniform signal resolutions across the spectrum. So, this is basically uniform resolution and this distinction should be made very clear in your head, this applies to the DFT and this is non uniform resolution and these diagrams are typically called disk partition diagrams.

But, let us imagine natural signals such as images speech etcetera. Take an example of an image very likely I mean you would first when you look at an object or a scene the first thing you see is an intensity integrated over that image you know, it is dark or bright image right, you just get an average information.

First, then maybe you segment you zoom into a particular object of your interest you go further deeper you just imagine a snapshot of a screen, you have a nice you know maybe you have a mountain, you have a river, you have some trees, etcetera. It is very unlikely somebody will start looking at what specific tree it is? What color the branch is? What the shape of the leaf is etcetera.

Though such details are visible in the portrait likely, you will see the snapshot and say yeah this is a natural scenery perhaps that is the first information that you want to grab grab out. Then maybe you look at probably the sky color is blue or a certain shade of blue is a next level of detail or how the valley is, what the water is, what the tree is then the leaf on the tree right?

So, you as you go through the decomposition of the scene right you place different sort of a different emphasis of the signal in different regions or certain frequencies you I mean what are these frequencies. In the image I mean these are basically the changes that happen right I mean, if you look at a transition between you know a an edge is a good example of a transition and that is a detail which is basically a high pass component.

So, first is you will get an average information then you will look at the high pass filtered image, which is the high pass filtered image you might want to go a further step down or in the low pass filtered image you want to go a further step down right. From average or an average then details and then again further details so on and so forth. So, there is a certain kind of resolution that you want to bring in as you decompose the signal. And, the DFT does not do that and it is probably not a good tool for decomposing a signal keeping in mind the non uniform resolution that you want to bring into different frequency bands.

And people in the bell half and the elsewhere were interested in compression for such signals way back in 70's 1970s and 80s. So, if I want to court the signal why should I give the emphasis, of having the same number of bits across all the frequency bands I am wasting my bandwidth I am wasting my power having the same representation across all the frequency bands.

So, that automatically led into the birth of tools transformations and techniques that placed non uniform resist that emphasis I mean at emphasized frequency bands with different signal resolutions, non uniformly across the entire spectrum. For efficient, representation, coding, compression etcetera as well as perhaps for denoising in certain applications such as case a where we perhaps would not identify such short bursts of high frequency signals riding over low frequency signal ok.

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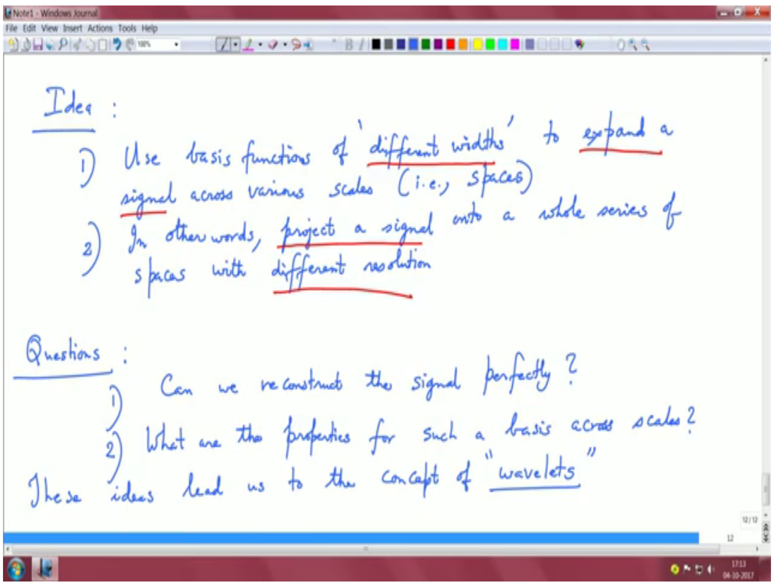

Now this in mind what is the idea? Central idea, well we know that if we were to expand a signal we would need to do this using a basis y basis, because we are invoking the framework of linear algebra into this you may question well I can also approximate signal using some non-linear components etcetera, but that is probably difficult or the mathematics could be formidable if you were to think about it in a non-linear space. So, basically it is easier to bring in linear algebraic framework to the signal space. So, therefore, we want to expand the signal using basis functions.

So, now, we use basis functions of different widths to expand a signal across various scales. The scales I would even call them as spaces I think these things will become very clear, when we get into more formal definition of what this scale means and you know interchangeably what space it corresponds to.

So, what we mean in other words project a signal onto a whole series of spaces signal spaces with different resolution, this is very important different resolution basis functions of different widths expanding a signal right and or possibly projecting a signal onto whole series of spaces. So, these are sort of ideas. Now with this idea in mind we may get a few questions into our mind, the first question is can we reconstruct the signal perfectly?

Second, what are the properties for such a basis across scales? And these ideas lead us to the concept of “wavelet us” right. These are important questions to ask can we reconstruct the signal perfectly if it is not perfect, what is the approximation error in what sense right, it is also very important in l 2 or l 1 or what sense? What are the properties for such a basis etcetera right? If you ask the right kind of questions that automatically leads us into devising this transformation that we need, which is basically the wavelet transformation.

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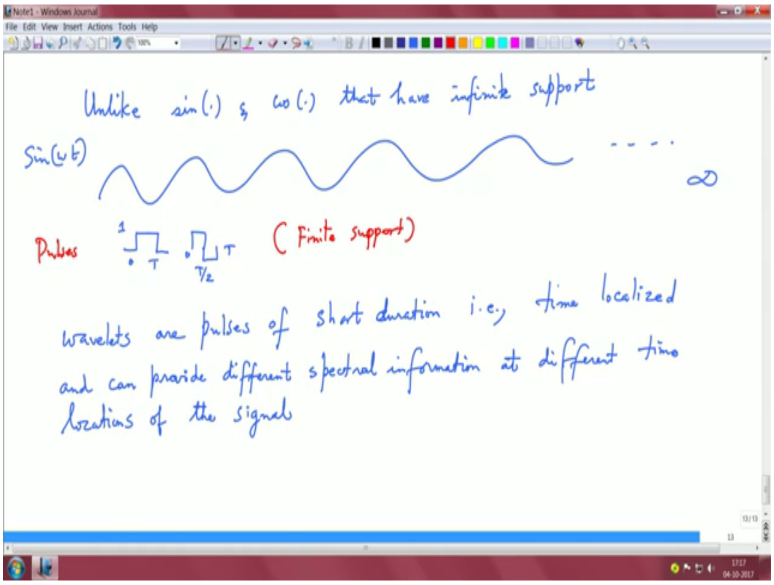

Now, a few things to note unlike sine and cos that have infinite support so, what do you mean by support right I mean if you take a sin function, this is sin omega T and it goes on. So, what is the support of the signal this is really infinity right. I do not want infinite support. For example, if I were to expand this I mean if I were to use a basis perhaps like this. So, this is and we see that is these are orthogonal signals this esteem up t upon 2 right, over the interval 0 to T both of these signals are orthogonal to each other because the pi here cancels, this you take inner product of this and this is 0 right, but now these are also basis in some signal space, but they are finite finitely supported and it did not go to infinity right.

So, unlike sin and cosine that have infinite support wavelet us are pulses of short duration, that is they are time localized and can provide different spectral information at different time locations of the signal.

So, now the concept is very clear to us we need different resolution at different scales and we want signal to be expanded using a set of basis functions, that have finite support, whose widths are different across different scales right. And they have to be sort of providing you some time localization property such that you can get different spectral information at different times. Now, if you put all these I would say qualifying statements they are not quantifications yet because this is just what is desired?

And if we take all these qualifying statements and then you have to embed them into the background from Hilbert spaces, that will give us the transformation that we need ok. So, but before we del delve into the details we have to understand something called the multi resolution property and that I will try to cover in the next lecture ok. So, this introduces let us we will stop here.