Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 51
Perfect reconstruction of signals

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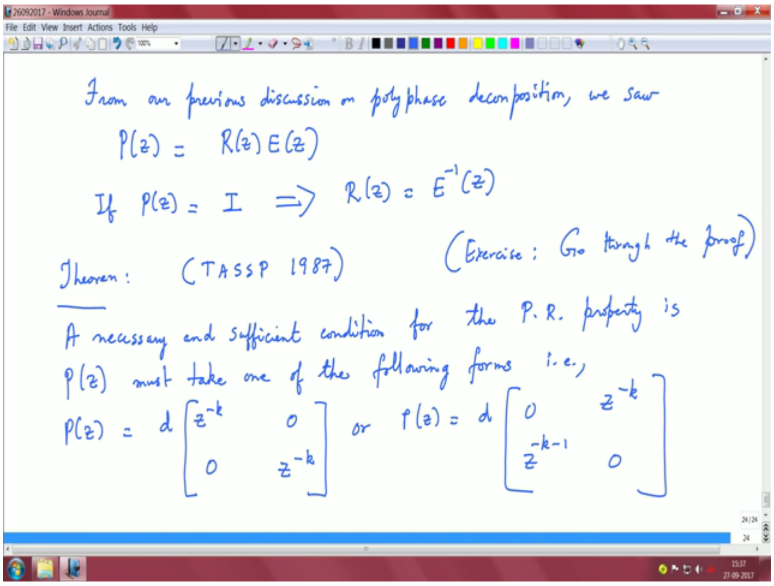

From our previous discussion on polyphase decomposition, we saw that P of z must be equal to R of z times E of z, where R of z is the matrix of all the type two components for the synthesis filters and E of z are the analysis filters. And we saw that, if P of z equals identity right, then R of z is E inverse of z.

Now, one of the questions that we have to ask is; are there necessary and sufficient conditions for perfect reconstruction property to be holding and if; so what should be the nature of P of z for a simple two channel filter bank and then we can we can extend this. So, this result was actually proved by P P Y Jonathan I think it is P P Y Jonathan and this appears in the transactions on acoustics, speech and signal processing in 1987.

So, I will just state the theorem; a necessary and sufficient condition for the perfect reconstruction property is P of z must take one of the following forms, that is P of z is some scalar d times is a delay here 0, 0 a delay or P of z equals 0 z power minus k, z power minus k minus 1 and 0. So, the proof is given in the paper, that you can refer through refer into and I will leave this as an exercise.

Now, this has some interesting consequences; that means, I have a sort of a structure I have a structure, that I have to impose on P of z right and this should be of this form and then, if we were to look into how E of z and R of z have to be are the restructures for E of z and R of z ok. So, let us start with the polyphase representation.

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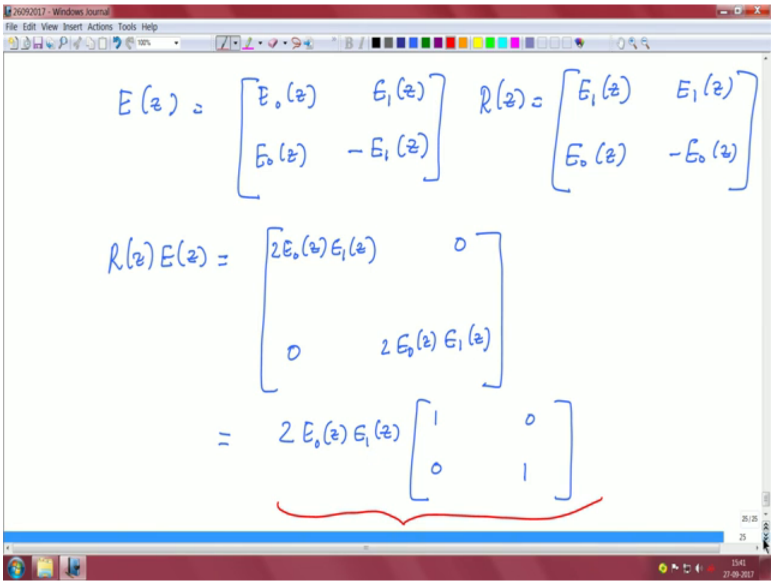

Now, we have this E of z matrix in the form E naught z, E 1 of z, and then this is E naught of z minus E 1 of z right. We saw how this came about, because if you have H naught and H 1 as a column stack; that was the matrix 1, 1 in the first row; 1 minus 1 in the second row times the components. Similarly we have R of z to be of the form, if we impose this structure, then we look at R of z times E of z that gives us this matrix to E naught of z, E 1 of z, 0, 0 to E naught of z, E 1 of z right.

This is just rewriting starting with the polyphase decomposition on the analysis bank and the synthesis bank and then writing what R of z times E of z has to be. So, this can be written in the form 2 times E naught of z, E 1 of z times an identity matrix ok. So, why do we need to do this algebra here? So, well we want ideally R of z to be an identity matrix, which we may not get right; it or it is too restrictive.

So, if I do not force it to be identity; let me have the QMF imposition on this structure, this is quadrature mirror property. Therefore, I can synthesize my analysis bank and synthesis bank and I can come up with a polyphase representation and from this I can simplify what R of z times E of z has to be and that can be expressed in terms of the polyphase components. If you closely observe this equation here, which is basically 2 times E naught of z E 1 of z times an identity matrix; we have a little more freedom in choice of E naught of z and E 1 of z; than just an identity matrix right.

So, one question that you may get is; a system is one question that you might get in get into your mind is how can we ensure that this is perfect reconstruction; that means, this E naught times E 1 should be some scale R times a delay right.

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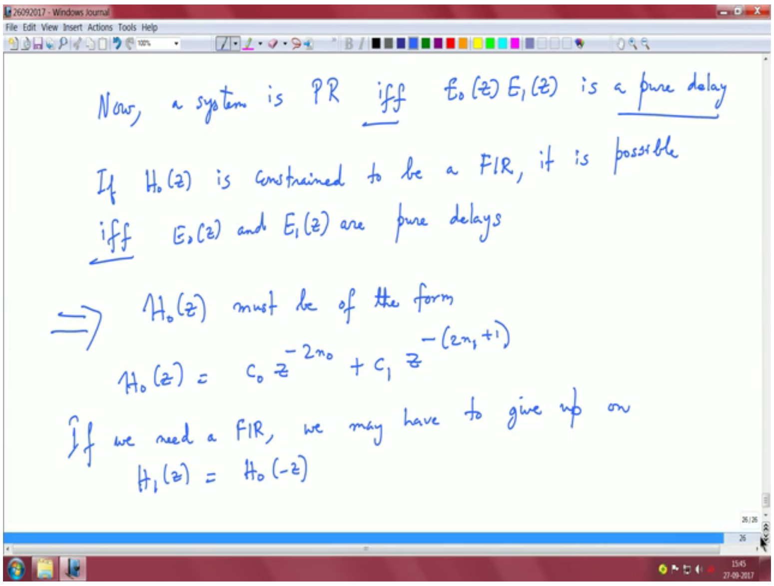

So, that is that is the inclusion and I think we start with that; now, the system satisfies perfect reconstruction property, if and only if E naught of z times E 1 of z is a pure delay. So, if H naught of z is constrained to be a finite impulse response filter it is possible, if and only if E naught of z and E 1 of z or pure delays; well sorted I want E naught times z E naught of z times even of z to be a pure delay.

So, if I constrain H naught to be FIR right. So, I mean if it is not FIR then I can think that I could cancel out somehow right; E naught can have a denominator polynomial and that denominator polynomial can be somehow factored into one of the factors in E 1 of z in the numerator. So, that is the reason why we are forcing it to be FIR. If it is FIR, then the components E naught of z and E 1 of z have to be individually delays right.

So, therefore, this implies; that H naught of z must be of the form H naught of z equals some C naught z power minus two times n naught plus C 1 times z power minus of 2 n 2 n plus 1, where n 1 and n 2 are n naught and n 1 are just some integers right. So, we are given the they have given the broad form here it can be even or it can be odd.

Now, if we need a FIR; we may have to give up on the condition H 1 of z is H naught of minus z, our filter banks allow us to do that right. I mean it is ideally easy for us to have one base filter H naught of z from, which we can get H 1 of z and from which we can synthesize the other filters.

But if we did not want to have a quadrature mirror property to get the high pass filter and I want to restrict it still to be FIR for the base filter, then it is difficult to get possibly an FIR filter, which satisfies this property right I mean you can plug in a minus z here and you will see what the effect is.

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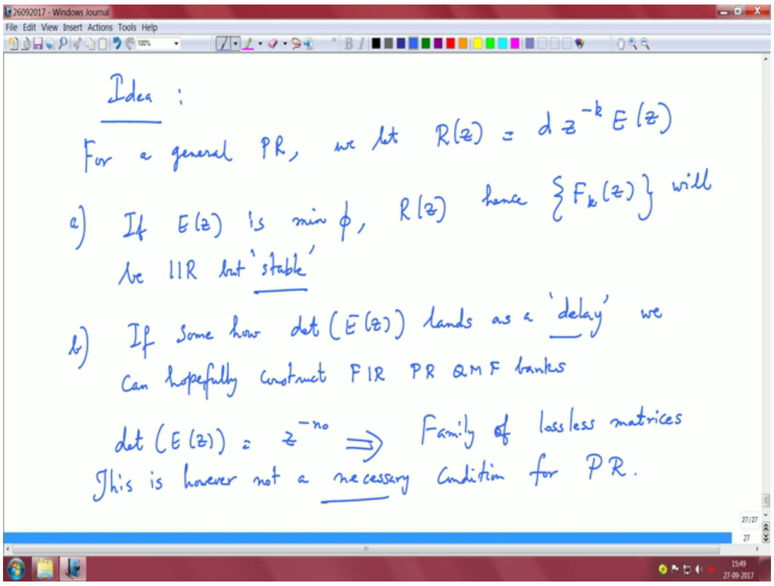

Now, the idea in general is as follows. For a general perfect reconstruction, we let R of z to be some d z power minus k, E of z we will let it in this form. So, you may wonder why this form I mean there; that means, E R of z times E dagger of; so E is basically E inverse in some sense can you think about having such self inverses in polynomials. If E of z is minimum phase a minimum phase filter is one in which the poles and zeros are inside the unit circle.

So, if it is minimum phase, then R of z hence the synthesis filters F k of z will be IIR, but stable right, because if you just invert them invert it the zeros will become the poles and the poles will be 0; so therefore, stability is ensured. The second point is if somehow determinant of E of z lands as a delay, we can hopefully construct FIR perfect reconstruction quadrature mirror filter banks.

So, basically we are looking at a family of lossless matrices that that can give us perfect deconstruction. So, determinant of E of z equals some delay put certain implied, that we have a family of lossless matrices this is; however, not a necessary condition for perfect reconstruction ok. So, the idea is very simple.

So, R of z times E of z is P of z, and if we let P of z is some z C z power minus n naught some delay; R of z is this delay with a gain with some E inverse. And the E inverse is 1 over determinant of E or z time the adjoint, and if the adjoint somehow works out to be of the form of E of z itself; then you know we are home for good right such matrices, which have this property are good.

Now, let us consider an example for the design of such synthesis filters small thing to note this has to be any inverse here before we wrap up this slide just a detail, because R times E of z must be in basically d times a delay. So, therefore, this has to be E inverse.

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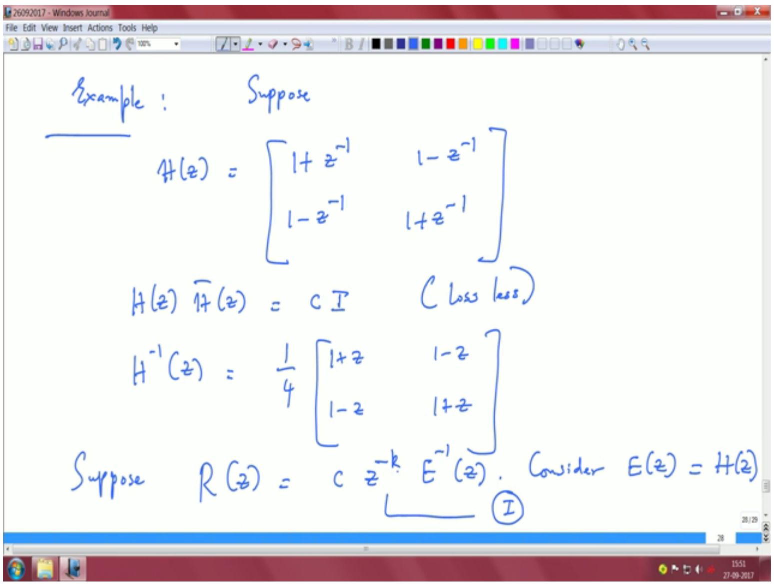

Suppose H of z is 1 plus z power minus 1, 1 minus z power minus 1, 1 minus z power minus 1, 1 plus z power minus 1 as shown. Now, H of z times H bar of z should be some

C times identity for lossless conditions, if we expand H inverse of z this is routine algebra you could do this it is 1 by 4, 1 plus z, 1 minus z, 1 minus z and 1 plus 1 plus z.

Now, suppose R of z is C times z power minus k times some E inverse of z right; where k is some delay that we can choose; so that system is also causal right you want to ensure the system is causal for certain reasons the gain is in your control. So, let us see what happens; when E of z is basically H of z right; that is u z is H of z.

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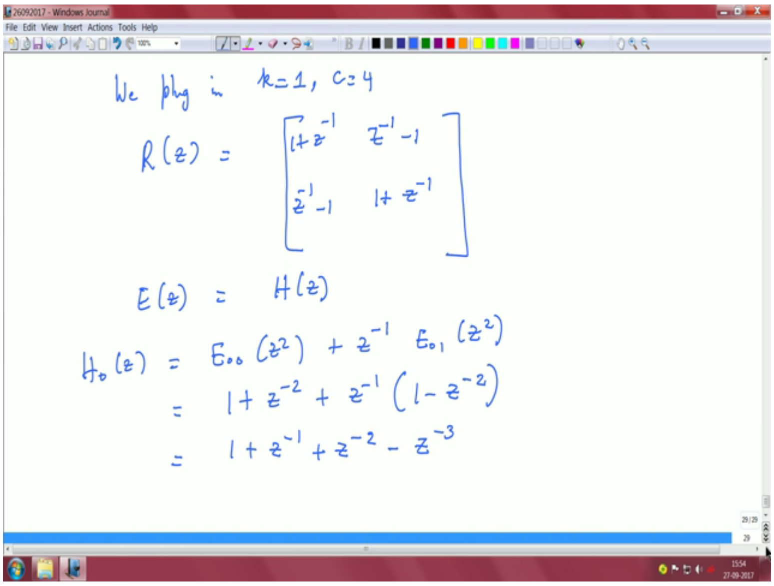

Now, we plug in k equals 1; let me call this equation say 1. In this equation I plug in k equals 1, I can choose whichever delay I want to choose it is an art right. Let me plug in k equals 1, and I choose a factor C appropriately right, then I can have R of z a is basically you multiply H of z by this delay right. So, basically it z power minus 1 times this z power minus 1 times this turn so on and so forth right.

So, I do this I get 1 plus z power minus 1, z power minus 1 minus 1, z power minus 1 minus 1, 1 plus z power minus 1. Now, I know the structure of E of z right, I know the structure of E of z, which is basically H of z right. So, just recall from your previous worksheet here E of z is H of z and I am I am rewriting it again. So, my H naught of z the first polyphase component is E naught naught z square plus z power minus 1 E naught from z square.

And here I think one thing that you have to notice C is chosen to be 4, I have just taken I just ignored it, but I think it is implied that I take C equals 4 k equals 1 and C equals 4 I get this. So, you do the math this comes out to be 1 plus z power minus 2 plus delay times 1 minus z power minus 2; simplify this is 1 plus z power minus 1 plus z power minus 2 minus z power minus 3 ok.

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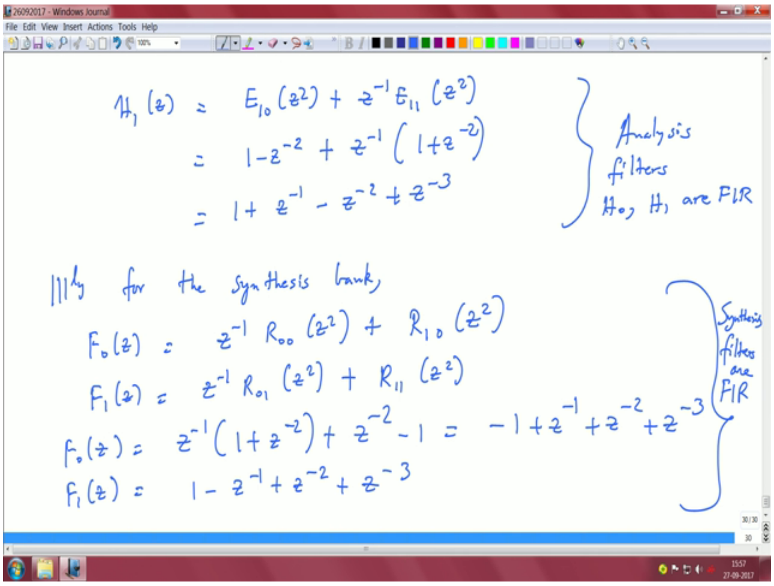

Similarly, we have H 1 of z, which is E 1 0 z square plus z power minus 1, E 1 1 of z square, which is 1 minus z power minus 2 plus z power minus 1 times 1 plus z power minus 2, which is equal to 1 plus z power minus 1 minus z power minus 2 plus z power minus 3 and analysis filters H naught H 1 are indeed F I R.

Now, we can do this for the synthesis bank. So, we get F naught z is z power minus 1 R naught naught z square plus R 1 0 z square, and F 1 of z is z power minus 1 R naught 1 z square plus R 11 z square; and you plug in you get F naught of z is z power minus 1 times 1 plus z power minus 2 plus z is power minus 2 minus 1, which is minus 1 plus z power minus 1 plus z power minus 2; that is z power minus 3 and F 1 of z is you do the math you will land up with this equation and we still have same this is filters to be FIR ok.

This is still good for alias cancellation, because F naught of z is H 1 of minus z and F 1 of z is minus H naught of minus z; and then this is still good for alias cancellation. So, it is not necessarily true that you have to start with quadrature mirror property you can relax that, if you want to have your choice of F naught and F 1 based upon what you would choose for your H naught and H 1 right; you have a choice that you could make for your high pass filter H 1, but ideally it is preferred, if we can get you know a one base filter and from which we can synthesize the rest of the filters else; if we want to keep the design a little more flexible then we have to basically have two different filters in the analysis bank and we can have similar filters in the synthesis bank ok.

And then of course, you could analyze the system for phase distortion, amplitude distortion and other artifacts ok. This we are done with perfect reconstruction property looking, we looked into the two channel filter bank, we looked at the generalization to m channel filter banks.

We saw how polyphase decomposition is very useful for this purpose and we also have ways in which we can construct FIR filters that satisfy perfect reconstruction property and this is this design is going to be very useful when you then you have to design filter banks and that that work with over sample or you know systems, which have assimilators and expanders along with filters ok.

So, we will discuss in the next class, special filters and some of the properties. And these filters would be very useful for designing these filter banks. So, there will be additional constraints we can end this lecture.