Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 49
M-channel filter banks

So, now that we have studied 2 channel filter bank, we can generalize this in a natural way to an arbitrary M-channel filter bank. So, the idea is very simple instead of two filters we have M such filters stacked, and they form the analysis bank if their analysis filters.

And these analysis filters will be followed by down sampling by M if you have M-channel filter banks then you will have up sampling by M to restore the sampling rate, and then you have synthesis filters that you combine all these are band signals and then you get the reconstructed signal at the output. So, now, this is the idea. So, let us sketch the schematic first.

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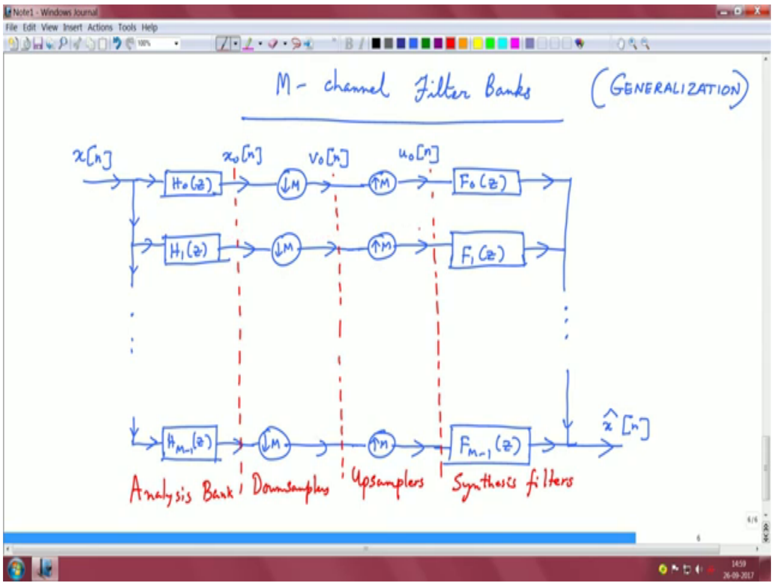

So, we have x of n, we have H naught of Z, this interim point is x naught of n, I have down sample by M. Let us say the signal here is v naught of n and remember at this stage you can do whatever you want for quantization compression etcetera. But if you did not do any quantization and compression we will just feed this signal to an up sampler by M the interim signal is v know v u naught of n post the up sampler followed by filtering through the synthesis filter F naught of Z. So, this is one branch. So, you can imagine M such branches.

We do this, similarly we have H 1 of Z you have a down sampler, I have an up sampler filtered through F 1. We have H M minus 1 of Z, a down sampler and up sampler is synthesis filter F M minus 1 of Z. It combine all the outputs and I get x hat of n which is the reconstructed signal. So, this is the analysis bank of filter filters, these are the down samplers, these are the up samplers, and these are our synthesis filters and this is this is collection of synthesis filters forms the synthesis bank and this forms the analysis bank, ok.

Now, I think with this picture in mind and you can think about interim signal if this is x naught this should be x 1 here this should be v 1 u 1 correspondingly so on. So, I think you can you can get the picture what is really happening here. So, let us define some notations that we will use towards the analysis.

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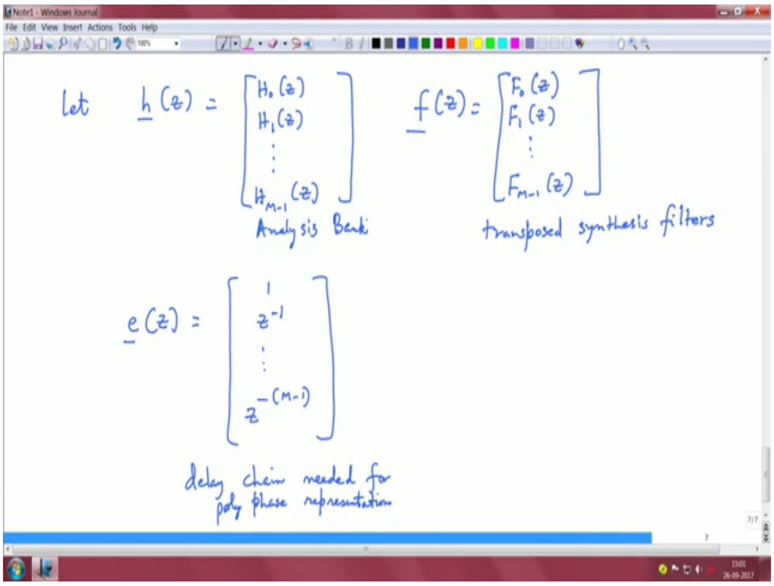

Let h of Z be a column vector comprising of the analysis bank, this is H naught of Z, H 1 of Z dot dot dot H M minus 1 of c this is a column vector similarly we have f of Z is a stack of the transposed synthesis filters.

And you may wonder again why we want to make a distinction of this transposition here is again type 1 and type 2 representation within the polyphase decomposition that that would play pivotal role in the way we would look at the structure for the analysis and the synthesis bank of filters. So, let e of Z be a delay chain which is required for polyphase and that is one a unit delay dot dot dot a delay of M minus 1 units this is a delay chain needed for polyphase representation, ok.

Now, with this we are sort of set. So, all we have look at is the Z transform of the interim variables x naught of n x 1 and so on and so forth, and then express it in a compact form, towards the analysis of what the overall transfer function is and how we can cancel the alias and if you want remove the alias what we need to do.

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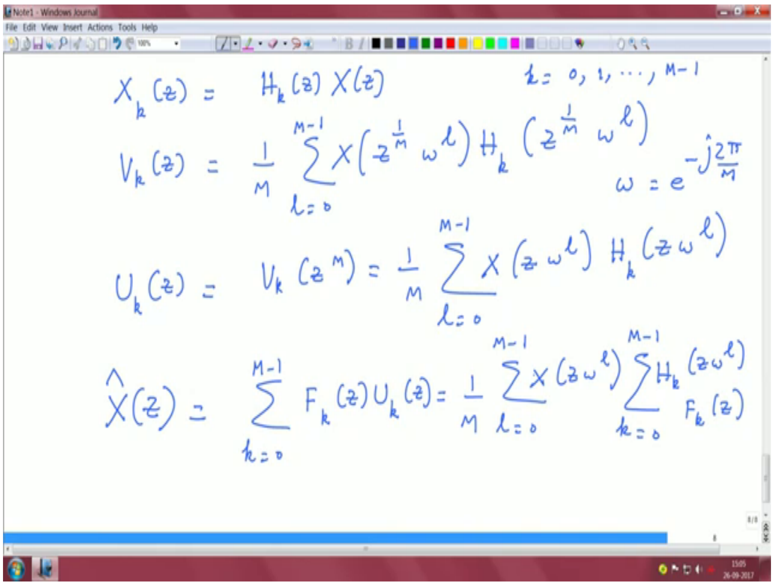

So, we start with the usual analysis that we are comfortable X k of Z is H k of Z times X of Z and of course, this is k equals 0 1 dot dot dot till M minus 1 right. If you recall for a 2 channel filter bank we had just k equals 0 and 1 just two components here we have M components 0 to M minus 1. So, therefore, X k of Z is this for k equals 0 to M minus 1.

Then we have V k of Z which is the output after the down sampler right and we know the equation for the output of the down sampler, and have conveniently ignored the subscript M here in omega and this is basically power minus j 2 pi upon M. Then we have U k which is V k of Z power M post expansion and you can write this as 1 upon M sum l equals 0 to M minus 1 X of Z omega omega l H k of Z omega power n. So, this step is also straightforward.

Now, we can get the relationship the expression for X hat of Z which is basically you are taking all the sub and signal post the up sampling process and filtering them through the bank of synthesis filters. So, this is summation k equals 0 to M minus 1 F k of Z, U k of Z and if you simplify this you get 1 upon M summa l equals 0 to M minus 1. So, we will put the U k here plug this in X of Z omega power l and summation k equals 0 to M minus 1 H k of Z omega power l with F k of Z right. The k equals 0 is what we want to bring in and as U V just express it out using this equation right and this is this X does not have any k dependency.

So, you pull this out and you can simplify this in this form, ok. Now, we can rewrite this messy equation slightly in a in a form that is presentable.

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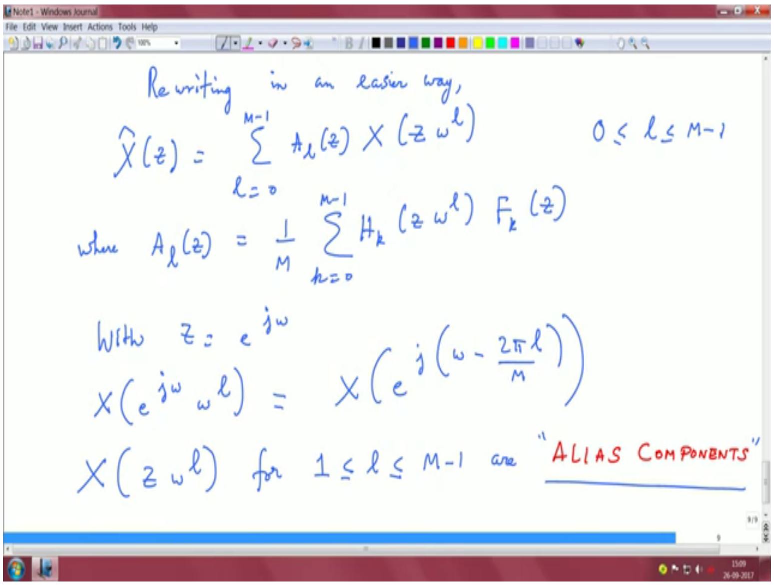

Rewriting in an easier way X hat of Z is summa l equals 0 to M minus 1, A l of Z times X Z omega power l, where A l of Z equals 1 upon M summa k equals 0 to M minus 1 H k of Z omega power l F k of Z for all the M components, ok.

Now, with Z equals e power j omega, X of e power j omega x times we take this omega power l and that is for all l from 0 to M minus 1 this is basically a shifted version of the spectrum which is basically translated version of the base spectrum at points 2 pi l upon M ok. So, l equals 0 1 2 3 so on till M minus 1 you plug this thing here and you get the translations of the spectrum that are uniformly shifted.

Now, these components X of Z omega power l for l equals 1 to M minus 1 are alias components right, X of Z is what we want and the rest x of Z omega power l for l from 1 to M minus 1 are alias components. It is exactly like what we had for the two channel bank, where when l equals 1 with M equals 2 we had the minus 1 that was appearing before Z right. And that was our alias component, this is just generalized version.

Of course, a little bit of detail before we wrap up with the slide, do not just confuse with the omega that is here which is a frequency, and this w which is basically the m-th route of unity, right. This is a small detail. So, I think you can.

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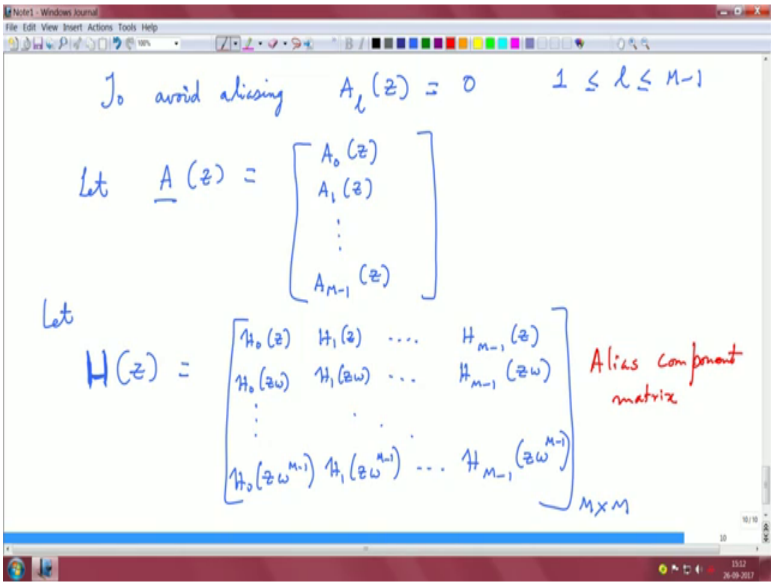

Now, to avoid or aliasing; to avoid aliasing A l of Z has to be 0 for all these components; l equals 1 to M minus 1 ok. So, let A of Z be a stack of all polynomials in Z starting with the unaliased component with the rest of the aliased components. Let H of Z is really bold matrix, be a stack of matrices given as follows: H naught of Z H 1 of Z dot dot dot H M minus 1 of Z, then you have H naught of Z omega H 1 of Z omega dot dot H M minus 1 of Z omega. So, you have H naught Z omega power M minus 1 Z omega power M minus 1.

So, this is our M by M Alias Component Matrix or our AC matrix. So, we had the AC matrix for a two channel filter bank which we had this two components, right. So, it was a two by two matrix. Now, generalizing this we have an M by M matrix which is the alias component matrix.

Now, if you look into this carefully, I mean if you want to avoid aliasing you have to set all the a i's i equals one to M minus 1 to 0, right. So, that is what you want to force. And therefore, your overall distortion function would be just a naught which would be your analyst version ok. So, we will go one step further.

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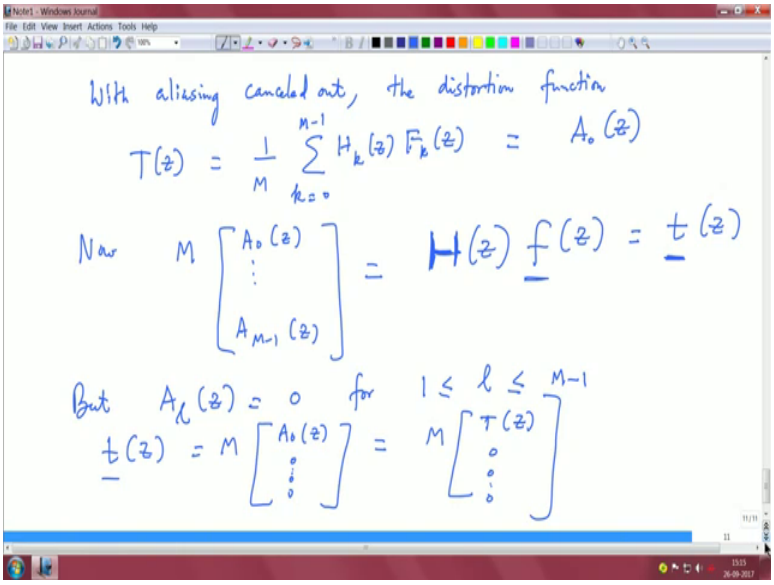

With aliasing cancelled out the distortion function t of Z is 1 upon M summa k equals 0 to M minus 1 H k of Z F k of Z; and we call this as a naught Z.

Now, M times this A of Z which is a naught Z through A M minus 1 of Z is essentially this alias component matrix H of Z times F of c. Right, you just rewriting this in a matrix form and we call this sum t of z. But, A l of Z is 0 because we want to remove the alias components for one less than or equal to l less than or equal to M minus 1. So, since this has to be forced to 0 our t of Z is essentially of the form M times a naught of Z the rest being 0s, and this is M times t of Z which is your overall distortion function with alias; aliasing cancelled out and the rest all being 0s, ok.

Now, with this in place we can write down the expression for X hat of Z for your reconstructed signal.

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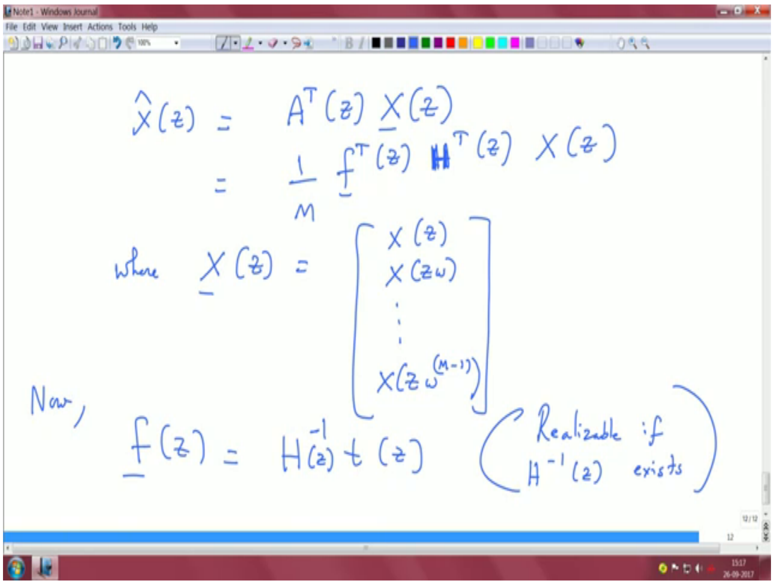

X hat of Z is basically a transpose Z times X of Z which is basically 1 upon M F transpose of Z H transpose of Z X of Z. Where, X of Z is our usual vector which has the analyst component of the signal with all the earliest versions. Now, F of Z is H inverse t of Z, because we said H of Z with F of Z is t of Z which is our transfer function, right. If you start here H F is t is what we start with. So therefore, the synthesis filters can be obtained the knowledge of the overall transfer function t of Z that we want with the alias component matrix, ok.

Now, this is realizable. So, at one shot we are getting all the synthesis filters this is in vector form: F naught F 1 F 2 so on till F M minus 1 we get all these interests filters at one shot. So, that is the reason why we have adopted the matrix form of representation. You may question. Now, I want to get F of z; that means H has to be invertible, right it is realizable. If H inverse exists, right. Otherwise we have difficulties.

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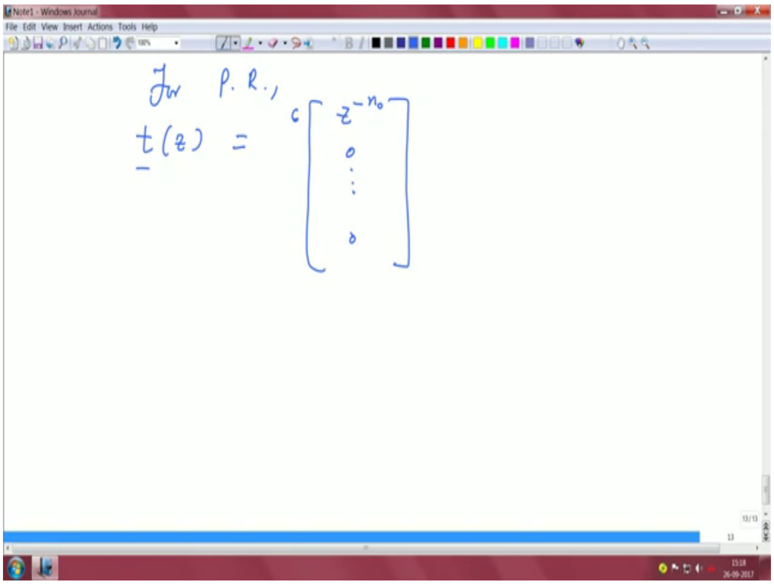

Now, for perfect reconstruction we force t of Z to take this form Z power minus n naught rest all being 0s if you want you can put a scalar here it is ok, if you want to bring in some scalar c, but we want it to basically be a delay with some gain and the rest of the components are all 0s.

Now, you may wonder at the moment what are the difficulties with the AC inversion matrix right, we said the synthesis filters can be realized by inverting the alias component matrix and that is a critical step and there would be difficulties if we cannot invert this. So, we will we will study some of the issues related to the inversion.

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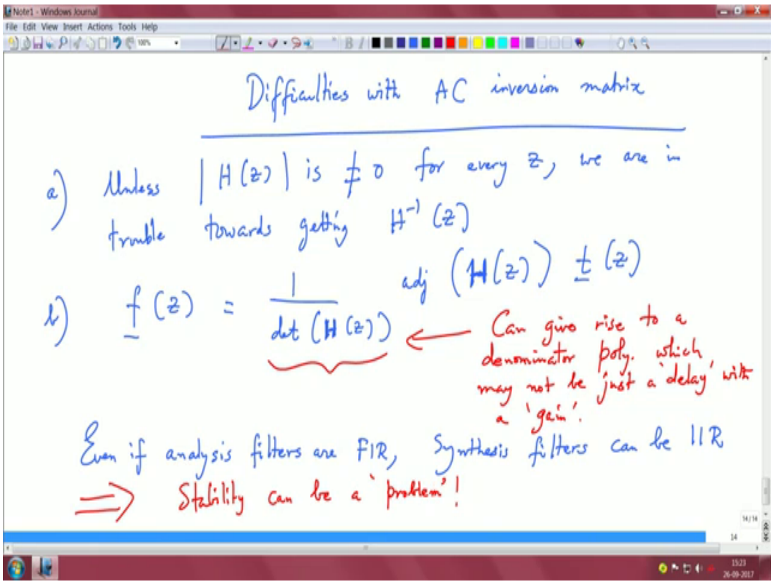

Unless the magnitude of H of Z is not equal to 0 for every Z we are in trouble towards getting H inverse of Z, right. I mean because if you think of realizing H of Z as magnitude of H of Z times some E power j phase and if you have inverted you are having a 1 over the magnitude of H of Z in the denominator right and that should not be 0.

Second we have f of Z that is the synthesis filters can be realized as one over determinant of H of Z adjoint of H of Z times t of Z. Now, you see the difficulty here. If the analysis filters are FIR synthesis filters can become IIR because we have a denominator polynomial that can occur because of the determinant. Can give rise to a denominator polynomial which may not be just a delay with a gain, ok. So, if you have to delay and a gain then there is no issue, but you may not guarantee the determinant to be just a delay with a gain. So, therefore, even if analysis filters are FIR synthesis filters can be IIR right.

And now, we know the trouble with IIR filters, round of noise stability and lot of these issues are there right and when you do quantization, even a small shift in the pole position right can alter your response significantly or you can drive the system to instability right. This is this is these are very very important considerations. I think as an offshoot of this I mentioned to you that this implies that stability can be a problem.

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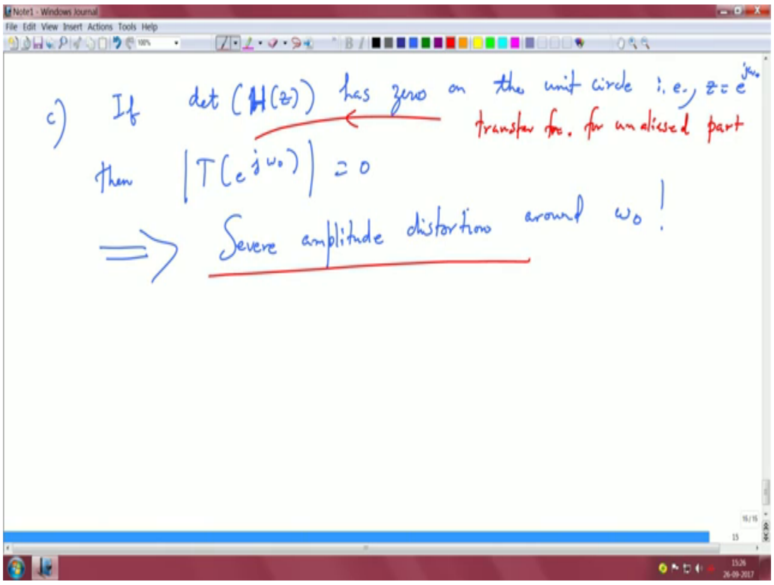

Then there is one other issue, if determinant of H of Z has 0s on the unit circle that is Z equals some e j omega naught then magnitude of t of e j omega naught E 0 right; if it has a 0 on the unit circle then magnitude of this t e j omega naught that becomes 0 which means severe amplitude distortion around omega naught.

Now, if T of Z, capital T of Z is the transfer function that that we saw right that is for the unaliased part right if you look at the unaliased part that is the transfer function. So, I think just to recall this is for unaliased part this is function ok, so that is an issue. If the magnitude of T of e j omega naught equal 0 at those positions omega naught right, those frequencies that that in this implies that you would have severe amplitude distortion around that frequency.

So, then you can imagine what should be the order of the filters etcetera and perhaps the issues that you will have to deal with then noise gets amplified around those frequencies etcetera. So, these are some practical issues that you will have to deal with.

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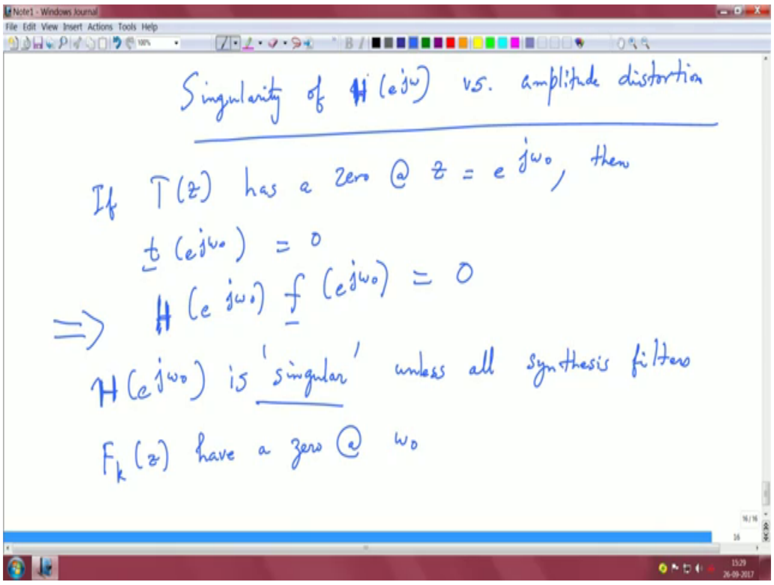

So, with this let us carefully look into the singularity issues in H of e j omega and linking it to the amplitude distortion. So, we saw that if T of Z has a 0 at Z equals e power j omega naught then t of e j omega naught equal 0, but this t is related to H and F because we can say that this is H of e j omega naught times F of e j omega naught and that is t of e j omega naught and that is essentially 0 which means H of e j omega naught is singular unless all synthesis filters F k of Z have a 0 at omega naught right.

And t of e j omega naught equals 0 in an alias free system and this implies that there is a singularity of the AC matrix at omega equals omega naught. So, this is what really happens. So, if you want the t of e j omega naught to be equal to 0 right then this would imply that this expression H of e j omega naught with F of e j omega naught is equal to 0 and we know that F is basically a transpose stack of all the synthesis filters and this is singular unless all synthesis filters have a 0 at this frequency right. And this is the implication of amplitude distortion which links the singularity in the alias component matrix, ok.

So, with this we are sort of set we have derived and an equation an expression for the reconstructed signal for an M-channel filter bank. So, we have analyzed how the distortion function looks, and the matrix formulation has been very convenient for us right by looking at the alias component matrix, then all the components in the signal the analyst components and the alias components, and the synthesis filters we can link all the quantities together towards a overall distortion function. And if you force the alias conditions to be 0 then you land up with the unaliased distortion function and then we see what is the relationship if you know we have to get realizable synthesis filters, and what are the issues when we have to consider difficulties with the AC inversion matrix ok. So, these are all important considerations.

So, hopefully with these set of tools we are set for analyzing an M-channel filter bank. So, the idea is again very simple we start with the base filter which is the mother filter H naught of Z and then we can realize all the synthesis filters I mean for AC cancellation as well as the analysis filters v r translation of the base filter, ok.

So, with this we complete this part and then we are ready to start with the polyphase representation of the M-channel filter bank.